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NEW CONSTRUCTIONS OF DEZA DIGRAPHS

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Abstract. Deza digraphs were introduced in 2003 by Zhang and Wang as directed graph version of Deza graphs, that also generalize the notion of directed strongly regular graphs. In this paper, we give several new constructions of Deza digraphs. Further, we introduce twin and Siamese twin (directed) Deza graphs and construct several examples. Finally, we study a variation of directed Deza graphs and provide a construction from finite fields.

1. Introduction

Deza graphs were introduced in [5], as a generalization of strongly regular graphs. For recent results on Deza graphs we refer the readers to [9, 10, 18]. Further, Duval in [4] introduced directed strongly regular graphs as directed graph version of strongly regular graphs. A type of digraphs which are close to directed strongly regular graphs, called normally regular digraphs, was introduced by Jørgensen in [16] and later studied in [17]. Deza digraphs, a directed graph version of Deza graphs, were introduced by Zhang and Wang in 2003 (see [26]). A different version of Deza digraphs was considered in [24, 25]. In this paper, we consider Deza digraphs in a sense of the definition given by Zhang and Wang in [26]. We give several new constructions of Deza digraphs. Further, we introduce twin and Siamese twin (directed) Deza graphs and construct several examples. Finally, we consider a variation of directed Deza graphs, called directed Deza graphs of type II, and provide a construction from finite fields. Directed Deza graphs of type II are related to a version of Deza digraphs introduced in [24]. Directed

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Deza graphs of type II include divisible design digraphs (see [2]). Note that a divisible design digraph is not necessarily a directed Deza graph.

Let $D = (V, A)$ be a directed graph (digraph), where V is the set of vertices and A is the set of arcs (directed edges). For $u, v \in V$ we will write $u \to v$ if there is an arc from u to v, and we will say that u dominates v or that v is dominated by u. We will also write $u \sim v$ if $u \to v$ and $v \to u$. In this case, we will count these as one undirected edge, and say that u is adjacent to v. A digraph D is called regular of degree k if each vertex of D dominates exactly k vertices and is dominated by exactly k vertices. The digraphs that we use will not have more than one arc from one vertex to another. Further, the digraphs will not have any arcs from a vertex to itself, except of the reflexive (di)graphs studied in Section 4. For a vertex u, the number of vertices that dominates u is called the in-degree of u, and the number of vertices that are dominated by u is called the out-degree of u.

A digraph D on n vertices is characterized by the $n \times n$ $(0,1)$-matrix $M = [m_{i,j}]$, where $m_{ij} = 1$ if and only if $i \to j$ (or $i \sim j$), called the adjacency matrix of D. If the adjacency matrix M of a digraph D has the property that $M + M^t$ is a $(0,1)$-matrix, then D is called asymmetric. If M satisfies the conditions

$$MJ_n = J_nM = kJ_n,$$

$$M^2 = tI_n + \lambda M + \mu(J_n - I_n - M),$$

where I_n is the identity matrix of order n, and J_n is the $n \times n$ matrix of all 1's, then D is called a directed strongly regular graph (DSRG) with parameters (n, k, λ, μ, t).

The rest of the paper is organized as follows. In Section 2, we give basic properties of Deza digraphs, in Section 3 we give constructions of directed Deza graphs, and construct several examples. In Section 4 twin and Siamese twin (directed) Deza graphs are introduced and constructed. In Section 5 we consider a variation of directed Deza graphs, called directed Deza graphs of type II, and show some of its construction.

2. Preliminaries

The following definition of directed Deza graphs was introduced by Zhang and Wang in 2003 (see [26]).

Definition 2.1. *Let n, k, b, a, and t be integers such that $0 \le a \le b \le k \le n$ and $0 \le t \le k$. A digraph $D = (V, A)$ is called a directed (n, k, b, a, t)-Deza graph if the following hold:*

1. $|V| = n$,
2. *every vertex has in-degree and out-degree k, and is adjacent to t vertices,*
3. *for any two distinct vertices u and v the number of vertices w such that $u \to w \to v$ is either a or b.*

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Example 2.2. *Let us define the matrices M_1 and M_2 as follows*

$$M_1 = \begin{bmatrix} 0&0&1&0&1&0&1&0\\ 0&0&0&1&0&1&0&1\\ 0&1&0&0&1&0&0&1\\ 1&0&0&0&0&1&1&0\\ 0&1&0&1&0&0&1&0\\ 1&0&1&0&0&0&0&1\\ 0&1&1&0&0&1&0&0\\ 1&0&0&1&1&0&0&0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0\\ 1&0&0&1&0&1&1&0\\ 0&1&1&0&1&0&0&1\\ 1&0&1&0&0&1&0&1\\ 0&1&0&1&1&0&1&0\\ 1&0&0&1&1&0&0&1\\ 0&1&1&0&0&1&1&0 \end{bmatrix}.$$

The matrix M_1 is the adjacency matrix of a directed $(8, 3, 3, 1, 0)$-Deza graph D_1. Since $t = 0$, the digraph D_1 is asymmetric. The matrix M_2 is the adjacency matrix of a directed $(8, 4, 3, 1, 1)$-Deza graph D_2.

