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THE INFLUENCE OF $\mathscr{H}$-SUBGROUPS ON p-NILPOTENCY AND p-SUPERSOLVABILITY OF FINITE GROUPS

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Abstract. Let G be a finite group. A subgroup H of G is an $\mathscr{H}$-subgroup in G if $N_G(H) \cap H^g \leq H$ for any $g \in G$. In this article, by using the concept of $\mathscr{H}$-subgroups, we study the influence of the intersection of $O^p(G_p^*)$ and the members of some fixed $\mathcal{M}_d(P)$ on the structure of the group G, where P is a Sylow p-subgroup of G. Some new criteria for a group to be p-nilpotent and p-supersolvable are given and some recent results are extended and generalized.

1. Introduction

Throughout this article, all groups are finite. Let G be a group and p a prime. Let $O^p(G)$ be the unique smallest normal subgroup of G for which the corresponding factor group is a p-group. Following Berkovich and Isaacs in [2], we let G_p^* be the unique smallest normal subgroup of G for which the corresponding factor group is abelian of exponent dividing $p-1$. A group G is called p-supersolvable if every chief factor of G either has order p or it has order coprime to p. It is known that G is p-supersolvable if and only if G_p^* is p-nilpotent (see [2, Lemma 3.6]). To state our results, we need to recall some concepts.

Goldschmidt, Flores and Foote in [5, 6, 7, 8] studied the following concept.

Keywords: $\mathscr{H}$-subgroup, weakly $\mathscr{H}$-subgroup, p-supersolvablility, p-nilpotency.

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Definition 1.1. *Let $K \leq G$. A subgroup H of K is called strongly closed in K with respect to G if for every element h of H, $h^G \cap K \subseteq H$, where h^G denotes the G-conjugacy class of h. In particular, when H is a subgroup of prime-power order and K is a Sylow subgroup containing it, H is simply said to be a strongly closed subgroup.*

This concept plays an important role in group theory and representation theory, particularly in the case when K is a Sylow subgroup of G. If H is a p-subgroup, then H is strongly closed in a Sylow p-subgroup if and only if it is strongly closed in $N_G(P)$, so this concept for a p-subgroup does not depend on the Sylow subgroup containing it. In 2000, Bianchi et al. in [3] introduced the definition of $\mathscr{H}$-subgroups which is related to strongly closed subgroups. A subgroup H of a group G is an $\mathscr{H}$-subgroup in G if $N_G(H) \cap H^g \leq H$ for any $g \in G$. Notice that if H is a subgroup of G of prime-power order, then H is an $\mathscr{H}$-subgroup in G if and only if H is a strongly closed subgroup. Asaad et al. in [1] introduced the following subgroup embedding property called weakly $\mathscr{H}$-subgroup, which is a generalization of $\mathscr{H}$-subgroup. A subgroup H of a group G is called a weakly $\mathscr{H}$-subgroup in G if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N \in \mathscr{H}(G)$, where $\mathscr{H}(G)$ denotes the set of all $\mathscr{H}$-subgroups in G. In addition, they proved the following result.

Theorem 1.2. [1] *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If all maximal subgroups of P are weakly $\mathscr{H}$-subgroups in G, then G is p-nilpotent.*

In general, a weakly $\mathscr{H}$-subgroup in G is not always an $\mathscr{H}$-subgroup. However, in Lemma 2.5, we can show that if H is a p-subgroup of G, then $H \cap O^p(G_p^*)$ is an $\mathscr{H}$-subgroup in G if and only if $H \cap O^p(G_p^*)$ is a weakly $\mathscr{H}$-subgroup in G. In Lemma 2.6, we prove that if H is a maximal subgroup of a Sylow p-subgroup of G, then $H \cap O^p(G)$ is an $\mathscr{H}$-subgroup in G if and only if H is a weakly $\mathscr{H}$-subgroup in G.

Let P be a nontrivial finite p-group and $\mathcal{M}(P)$ be the set of all maximal subgroups of P. Li et al. in [12] introduced a subset of $\mathcal{M}(P)$ as follows: Let d be the smallest generator number of a p-group P and $\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$ be a set of maximal subgroups of P such that $\Phi(P) = \bigcap_{i=1}^{d} P_i$. For a fixed P, $\mathcal{M}_d(P)$ is not unique in general. We know that $|\mathcal{M}(P)| = \frac{p^d-1}{p-1}$ and $|\mathcal{M}_d(P)| = d$. Hence, it is easy to see that $\mathcal{M}_d(P)$ is a very small subset of $\mathcal{M}(P)$ while d is large enough. In [13], Shen et al. obtained the following criteria of p-nilpotency by using the set $\mathcal{M}_d(P)$ and the concept of $\mathscr{H}$-subgroups.

