http://ijgt.ui.ac.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 13 No. 1 (2024), pp. 97-114.

www.ui.ac.ir

NON-INNER AUTOMORPHISMS OF ORDER p IN FINITE p-GROUPS OF COCLASS 4 AND 5

KOMMA PATALI

Abstract. A long-standing conjecture asserts that every finite nonabelian p-group has a non-inner automorphism of order p. This paper proves the conjecture for finite p-groups of coclass 4 and 5 ($p \geq 5$). We also prove the conjecture for an odd order nonabelian p-group G with cyclic center satisfying $C_G(G^p\gamma_3(G)) \cap Z_3(G) \leq Z(\Phi(G))$.

1. Introduction

Let p be a prime number and let G be a finite nonabelian p-group. By a celebrated theorem of Gaschütz [10], G admits a non-inner automorphism of p-power order. In 1973, Berkovich [19, Problem 4.3] proposed the following conjecture:

Conjecture. *Every finite nonabelian p-group admits a non-inner automorphism of order p.*

This is a simple to state and notoriously hard problem in group theory. The validity of the conjecture for regular p-groups follows from a cohomological result of Schmid [22] and [8]. Deaconescu and Silberberg [8] proved that a finite nonabelian p-group G satisfying the condition $C_G(Z(\Phi(G))) \neq \Phi(G)$ has a non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed. Liebeck [17] proved that odd order p-groups of class 2 admits a non-inner automorphism of order p leaving $\Phi(G)$ elementwise fixed. Abdollahi [1, 3] proved the conjecture for 2-groups of class 2, p-groups for which $G/Z(G)$ is powerful, and p-groups of maximal class.

Keywords: Finite p-groups, Non-inner automorphisms, Coclass.

MSC(2010): Primary: 20D15; Secondary: 20D45.

Communicated by Hamid Mousavi.

Article Type: Research Paper.

Received: 22 November 2022, Accepted: 04 April 2023.

Cite this article: K. Patali, Non-inner automorphisms of order p in finite p-groups of coclass 4 and 5, Int. J. Group Theory, **13** no. 1 (2024) 97–114. http://dx.doi.org/10.22108/ijgt.2023.135843.1816 .

Abdollahi, Ghoraishi, and Wilkens [4] proved the conjecture for finite p-groups of class 3, and Abdollahi et al. [5] proved the conjecture for p-groups of coclass 2. Ruscitti et al. [21] proved the conjecture for p-groups of coclass 3 with the exception of $p = 3$. Ghoraishi [11, 12] proved the conjecture for p-groups not satisfying the condition $Z_2^*(G) \leq C_G(Z_2^*(G)) = \Phi(G)$, and for an odd order p-group G for which $(G, Z(G))$ is a Camina pair. Abdollahi and Ghoraishi [2] proved the conjecture for 2-generator finite p-groups with abelian Frattini subgroup. Jamali and Viseh [15] proved the conjecture for finite p-groups with cyclic commutator subgroup. Shabani-Attar [23] proved the conjectutre for p-groups of order p^m and exponent p^{m-2}.

The objectives of this paper are twofold. Firstly we obtain results of independent interest (See Section 2). In doing this, we look for criteria for the existence of a derivation $\delta : G \to A$ such that $\delta(\gamma_2(G)) \neq 1$ when G is an extra-special group of exponent p. Moreover, we prove a structure theorem for finite p-groups.

Theorem 2.5. *Let p be an odd prime and let G be a finite p-group. If G is not powerful, then a normal subgroup N of G exists such that either G/N is an extra-special group of exponent p or $G/N = U \times V$ where $U \leq Z(G/N)$ is elementary abelian and V is an extra-special group of exponent p.*

Other than giving the structure of a finite p-group, Theorem 2.5 allows us to construct derivations on every finite p-group when the above-indicated criteria are satisfied.

The second objective of this paper is to prove the conjecture for the classes of finite p-groups given in the abstract. Section 2 provides tools for constructing automorphisms of order p in finite p-groups with a cyclic center, and the results below are consequences.