Let M be the adjacency matrix of a directed graph D on n vertices. Then D is a directed (n, k, b, a, t)-Deza graph if and only if $MJ_n = J_nM = kJ_n$ and $M^2 = aX + bY + tI_n$ for some (0,1)-matrices X and Y such that $X + Y + I_n = J_n$. Note that D is a directed strongly regular graph if X or Y is equal to M.

In case $t = k$, we have a Deza graph, and that case is excluded here. This means that we require that $t < k$.

Let $D = (V, A)$ be a directed (n, k, b, a, t)-Deza graph. For vertices u and v, let N_{uv} be the number of vertices w such that $u \to w \to v$. Further, for a vertex u we define

$$\alpha = |\{v \in V : N_{uv} = a\}|, \qquad \beta = |\{v \in V : N_{uv} = b\}|.$$

The following statement can be found in [26, Proposition 1.1.].

Proposition 2.3. *Let D be a directed (n, k, b, a, t)-Deza graph. The numbers α and β do not depend on the vertex u and*

$$\alpha = \begin{cases} \dfrac{b(n-1) - k^2 + t}{b - a}, & a \neq b, \\ \dfrac{k^2 - t}{a}, & a = b. \end{cases}$$

$$\beta = \begin{cases} \dfrac{a(n-1) - k^2 + t}{a - b}, & a \neq b, \\ \dfrac{k^2 - t}{a}, & a = b. \end{cases}$$

Remark 2.4. *For $k = t$ we have a Deza graph for which conditions given in* [5, Proposition 1.1] *follow from the conditions given in Proposition 2.3.*

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If $a = b$, then a directed (n, k, b, a, t)-Deza graph is a directed strongly regular graph with $\lambda = \mu = a$. Hence, we are interested in the case $a \neq b$. The following corollary is a direct consequence of Proposition 2.3.

Corollary 2.5. *Let D be a directed (n, k, b, a, t)-Deza graph. If $a \neq b$, then $b-a$ divides $b(n-1)-k^2+t$ and $a(n-1)-k^2+t$. If $a < b$ and $\alpha, \beta \neq 0$, then*

$$a(n-1) < k^2 - t < b(n-1).$$

3. Constructions of directed Deza graphs

In this section, we give some new constructions of directed Deza graphs.

3.1. **A construction from the lexicographical product.** For two digraphs $D = (V_1, A_1)$ and $H = (V_2, A_2)$ the lexicographical product digraph (or the composition) $D[H]$ is the digraph with vertex set $V_1 \times V_2$ and arc set defined as follows. There is an arc from a vertex (x_1, y_1) to a vertex (x_2, y_2) in $D[H]$ if and only if either $x_1 \to x_2$ in D or $x_1 = x_2$ and $y_1 \to y_2$ in H.

Let M_1 be the adjacency matrix of D and M_2 be the adjacency matrix of H. Then $M_1 \otimes J_{|V_2|} + I_{|V_1|} \otimes M_2$ is the adjacency matrix of $D[H]$, where $\otimes$ denotes the Kronecker product of matrices and the vertices of $D[H]$ are ordered lexicographically. The following theorem is given in [26, Theorem 2.2.].

Theorem 3.1. *Let $D_1 = (V_1, A_1)$ be a DSRG with parameters (n, k, λ, μ, t) and $D_2 = (V_2, A_2)$ be a directed (n', k', b, a, t')-Deza graph. Then $D_1[D_2]$ is a $(k' + kn')$-regular digraph on nn' vertices. It is a directed Deza graph if and only if*

$$|\{a + kn', b + kn', \mu n', \lambda n' + 2k'\}| \leq 2.$$

Example 3.2. *Let $D_1 = (V_1, A_1)$ be a DSRG with parameters $(n, k, \lambda, \lambda, t)$, and $D_2 = (V_2, A_2)$ be the empty digraph on n' vertices (i.e., $A_2 = \emptyset$). Then $D_1[D_2]$ is a directed Deza graph with parameters $(nn', kn', tn', \lambda n', tn')$.*

3.2. **A construction from association schemes.** We assume that the reader is familiar with the basic facts of theory of association schemes. For background reading in theory of association schemes we refer the reader to [1].

Let X be a finite set of size n and $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ be relations defined on the set X. Let $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ be the set of $(0,1)$-adjacency matrices such that $[A_i]_{xy} = 1$ if and only if $(x, y) \in R_i$. Then a pair $(X, \mathcal{R})$ is called an *association scheme* with d classes if

(1) $A_0 = I$,
(2) $\sum_{i=0}^{d} A_i = J$,
(3) $A_i^t \in \mathcal{A}$, for all $i \in \{0, 1, \ldots, d\}$,

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(4) For any $i, j, k \in \{0, 1, \ldots, d\}$, there exists a non-negative integer $p_{i,j}^k$ such that $A_i A_j = \sum_{k=0}^{d} p_{i,j}^k A_k$.

If in addition $A_i A_j = A_j A_i$, for any $i, j \in \{1, \ldots, d\}$, then the association scheme is called commutative, and otherwise it is called non-commutative. We view $A_1, \ldots, A_d$ as adjacency matrices of directed graphs $D_1, \ldots, D_d$, with common vertex set. A commutative association scheme is symmetric if each matrix in it is symmetric, and non-symmetric otherwise. A method of constructing Deza graphs from symmetric association schemes is given in [5, Theorem 4.2.]. Deza digraphs can be constructed from association scheme in a similar way, as shown in the following theorem.