Theorem 1.3. [13] *Let p be a prime dividing the order of a group G, and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and every member of some fixed $\mathcal{M}_d(P)$ is an $\mathscr{H}$-subgroup of G.*

Theorem 1.4. [13] *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and let P be a Sylow p-subgroup of G. Then the following statements are equivalent:*

(1) *G is p-nilpotent.*

(2) *For every member of $\mathcal{M}(P)$ is an $\mathscr{H}$-subgroup of G.*

(3) *For every member of some fixed $\mathcal{M}_d(P)$ is an $\mathscr{H}$-subgroup of G.*

Our goal in this article is to improve and extend the results we mentioned above. Our basic idea is from several recent articles (for example, see [10, 15, 16, 17]) where p-supersolvability or p-nilpotency of a group G is obtained by assuming that some p-subgroups of $O^p(G)$ or $O^p(G_p^*)$ satisfy some embedding property in G. By using the concept of $\mathscr{H}$-subgroups, we investigate the influence of the intersection of $O^p(G_p^*)$ and the members of some fixed $\mathcal{M}_d(P)$ on the structure of the group G, where P is a Sylow p-subgroup of G. We obtain the following theorems.

Theorem A. *Let G be a p-solvable group and P be a Sylow p-subgroup of G, where p is a prime divisor of the order of G. Then G is p-supersolvable if and only if for every member P_i of some fixed $\mathcal{M}_d(P)$, $P_i \cap O^p(G_p^*)$ is an $\mathscr{H}$-subgroup in G.*

Theorem B. *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. Then the following statements are equivalent:*

(1) *G is p-nilpotent.*

(2) *$N_G(P)$ is p-nilpotent and for every member P_i of some fixed $\mathcal{M}_d(P)$, P_i is a weakly $\mathscr{H}$-subgroup in G.*

(3) *$N_G(P)$ is p-nilpotent and for every member P_i of some fixed $\mathcal{M}_d(P)$, $P_i \cap O^p(G_p^*)$ is an $\mathscr{H}$-subgroup in G.*

Theorem C. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and let P be a Sylow p-subgroup of G. Then the following statements are equivalent:*

(1) *G is p-nilpotent.*

(2) *For every member P_i of some fixed $\mathcal{M}_d(P)$, P_i is a weakly $\mathscr{H}$-subgroup of G.*

(3) *For every member P_i of some fixed $\mathcal{M}_d(P)$, $P_i \cap O^p(G_p^*) = P_i \cap O^p(G)$ is an $\mathscr{H}$-subgroup in G.*

2. Preliminaries

Lemma 2.1. *Let G be a group and let H, K, N be subgroups of G satisfying $H \in \mathscr{H}(G)$, $H \leq K$, and N is normal in G. Then the following hold:*

(1) *$H \in \mathscr{H}(K)$;*

(2) *If P is a Sylow p-subgroup of G, then $P \cap N \in \mathscr{H}(G)$;*

(3) *If H is subnormal in K, then H is normal in K;*

(4) *If $N \leq H$, then $H \in \mathscr{H}(G)$ if and only if $H/N \in \mathscr{H}(G/N)$;*

(5) *If H is a p-subgroup of G such that $(|H|, |N|) = 1$, then $HN \in \mathscr{H}(G)$and $HN/N \in \mathscr{H}(G/N)$.*

(6) *If P is a Sylow p-subgroup of N, then $P \in \mathscr{H}(G)$.*

Proof. For (1)-(5), see [1, Lemma 2.1]. For (6), see [3, Proposition 3(2)]. □

Lemma 2.2. [15, Lemma 2.9][16, Lemma 2.4] *Let p be a prime dividing the order of a finite group G, $H \leq G$ and $N \trianglelefteq G$. Then $O^p(H_p^*) \leq O^p(G_p^*), O^p(H) \leq O^p(G)$, $(G/N)_p^* = G_p^*N/N, O^p(G/N) = O^p(G)N/N$ and $O^p((G/N)_p^*) = O^p(G_p^*)N/N$.*

Lemma 2.3. *Let P be a Sylow p-subgroup of G for a prime p. If H and K are strongly closed in P with respect to G, then $H \cap K$ is strongly closed in P with respect to G.*

Proof. Let $a \in H \cap K$. By assumption, $a^G \cap P \subseteq H$ and $a^G \cap P \subseteq K$. Hence, $a^G \cap P \subseteq H \cap K$. It follows that $H \cap K$ is strongly closed in P with respect to G. □