Theorem 3.3. *Let $(X, \mathcal{R})$ be an association scheme, and $F \subset \{1, 2, \ldots, d\}$. Let D be the directed graph with adjacency matrix $\sum_{f \in F} A_f$. Then D is a directed Deza graph if and only if*

$$\sum_{f,g \in F} p_{f,g}^k$$

takes on at most two values, when $k \in \{1, \ldots, d\}$.

Proof. Each A_i can be regarded as an adjacency matrix of a regular digraph (see [12]). Since $\sum_{i=0}^{d} A_i = J$, the sum of A_i's over any subset F of $\{1, \ldots, d\}$ is the adjacency matrix of a regular digraph, i.e., every vertex has constant in-degree and out-degree. Moreover, every vertex of D is adjacent with

$$t = \sum_{f \in F} p_{f,f}^0$$

vertices. Let $k \in \{1, \ldots, d\}$ and $u, v \in V(D)$ be vertices such that the distance $d(u, v) = k$. Then

$$N_{uv} = \sum_{f,g \in F} p_{f,g}^k.$$

When these numbers take on at most two values D is a directed Deza graph, which completes the proof. □

Example 3.4. *The adjacency matrix A of a doubly regular tournament with $4t + 3$ vertices satisfies that $A^2 = (t + 1)(J - I) - A$, which implies that a doubly regular tournament is a directed $(4t + 3, 2t + 1, t + 1, t, 0)$-Deza graph. Moreover, this doubly regular tournament is a DSRG. Note that a doubly regular tournament is equivalent to a non-symmetric association scheme with 2 classes.*

Example 3.5. *For a non-symmetric association scheme with 3 classes such that $A_1^t = A_2$, it holds that $p_{1,1}^1 = p_{2,2}^2$. Therefore, the digraph D_1 with adjacency matrix A_1 is a directed $(|X|, p_{1,2}^0, b, a, p_{1,1}^0)$-Deza graph where $\{a, b\} = \{p_{1,1}^1, p_{1,1}^3\}$. For construction, see [8, 23].*

Example 3.6. *For a non-symmetric association scheme with 5 classes constructed in* [22, Theorem 6.3] *with $A_1^t = A_2$, it holds that $\{p_{1,1}^k : 1 \le k \le 5\} = \{(2n - 1)^2(n - 1)n, (2n - 1)^2 n^2\}$. Therefore, A_1 is a directed Deza graph.*

3.3. **A construction from Hadamard matrices.** We say that a $(0,1)$-matrix X is skew if $X+X^t$ is a $(0,1)$-matrix. A Hadamard matrix H of order n is called skew-type if $H+H^t=2I_n$.

Let D be a regular asymmetric digraph of degree k on v vertices. D is called a divisible design digraph (DDD for short) with parameters $(v,k,\lambda_1,\lambda_2,m,n)$ if the vertex set can be partitioned into m classes of size n, such that:

- for any two distinct vertices x and y from the same class, the number of vertices z that dominates both x and y is equal to λ_1, and for any two distinct vertices x and y from different classes, the number of vertices z that dominates both x and y is equal to λ_2,
- for any two distinct vertices x and y from the same class, the number of vertices z being dominated by both x and y is equal to λ_1, and for any two distinct vertices x and y from different classes, the number of vertices z being dominated by both x and y is equal to λ_2.

An incidence structure with v points and the constant block size k is a (group) divisible design with parameters $(v,k,\lambda_1,\lambda_2,m,n)$ whenever the point set can be partitioned into m classes of size n, such that two points from the same class are incident with exactly λ_1 common blocks, and two points from different classes are incident with exactly λ_2 common blocks. A divisible design D is said to be symmetric (or to have the dual property) if the dual of D is a divisible design with the same parameters as D. If D is a divisible design digraph with parameters $(v,k,\lambda_1,\lambda_2,m,n)$ then its adjacency matrix is the incidence matrix of a symmetric divisible design $(v,k,\lambda_1,\lambda_2,m,n)$.

The following construction of directed Deza graphs follows from the construction of divisible design digraphs given in [2, Theorem 4.3.]. By O_n we denote the $n\times n$ zero-matrix.

Theorem 3.7. *If there exists a skew-type Hadamard matrix of order* $4u$*, then there exists a directed Deza graph with parameters* $(8u,4u-1,4u-1,2u-1,0)$*.*

Proof. Let H be a skew-type Hadamard matrix of order $4u$. Replace each diagonal entry of H by O_2, each entry value 1 of H by I_2, and each entry value -1 by J_2-I_2. By [2, Theorem 10.], the obtained matrix is the adjacency matrix of a DDD with parameters $(8u,4u-1,0,2u-1,4u,2)$. By the definition of a DDD, the obtained digraph is asymmetric. By the construction, each row (and column) of the Hadamard matrix correspond to a class of size two of vertices of the DDD. Let r_i be the row of the adjacency matrix of the DDD corresponding to the vertex v_i, and c_i be the column of the adjacency matrix corresponding to v_i. Since the digraph is asymmetric, the dot product $r_i\cdot c_i=0$. If the vertex v_j belongs to the same class as v_i, then $r_i\cdot c_j=4u-1$, i.e. $r_i=c_j^t$. Since the adjacency matrix M of D is the incidence matrix of a symmetric divisible design and there is a column of M equal to r_i, the statement of the theorem follows. □

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Example 3.8. *The matrix M_1 from Example 2.2 can be obtained by Theorem 3.7 using the Hadamard matrix*

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ - & 1 & 1 & - \\ - & - & 1 & 1 \\ - & 1 & - & 1 \end{bmatrix},$$

where - stands for -1.

4. Twin and Siamese twin (directed) Deza graphs

Twin designs and Siamese twin designs were studied in [19, 20]. The concepts of twin and Siamese twin structures are developed for strongly regular graphs too (see [14]). In this paper, we introduce twin and Siamese twin (directed) Deza graphs and construct several examples.

A $(0, \pm 1)$-matrix T is called a twin (directed) Deza graph, if $T = K - L$, where K, L are non-zero disjoint $(0,1)$-matrices and both K and L are adjacency matrices of (directed) Deza graphs with the same parameters. Note that $(0,1)$-matrices A and B of the same dimension are called disjoint if $A + B$ is a $(0,1)$-matrix.

A $(0, \pm 1)$-matrix S is called a Siamese twin (directed) Deza graph sharing the entries of N, if $S = N + K - L$, where N, K, L are non-zero disjoint $(0,1)$-matrices and both $N + K$ and $N + L$ are adjacency matrices of (directed) Deza graphs with the same parameters.

Let G be a graph on v vertices. A reflexive graph RG is obtained from G by including a loop at every vertex. If A is the adjacency matrix of the graph G, then $A + I_v$ is the adjacency matrix of RG. Reflexive graphs have been studied, e.g., in [6, 7, 15]. We define Deza reflexive graphs as follows.

Definition 4.1. *Let n, k, b, and a be integers such that $0 \leq a \leq b \leq k \leq n$. A reflexive graph RG is called a (n, k, b, a)-Deza reflexive graph if the following hold:*

1. *the graph RG has n vertices,*
2. *every vertex is adjacent to k vertices (including the vertex itself),*
3. *for any two distinct vertices u and v the number of vertices w that are adjacent to both u and v is either a or b.*

In the following theorem, we give constructions of twin Deza graphs and Siamese twin Deza reflexive graphs.

Theorem 4.2. *If there exists a Hadamard matrix of order n, then there exist a twin Deza graph with parameters $((2n-1)n, (n-1)n, \frac{n(n-1)}{2}, \frac{n(n-2)}{2})$, and a Siamese twin Deza reflexive graph with parameters $((2n-1)n, n^2, \frac{n(n+1)}{2}, \frac{n^2}{2})$.*

Proof. Let H be a normalized Hadamard matrix of order n. Let r_i be the i-th row of the matrix H and let $C_i = r_i^t r_i$, $i = 1, 2, 3, \ldots, n$. Further, let C be the circulant matrix of order $2n - 1$ with the first row $(1, 2, 3, \ldots, n, n, n-1, \ldots, 2)$. Note that every pair of rows of this matrix have exactly one common entry in the same column. Now replace i with C_i in C. We get a $(2n-1)n \times (2n-1)n$ matrix K, which is a $(1, -1)$-matrix such that K^2 has all its off-diagonal entries n or $-n$. By changing the diagonal blocks of size $n \times n$ in K to the zero matrices, and split the remaining matrix into $A - B$, where A and B are disjoint $(0, 1)$-matrices, we obtain the adjacency matrices A and B of two graphs. The rows and the columns of A an B are divided into $2n-1$ groups of size n, according to the circulant matrix of order $2n - 1$. From [20, Lemma 3.] it follows that for two rows r_i and r_j, $i \neq j$, of A (or B) belonging to the same group $r_i \cdot r_j = \frac{n(n-2)}{2}$. The properties of the matrices C_i, $i = 1, 2, 3, \ldots, n$, given in the proof of [20, Theorem 1.] lead us to conclusion that for two rows r_i and r_j from different groups the dot product $r_i \cdot r_j$ is equal to $\frac{n(n-2)}{2}$ or $\frac{n(n-1)}{2}$. Hence, A and B are adjacency matrices of Deza graphs with parameters $((2n-1)n, (n-1)n, \frac{n(n-1)}{2}, \frac{n(n-2)}{2})$ and K is the adjacency matrix of a twin Deza graph.

If we now add the matrix $I_{2n-1} \otimes C_1$ to each of A or B, we get two Deza reflexive graphs with parameters $((2n-1)n, n^2, \frac{n(n+1)}{2}, \frac{n^2}{2})$ sharing the cliques of size n. □

Example 4.3. *Let us take the matrix*

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & 1 & - \\ 1 & - & - & 1 \end{bmatrix},$$

which is a normalized Hadamard matrix of order 4. *Then* $r_1 = (1, 1, 1, 1)$, $r_2 = (1, 1, -, -)$, $r_3 = (1, -, 1, -)$, $r_4 = (1, -, -, 1)$, *and*

$$C_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1 & 1 & - & - \\ 1 & 1 & - & - \\ - & - & 1 & 1 \\ - & - & 1 & 1 \end{bmatrix},$$

$$C_3 = \begin{bmatrix} 1 & - & 1 & - \\ - & 1 & - & 1 \\ 1 & - & 1 & - \\ - & 1 & - & 1 \end{bmatrix}, \qquad C_4 = \begin{bmatrix} 1 & - & - & 1 \\ - & 1 & 1 & - \\ - & 1 & 1 & - \\ 1 & - & - & 1 \end{bmatrix}.$$

Let C be the circulant matrix of order 7 *with the first row* $(1, 2, 3, 4, 4, 3, 2)$. *By applying Theorem* 4.2 *one gets a twin Deza graph with parameters* $(28, 12, 6, 4)$, *and a Siamese twin Deza reflexive graph with parameters* $(28, 16, 10, 8)$.