Lemma 2.4. *Let P be a Sylow p-subgroup of G for a prime p and let N be a normal subgroup of G. If $H \leq P$ is an $\mathscr{H}$-subgroup in G, then $H \cap N \in \mathscr{H}(G)$.*

Proof. Notice that if H is a p-subgroup of G, then H is an $\mathscr{H}$-subgroup in G if and only if H is a strongly closed subgroup. By Lemma 2.1(2), $P \cap N \in \mathscr{H}(G)$ and so $P \cap N$ is strongly closed subgroup. By Lemma 2.3, $H \cap N = (P \cap N) \cap H$ is strongly closed subgroup and hence $H \cap N \in \mathscr{H}(G)$, as wanted. □

By the above lemma, if a p-subgroup H of G is an $\mathscr{H}$-subgroup in G, then $H \cap O^p(G_p^*)$ is an $\mathscr{H}$-subgroup in G. But the converse is not true. Consider $G = S_4$ and its 2-subgroup $H = \langle(13)\rangle$ for instance. It is clear that $G_2^* = G$ and so $O^2(G_2^*) = A_4$. Hence, $H \cap O^2(G_2^*) = 1$ is an $\mathscr{H}$-subgroup in G. Let P be a Sylow 2-subgroup of G containing H. If $H \in \mathscr{H}(G)$, then by Lemma 2.1(1)(3), H is normal in P. This is a contradiction. Hence, H is not an $\mathscr{H}$-subgroup in G. However, we have the following:

Lemma 2.5. *Let p be a prime dividing the order of a finite group G and H be a p-subgroup of G. Then $H \cap O^p(G_p^*)$ is an $\mathscr{H}$-subgroup in G if and only if $H \cap O^p(G_p^*)$ is a weakly $\mathscr{H}$-subgroup in G.*

Proof. We only need to prove the sufficiency. Assume that $H \cap O^p(G_p^*)$ is a weakly $\mathscr{H}$-subgroup in G. Then by definition, there is a normal subgroup N of G such that $G = (H \cap O^p(G_p^*))N$ and $H \cap O^p(G_p^*) \cap N$ is an $\mathscr{H}$-subgroup in G. Notice that $G/N \cong H \cap O^p(G_p^*)/H \cap O^p(G_p^*) \cap N$ is a p-group. It follows that $O^p(G) \leq N$. By Lemma 2.2, we have $O^p(G_p^*) \leq O^p(G)$. Since $O^p(G_p^*) \trianglelefteq G$, by Lemma 2.4, $H \cap O^p(G_p^*) = H \cap O^p(G_p^*) \cap N$ is an $\mathscr{H}$-subgroup in G, as wanted. □

Lemma 2.6. *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. If H is a maximal subgroup of P, then $H \cap O^p(G)$ is an $\mathscr{H}$-subgroup in G if and only if H is a weakly $\mathscr{H}$-subgroup in G.*

Proof. If H is a weakly $\mathscr{H}$-subgroup in G, then there exists a normal subgroup N of G such that $G = HN$ and $H \cap N$ is an $\mathscr{H}$-subgroup in G. Notice that $O^p(G) \leq N$. Then by Lemma 2.4, $H \cap O^p(G)$ is an $\mathscr{H}$-subgroup in G. Now assume that $H \cap O^p(G)$ is an $\mathscr{H}$-subgroup of G. Consider $M = HO^p(G)$. If $G = M$, then by definition, H is a weakly $\mathscr{H}$-subgroup in G. If $M < G$, then it is clear that

$|G : M| = p$ and H is a Sylow p-subgroup of M. Notice that $M \trianglelefteq G$. Hence, H is an $\mathscr{H}$-subgroup in G by Lemma 2.1(6). Clearly, H is a weakly $\mathscr{H}$-subgroup in G. □

Lemma 2.7. [15, Lemma 2.8] *Let p be a prime dividing the order of a finite group G and H be a p-subgroup of G. Let $L \trianglelefteq G$ and N be a normal p'-subgroup of G. Then $HN/N \cap LN/N = (H \cap L)N/N$.*

3. Main results

Proof of Theorem A. Suppose that G is p-supersolvable. Then G_p^* is p-nilpotent and so G_p^* has a normal subgroup K such that $G_p^* = P \ltimes K$ and $P \cap K = 1$. It is clear that $K = O^p(G_p^*)$. Hence, for any $P_i \in \mathcal{M}_d(P)$, $P_i \cap O^p(G_p^*) = 1$ is an $\mathscr{H}$-subgroup in G.