Theorem 4.4. *If there exists a Hadamard matrix of order n, then there exists a Siamese twin directed Deza reflexive graph with parameters* $((2n-1)n, n^2, \frac{n(n+1)}{2}, \frac{n^2}{2}, n)$.

Proof. Let H be a normalized Hadamard matrix of order n and let the matrices C_i, $i = 1, 2, \dots, n$ be defined as in the proof of Theorem 4.2. Further, let D be the circulant matrix with the first row $(1, 2, 3, \dots, n, -n, -n + 1, \dots, -2)$. Now replace i with C_i in C if $i > 0$, and replace i with $-C_{|i|}$ if $i < 0$. We get a $(2n - 1)n \times (2n - 1)n$ $(1, -1)$-matrix K' such that $K'K'^t$ has all its off-diagonal entries n or $-n$. By changing the block diagonals of K' to the zero matrices, we get a matrix which splits as $A - B$, where A and B are $(0, 1)$-matrices. If we now add the matrix $I_{2n-1} \otimes C_1$ to each of A or B, we get adjacency matrices of two directed Deza reflexive graphs with parameters $((2n - 1)n, n^2, \frac{n(n+1)}{2}, \frac{n^2}{2}, n)$. □

Remark 4.5. *The matrices A and B from the proof of Theorem 4.4 are the adjacency matrices of divisible design digraphs with parameters $((2n - 1)n, (n - 1)n, \frac{n(n-2)}{2}, \frac{n(n-1)}{2}, 2n - 1, n)$.*

5. A variation of directed Deza graphs

Finally, we study a slightly different variation of directed Deza graphs. The motivation for investigation of this variation of directed Deza graphs, called directed Deza graph of type II, is their relation to doubly regular asymmetric digraphs, divisible design digraphs and divisible designs with the dual property.

Definition 5.1. *Let n, k, b, and a be integers such that $0 \le a \le b \le k \le n$. A digraph $D = (V, A)$ is a directed (n, k, b, a)-Deza graph of type II if*

(1) $|V| = n$.
(2) *Every vertex has in-degree and out-degree k.*
(3) *Let u and v be distinct vertices. The number of vertices w such that $u \to w \leftarrow v$ and the number of vertices w such that $u \leftarrow w \to v$ coincide and are a or b.*

Let M be the adjacency matrix of a directed graph D on n vertices. Then D is a directed (n, k, b, a)-Deza graph of type II if and only if

$$MJ_n = J_nM = kJ_n$$

and

$$MM^t = M^tM = aX + bY + kI_n,$$

for some (0,1)-matrices X and Y such that $X + Y + I_n = J_n$.

Remark 5.2. *In [24] the authors introduced a notion of Deza digraphs that is different than the one given in [26] by Zhang and Wang. Wang and Feng in [24] defined an (n, k, b, c)-Deza digraph to be a regular digraph on n vertices with adjacency matrix A satisfying*

$$AA^t = kI + bX + cY, (b \le c),$$

for some symmetric non-zero $(0, 1)$-matrices X and Y such that $X + Y + I_n = J_n$. The difference between our definition of a directed Deza graph of type II and the definition of a Deza digraph given in

[24] *is that we require that $AA^t = A^tA$. A digraph with an adjacency matrix A such that $AA^t = A^tA$ is said to be normal. So, a directed (n, k, b, a)-Deza graph of type II is a normal (n, k, b, a)-Deza digraph in the sense of the definition of Deza digraphs given by Wang and Feng in* [24].

Note that a doubly regular asymmetric digraph (DRAD) with parameters (v, k, λ) (see [13]) is a directed (v, k, λ, λ)-Deza graph of type II. Divisible design digraphs (see [2]) are natural generalization of doubly regular asymmetric digraphs. A divisible design digraph with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ is a directed $(v, k, \lambda_1, \lambda_2)$-Deza graph of type II. A divisible design admitting a symmetric incidence matrix with zero diagonal is the neighborhood design of a divisible design graph (see [11]). Moreover, a divisible design with the dual property (meaning that its dual is a divisible design with the same parameters) is the neighborhood design of a directed Deza graph of type II.

Let $D = (V, A)$ be a directed (n, k, b, a)-Deza graph of type II. For vertices u and v, let N_{uv} be the number of vertices w such that $u \to w \leftarrow v$. Further, for a vertex u we define

$$\alpha = |\{v \in V : N_{uv} = a\}|, \qquad \beta = |\{v \in V : N_{uv} = b\}|.$$

The following statement follows from [24, Proposition 1.2.].

Proposition 5.3. *Let D be a directed (n, k, b, a)-Deza graph of type II. The numbers α and β do not depend on the vertex u and*

$$\alpha = \begin{cases} \dfrac{b(n-1) - k^2 + k}{b - a}, & a \neq b, \\ \dfrac{k^2 - k}{a}, & a = b. \end{cases}$$

$$\beta = \begin{cases} \dfrac{a(n-1) - k^2 + k}{a - b}, & a \neq b, \\ \dfrac{k^2 - k}{a}, & a = b. \end{cases}$$

The following theorem can be proved in a similar way as Theorem 3.1 and [5, Proposition 2.3.].

Theorem 5.4. *Let G_1 be an (n, k, λ, μ)-SRG and $D_2 = (V_2, A_2)$ be a directed (n', k', b, a)-Deza graph of type II. Then $G_1[D_2]$ is a $(k' + kn')$-regular digraph on nn' vertices. It is a directed Deza graph of type II if and only if*

$$|\{a + kn', b + kn', \mu n', \lambda n' + 2k'\}| \leq 2.$$

It is straightforward to check that the following statement holds.

Theorem 5.5. *Let $D_1 = (V_1, A_1)$ be a digraph with the adjacency matrix M which is the incidence matrix of a symmetric (n, k, λ) design. Further, let $D_2 = (V_2, A_2)$ be the empty digraph on n' vertices (i.e., $A_2 = \emptyset$). Then $D_1[D_2]$ is a directed Deza graph of type II with parameters $(nn', kn', kn', \lambda n')$.*

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Directed Deza graph of type II with parameters (n, k, k, a) are recognizable by their parameters, as is seen in the following theorem. Theorem 5.6 can be proved in a similar way as [5, Theorem 2.6.], and therefore we omit its proof.

Theorem 5.6. *Let D be a directed (n, k, b, a)-Deza graph of type II. Then $b = k$ if and only if D is isomorphic to $D_1[D_2]$, where D_1 is a digraph with the adjacency matrix which is the incidence matrix of a symmetric (n_1, k_1, a_1) design, and D_2 is an empty digraph of n_2 vertices. Moreover, the parameters of D satisfy*

$$n = n_1 n_2, \qquad k = b = k_1 n_2, \qquad a = \lambda n_2.$$

Finally, we introduce construction of a (undirected) Deza graph or a directed Deza graph of type II from the finite fields. Note that the case of characteristic two corresponds to undirected Deza graphs and the case of odd characteristic directed Deza graphs of type II.

From now on let $q = p^m$ be a power of a prime p. We denote by $\mathbb{F}_q$ the finite field of q elements. Let H_q be the multiplicative table of $\mathbb{F}_q$, i.e., H_q is a $q \times q$ matrix with rows and columns indexed by the elements of $\mathbb{F}_q$ with (α, β)-entry equal to $\alpha \cdot \beta$. Then the matrix H_q is a generalized Hadamard matrix with parameters $(q, 1)$ over the additive group of $\mathbb{F}_q$. Letting G be an additively written finite abelian group of order g, a square matrix $H = (h_{ij})_{i,j=1}^{g\lambda}$ of order $g\lambda$ with entries from G is called a *generalized Hadamard matrix with the parameters* (g, λ) over G if for all distinct $i, k \in \{1, 2, \ldots, g\lambda\}$, the multiset $\{h_{ij} - h_{kj} : 1 \leq j \leq g\lambda\}$ contains exactly λ times of each element of G.

Let ϕ be a permutation representation of the additive group of $\mathbb{F}_q$ defined as follows. Since $q = p^m$, we view the additive group of $\mathbb{F}_q$ as $\mathbb{F}_p^m$. Define $U = \mathrm{circ}(0, 1, 0, \ldots, 0)$, a circulant matrix with the first row $(0, 1, 0, \ldots, 0)$, and a group homomorphism $\phi : \mathbb{F}_q \to GL_q(\mathbb{R})$ as $\phi((x_i)_{i=1}^m) = \otimes_{i=1}^m U^{x_i}$, where $\otimes$ denotes the tensor product.

From the generalized Hadamard matrix H_q and the permutation representation ϕ, we construct q^2 auxiliary matrices; for each $\alpha, \alpha' \in \mathbb{F}_q$, define a $(0, 1)$-matrix $C_{\alpha,\alpha'}$ to be a $q \times q$ block matrix, where $\phi(\alpha(-\beta + \beta') + \alpha')$ is placed in its (β, β')-entry as a submatrix:

$$C_{\alpha,\alpha'} = (\phi(\alpha(-\beta + \beta') + \alpha'))_{\beta,\beta' \in \mathbb{F}_q}.$$

Further, let x, y be indeterminates. We define $C_{x,\alpha}, C_{y,\alpha}$ by $C_{x,\alpha} = O_{q^2}$ and $C_{y,\alpha} = \phi(\alpha) \otimes J_q$ for $\alpha \in \mathbb{F}_q$, where O_{q^2} denotes the zero matrix of order q^2.

Let L be a $(2q + 3) \times (2q + 3)$ circulant matrix with the first row $(a_i)_{i=1}^{2q+3}$ satisfying

$$a_1 = x, a_2 = y, \{a_i : 3 \leq i \leq q + 2\} = \mathbb{F}_q, a_{2q+4-i} = a_{i+1} \text{ for } i \in \{1, \ldots, q + 2\}.$$

Let $S = \mathbb{F}_q \cup \{x, y\}$. Write L as $L = \sum_{a \in S} a \cdot P_a$. Note that $P_x = I_{2q+3}$.