Conversely, suppose that the theorem is not true and let G be a counterexample with minimal order. Set $\mathcal{M}_d(P) = \{P_1, P_2, \ldots, P_d\}$. We will obtain a contradiction by the following steps.

Step 1. $O_{p'}(G) = 1$.

Assume $O_{p'}(G) > 1$ and consider $G/O_{p'}(G)$. By Lemma 2.1(5), Lemma 2.2 and Lemma 2.7, $G/O_{p'}(G)$ satisfies the hypothesis of this theorem. By the minimal choice of G, we have that $G/O_{p'}(G)$ is p-supersolvable, and thus G is p-supersolvable, which is a contradiction. Hence, $O_{p'}(G) = 1$.

Step 2. Let $M = O^p(G_p^*)$ and $N = O_p(M)$. Then $N > 1$ and $C_M(N) \leq N$.

Since G is not p-supersolvable, it follows that G_p^* is not p-nilpotent. If $M = 1$, then G_p^* is a p-group, a contradiction. Hence, $M > 1$. Since $1 < M \leq G$ is p-solvable and $O_{p'}(M) \leq O_{p'}(G) = 1$, we have $N > 1$ and $C_M(N) \leq N$ by Hall-Higman's Lemma (see [11, Theorem 3.21]).

Step 3. $G_p^* \leq C_G(N)$.

Notice that for each $i \in \{1, \ldots, d\}$, $P_i \cap O^p(G_p^*)$ is an $\mathscr{H}$-subgroup in G. Since $N \leq O^p(G_p^*)$, by Lemma 2.4, for each i, $P_i \cap N = (P_i \cap O^p(G_p^*)) \cap N \in \mathscr{H}(G)$. Since $P_i \cap N \trianglelefteq \trianglelefteq N \trianglelefteq G$, by Lemma 2.1(3), we have that $P_i \cap N \trianglelefteq G$. Hence, $\Phi(P) \cap N = \bigcap_{i=1}^{d}(P_i \cap N) \trianglelefteq G$.

Let $L = \Phi(P) \cap N$. If $L > 1$, then it is clear that G/L satisfies the hypothesis of this theorem. Hence, G/L is p-supersolvable. As $L \trianglelefteq G$ and $L \leq \Phi(P)$, we have $L \leq \Phi(G)$. Hence, $G/\Phi(G)$ is p-supersolvable. Since the class of p-supersolvable groups is a saturated formation, it follows that G is p-supersolvable, a contradiction. Thus, $L = 1$.

If $N \leq P_i$, then $[N, G_p^*] \leq N = P_i \cap N$. Assume now $N \not\leq P_i$. Then $P = P_i N$ and so $|P : P_i| = |N : P_i \cap N| = p$. Consider $\overline{G} = G/P_i \cap N$. Notice that $\overline{G}/C_{\overline{G}}(\overline{N})$ is isomorphic to a subgroup of $\mathrm{Aut}(\overline{N}) \cong C_{p-1}$. It follows that $(\overline{G})_p^* \leq C_{\overline{G}}(\overline{N})$. By Lemma 2.2, we have $(\overline{G})_p^* = \overline{G_p^*}$ and so $[G_p^*, N] \leq P_i \cap N$. Hence, $[N, G_p^*] \leq P_i \cap N$ for each i. It follows that $[N, G_p^*] \leq \Phi(P) \cap N = 1$ and so $G_p^* \leq C_G(N)$.

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Step 4. Final contradiction.

By Step 2, $C_M(N) \leq N$. Notice that $M \leq G_p^*$. Then by Step 3, $M \leq C_M(N)$ and hence $N = M$. Since $O^p(G_p^*)/N = O^p(G_p^*)/M \cong O^p(G_p^*)O^p(G)/O^p(G) \leq G/O^p(G)$ and N are p-groups, it follows that $O^p(G_p^*)$ is a p-group and so is G_p^*, a contradiction. □

Proof of Theorem B. Clearly, we only need to show that (3) implies (1). We always set $\mathcal{M}_d(P) = \{P_1, P_2, \dots, P_d\}$. Suppose that it is not true and let G be a counterexample with minimal order. We work in the following steps to obtain a contradiction.

Step 1. $O_{p'}(G) = 1$.

Assume that $O_{p'}(G) > 1$. Consider $\overline{G} = G/O_{p'}(G)$. Notice that $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ is p-nilpotent. Then by Lemma 2.1(5), Lemma 2.2 and Lemma 2.7, $G/O_{p'}(G)$ satisfies the hypotheses of this theorem. By the minimal choice of G, we have that $G/O_{p'}(G)$ is p-nilpotent, and thus G is p-nilpotent, which is a contradiction. Hence, $O_{p'}(G) = 1$.