From the $(0, 1)$-matrices $C_{\alpha,\alpha'}$'s and the array L, we construct (directed) Deza graphs as follows. For $\alpha \in \mathbb{F}_q$, we define a $(2q + 3)q^2 \times (2q + 3)q^2$ $(0, 1)$-matrix N_α to be

$$N_\alpha = (C_{L(a,a'),\alpha})_{a,a' \in S} = \sum_{a \in \mathbb{F}_q \cup \{y\}} P_a \otimes C_{a,\alpha}.$$

In order to calculate $N_\alpha N_\alpha^t$ and study more properties, we prepare a lemma on $C_{\alpha,\alpha'}$ and P_a. We fix a bijection $\varphi : \mathbb{F}_q \cup \{y\} \to \{1, \dots, q+1\}$ such that $P_a = V^{\varphi(a)} + V^{-\varphi(a)}$ where V is the shift matrix of order $2q+3$.

Lemma 5.7. (1) *For $a \in \mathbb{F}_q \cup \{y\}$ and $\alpha \in \mathbb{F}_q$, $C_{a,\alpha}^t = C_{a,-\alpha}$.*

(2) *For $\alpha \in \mathbb{F}_q$, $\sum_{a\in\mathbb{F}_q\cup\{y\}} C_{a,\alpha} = qI_q \otimes \phi(\alpha) + (J_q + \phi(\alpha) - I_q) \otimes J_q$.*

(3) *For $a \in \mathbb{F}_q \cup \{y\}$ and $\alpha, \alpha' \in \mathbb{F}_q$, $C_{a,\alpha} C_{a,\alpha'} = qC_{a,\alpha+\alpha'}$.*

(4) *For distinct $a, a' \in \mathbb{F}_q \cup \{y\}$ and $\alpha, \alpha' \in \mathbb{F}_q$, $C_{a,\alpha} C_{a',\alpha'} = J_{q^2}$.*

(5) *For $\alpha, \alpha', \alpha'' \in \mathbb{F}_q$, $(I_q \otimes \phi(\alpha''))C_{\alpha,\alpha'} = C_{\alpha,\alpha'+\alpha''}$.*

(6) *For $\alpha, \alpha' \in \mathbb{F}_q$, $(I_q \otimes \phi(\alpha))C_{y,\alpha'} = C_{y,\alpha'}$.*

(7) *$\sum_{a,b\in\mathbb{F}_q\cup\{y\},a\neq b} P_a P_b = 2q(J_{2q+3} - I_{2q+3})$.*

Proof. (1) is easy to see. (2): For $\alpha, \beta, \beta' \in \mathbb{F}_q$, the (β, β')-entry of $\sum_{\gamma\in\mathbb{F}_q} C_{\gamma,\alpha}$ is

$$\sum_{\gamma\in\mathbb{F}_q} \phi(\gamma(-\beta+\beta')+\alpha) = \begin{cases} \sum_{\gamma\in\mathbb{F}_q} \phi(\alpha) & \text{if } \beta = \beta' \\ \sum_{\gamma'\in\mathbb{F}_q} \phi(\gamma'+\alpha) & \text{if } \beta \neq \beta' \end{cases}$$
$$= \begin{cases} q\phi(\alpha) & \text{if } \beta = \beta', \\ J_q & \text{if } \beta \neq \beta', \end{cases}$$

which yields $\sum_{\gamma\in\mathbb{F}_q} C_{\gamma,\alpha} = qI_q \otimes \phi(\alpha) + (J_q - I_q) \otimes J_q$. Therefore,

$$\sum_{a\in\mathbb{F}_q\cup\{y\}} C_{a,\alpha} = \sum_{\gamma\in\mathbb{F}_q} C_{\gamma,\alpha} + C_{y,\alpha} = qI_q \otimes \phi(\alpha) + (J_q + \phi(\alpha) - I_q) \otimes J_q.$$

(3): For $a = y$, $C_{y,\alpha} C_{y,\alpha'} = (\phi(\alpha) \otimes J_q)(\phi(\alpha') \otimes J_q) = q\phi(\alpha+\alpha') \otimes J_q = qC_{y,\alpha+\alpha'}$. For $a, \beta, \beta' \in \mathbb{F}_q$, the (β, β')-entry of $C_{a,\alpha} C_{a,\alpha'}$ is

$$\sum_{\gamma\in\mathbb{F}_q} \phi(a(-\beta+\gamma)+\alpha)\phi(a(-\gamma+\beta')+\alpha') = \sum_{\gamma\in\mathbb{F}_q} \phi(a(-\beta+\beta')+\alpha+\alpha')$$
$$= q\phi(a(-\beta+\beta')+\alpha+\alpha').$$

Thus we have $C_{a,\alpha} C_{a,\alpha'} = qC_{a,\alpha+\alpha'}$.

(4): The case of $a \in \mathbb{F}_q$ and $a' = y$ follows from the fact that $C_{a,\alpha}$ is a block matrix whose $q \times q$ sub-block is a permutation matrix. The case of $a, a' \in \mathbb{F}_q, a \neq a'$ follows from a similar calculation to (2) with the fact that $\{(a-a')\gamma : \gamma \in \mathbb{F}_q\} = \mathbb{F}_q$.

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(5) and (6) are routine, and (7) follows from the equations below. Recall that $S = \mathbb{F}_q \cup \{x, y\}$.

$$\begin{aligned}\sum_{a,b\in\mathbb{F}_q\cup\{y\},a\neq b} P_aP_b &= \sum_{a,b\in\mathbb{F}_q\cup\{y\}} P_aP_b - \sum_{a\in\mathbb{F}_q\cup\{y\}} P_a^2\\ &= (\sum_{a\in\mathbb{F}_q\cup\{y\}} P_a)^2 - \sum_{a\in\mathbb{F}_q\cup\{y\}}(2I_{2q+3} + V^{2\varphi(a)} + V^{-2\varphi(a)})\\ &= (J_{2q+3} - I_{2q+3})^2 - 2(q+1)I_{2q+3} - (J_{2q+3} - I_{2q+3})\\ &= 2q(J_{2q+3} - I_{2q+3}).\end{aligned}$$

This completes the proof. □

We are now ready to prove the results for N_α's.

Theorem 5.8. (1) *For any* $\alpha \in \mathbb{F}_q$, $N_\alpha^t = N_{-\alpha}$.

(2) *For any* $\alpha, \beta \in \mathbb{F}_q$,

$$\begin{aligned}N_\alpha N_\beta &= 2q^2 I_{q(2q+3)} \otimes \phi(\alpha+\beta) + 2qI_{2q+3} \otimes (\phi(\alpha+\beta) - I_q) \otimes J_q + 2qJ_{q^2(2q+3)}\\ &\quad + q\sum_{a\in\mathbb{F}_q\cup\{y\}}(V^{2\varphi(a)} + V^{-2\varphi(a)}) \otimes C_{a,\alpha+\beta}.\end{aligned}$$

Proof. (1): It follows from Lemma 5.7 (1) and the properties there that the matrices L are symmetric.

(2): We use Lemma 5.7 to obtain:

$$\begin{aligned}N_\alpha N_\beta &= \sum_{a,b\in\mathbb{F}_q\cup\{y\}} P_aP_b \otimes C_{a,\alpha}C_{b,\beta}\\ &= \sum_{a\in\mathbb{F}_q\cup\{y\}} P_a^2 \otimes C_{a,\alpha}C_{a,\beta} + \sum_{a,b\in\mathbb{F}_q\cup\{y\},a\neq b} P_aP_b \otimes C_{a,\alpha}C_{b,\beta}\\ &= \sum_{a\in\mathbb{F}_q\cup\{y\}}(2I_{2q+3} + V^{2\varphi(a)} + V^{-2\varphi(a)}) \otimes qC_{a,\alpha+\beta} + \sum_{a,b\in\mathbb{F}_q\cup\{y\},a\neq b} P_aP_b \otimes J_{q^2}\\ &= 2qI_{2q+3} \otimes (qI_q \otimes \phi(\alpha+\beta) + (J_q + \phi(\alpha+\beta) - I_q) \otimes J_q)\\ &\quad + q\sum_{a\in\mathbb{F}_q\cup\{y\}}(V^{2\varphi(a)} + V^{-2\varphi(a)}) \otimes C_{a,\alpha+\beta}\\ &\quad + 2q(J_{2q+3} - I_{2q+3}) \otimes J_{q^2}\\ &= 2q^2 I_{q(2q+3)} \otimes \phi(\alpha+\beta) + 2qI_{2q+3} \otimes (\phi(\alpha+\beta) - I_q) \otimes J_q + 2qJ_{q^2(2q+3)}\\ &\quad + q\sum_{a\in\mathbb{F}_q\cup\{y\}}(V^{2\varphi(a)} + V^{-2\varphi(a)}) \otimes C_{a,\alpha+\beta}.\end{aligned}$$

This completes the proof. □

Corollary 5.9. (1) *For* $\alpha \in \mathbb{F}_q$, $N_\alpha N_\alpha^t = 2q^2 I_{q^2(2q+3)} + 2qJ_{q^2(2q+3)} + q\sum_{a\in\mathbb{F}_q\cup\{y\}}(V^{2\varphi(a)} + V^{-2\varphi(a)}) \otimes C_{a,0}$.

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(2) *If either p is odd and $\alpha = 0$, or $p = 2$ and $\alpha \in \mathbb{F}_q$, then N_α is the adjacency matrix of a Deza graph with parameters $(q^2(2q+3), 2q^2+2q, 3q, 2q)$.*

(3) *If p is odd and $\alpha \in \mathbb{F}_q^*$, then N_α is the adjacency matrix of a directed Deza graph of type II with parameters $(q^2(2q+3), 2q^2+2q, 3q, 2q)$.*

Furthermore, the resulting Deza graphs or directed Deza graphs of type II are commuting with each other. Similar construction and a result for prime power 2^m were given in [21].

Remark 5.10. *For $q = 2$, the Deza graphs given by Corollary 5.9, part (2), have parameters $(28, 12, 6, 4)$. In [3], a database of small strictly Cayley Deza graphs is given. Two of the four Cayley Deza graphs with parameters $(28, 12, 6, 4)$ listed in [3] can be obtained by the construction given in Corollary 5.9, part (2).*

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