Step 2. If $P \leq H < G$, then H is p-nilpotent.

Noting that $N_H(P) \leq N_G(P)$, we have $N_H(P)$ is p-nilpotent. By Lemma 2.1(1), $P_i \cap O^p(G_p^*) \in \mathscr{H}(H)$ for each i. It is clear that $O^p(H_p^*) \trianglelefteq H$ and $O^p(H_p^*) \leq O^p(G_p^*)$ by Lemma 2.2. Thus, $P_i \cap O^p(H_p^*) \in \mathscr{H}(H)$ by Lemma 2.4. Hence, by induction, H is p-nilpotent.

Step 3. G is p-solvable.

Since G is not p-nilpotent, by a result of Thompson [14, Corollary], there exists a non-trivial characteristic subgroup T of P such that $N_G(T)$ is not p-nilpotent. Choose T such that the order of T is as large as possible. Since $N_G(P)$ is p-nilpotent, $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P satisfying $T < K \leq P$. Notice that T char $P \trianglelefteq N_G(P)$. Then $T \trianglelefteq N_G(P)$ and so $N_G(P) \leq N_G(T)$. By Step 2, $N_G(T) = G$ and hence $T \leq O_p(G)$. By the choice of T, we have $T = O_p(G)$. Now applying the result of Thompson again, we have that $G/O_p(G)$ is p-nilpotent, therefore G is p-solvable.

Step 4. Final contradiction.

By Theorem A, G is p-supersolvable. In particular, G is p-solvable with p-length 1. By Step 1, we have $P \trianglelefteq G$ and so $G = N_G(P)$ is p-nilpotent, a contradiction. □

Proof of Theorem C. We only need to show that (3) implies (1). Since $(|G|, p-1) = 1$, we have that $G_p^* = G$ and $O^p(G_p^*) = O^p(G)$. It is easy to see that the hypotheses are inherited by $N_G(P)$. If $N_G(P) < G$, then $N_G(P)$ is p-nilpotent by induction. It follows from Theorem B that G is p-nilpotent, as wanted. Assume that $N_G(P) = G$. Then $P \trianglelefteq G$ and so G is p-solvable. By Theorem A, we have that G is p-supersolvable. Hence, $G = G_p^*$ is p-nilpotent, as wanted. □

4. **Applications**

Recently, Chen et al in [4] proved the following theorem.

Theorem 4.1. [4] *Let G be a group and let P be a Sylow p-subgroup of G, where p is the smallest prime dividing the order of G. If all maximal subgroups of P are weakly $\mathscr{H}$-subgroups in $N_G(P)$ and P' is normal in G, then G is p-nilpotent.*

In view of Theorem A, Theorem B and Theorem C, we extend the above theorem and obtain the following result.

Corollary 4.2. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and $P \in \mathrm{Syl}_p(G)$. If for every member P_i of some fixed $\mathcal{M}_d(P)$, $P_i \cap O^p(N_G(P))$ is an $\mathscr{H}$-subgroup in $N_G(P)$ and P' is normal in G, then G is p-nilpotent.*

Proof. By Theorem C, we have that $N_G(P)$ is p-nilpotent and so $N_G(P) = P \times O_{p'}(N_G(P)) = P \times O^p(N_G(P))$. Hence, $P_i \cap O^p(N_G(P)) = 1$. If $P' = 1$, then $C_G(P) = N_G(P)$. By Burnside's theorem ([9, Theorem 7.4.3]), G is p-nilpotent. Now assume $P' > 1$. Consider $\overline{G} = G/P'$. Clearly, $P' \leq \Phi(P)$ and $O^p(N_{\overline{G}}(\overline{P}))$ is a p'-group. Then $\overline{P_i} \cap O^p(N_{\overline{G}}(\overline{P})) = 1$ is an $\mathscr{H}$-subgroup of $N_{\overline{G}}(\overline{P})$. It follows that $\overline{G}'$ satisfies the hypotheses. By induction, G/P' is p-nilpotent. Notice that $P' \leq \Phi(P)$ and $P' \trianglelefteq G$. Then $P' \leq \Phi(G)$ and so $G/\Phi(G)$ is p-nilpotent. It follows that G is p-nilpotent. □

Corollary 4.3. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and $P \in \mathrm{Syl}_p(G)$. If for every member P_i of some fixed $\mathcal{M}_d(P)$, P_i is a weakly $\mathscr{H}$-subgroup in $N_G(P)$ and P' is normal in G, then G is p-nilpotent.*

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