Theorem 4.2. *Let p be an odd prime and let G be a finite nonabelian p-group with cyclic center. Suppose that all the automorphisms of G of order p fixing $G^p\gamma_3(G)$ elementwise are inner, then the following holds:*

(i) $C_G(G^p\gamma_3(G)) \cap Z_3(G) \not\leq Z(\Phi(G))$.

(ii) $\binom{d(G)+1}{2} \leq r$, *where r is the coclass of G.*

Theorem 5.4. *Let $p \geq 5$ and let G be a finite nonabelian p-group.*

(i) If G is of coclass 4, then G admits a non-inner automorphism of order p fixing $G^p\gamma_3(G)$ elementwise.

(ii) If G is of coclass 5, then G admits a non-inner automorphism of order p fixing $G^p\gamma_4(G)$ elementwise.

The outline of the paper is as follows.

In Section 2, we study the construction of derivations on finite p-groups.

In Section 3, we recall some well-known results about the existence of non-inner automorphisms of order p in finite p-groups.

Sections 4 and 5 are devoted to prove Theorem 4.2 and Theorem 5.4, respectively.

For a finite group G, $|G|, \exp(G), Z(G), Z_i(G)$, and $\Phi(G)$ denote the order, the exponent, the center, the i-th center, and the Frattini subgroup of G. For a finite p-group G, $d(G)$ and $\Omega_1(G)$ denote the minimal number of generators of G and the subgroup of G generated by all the elements of order p in G.

2. Finite p-groups and derivations

Let G be a group and let M be a right G-module. A derivation $\delta : G \to M$ is a function such that

$$\delta(gh) = \delta(g)^h\ \delta(h) \text{ for all } g, h \in G.$$

And δ is a principal derivation if there exists $m \in M$ such that $\delta(g) = m^{-1}m^g$ for all $g \in G$. Let $Z^1(G, M)$ denote the abelian group of all derivations of G to M and $B^1(G, M)$ denote all principal derivations.

Note that the values of a derivation δ over a set of generators of G will uniquely determine δ. We set up the following notations: Let F be a free group generated by a finite subset X and let $G = \langle X \mid r_1, \dots, r_n\rangle$ be a group whose free presentation is F/R, where R is the normal closure in F of the set of relations $\{r_1, \dots, r_n\}$ of G. Let $\pi : F \to G$ be the canonical homomorphism. We have that M is a G-module if and only if M is an F-module on which R acts trivially. Moreover, we have (cf. [13]):

Lemma 2.1. *(i) Let M be an F-module. Then every function $f : X \to M$ extends in a unique way to a derivation $\delta : F \to M$.*

(ii) Let M be a G-module and let $\delta : G \to M$ be a derivation. Then $\bar\delta : F \to M$ given by the composition $\bar\delta(f) = \delta(\pi(f))$ is a derivation such that $\bar\delta(r_i) = 0$ for all $i \in \{1, \dots, n\}$. Conversely, if $\bar\delta : F \to M$ is a derivation such that $\bar\delta(r_i) = 0$ for all $i \in \{1, \dots, n\}$, then $\delta(fR) = \bar\delta(f)$ defines, uniquely, a derivation on $G = F/R$ to M such that $\bar\delta = \delta \circ \pi$.

For a G-module M, M^G denote the submodule $\{m \in M | m^g = m \text{ for all } g \in G\}$ and $[M, G]$ denote the submodule generated by the elements $m^{-1}m^g$ for $m \in M, g \in G$.

Lemma 2.2. *Let p be an odd prime and let G be an extra-special group of exponent p. Let M be an elementary abelian p-group which is also a G-module. Suppose that M^G and $[M, G]$ coincides and have order p, and that $d(M) \geq d(G)$. Then there exists a derivation $\delta : G \to M$ such that $\delta(\gamma_2(G)) \neq 1$.*

Proof. Recall that the order of an extra-special group is p^{2n+1}, and for every integer $n \geq 1$ and every odd prime p there is only one isomorphism class for extra-special groups of order p^{2n+1} and exponent p (cf. [16, p. 34]). Thus $d(G) = 2n$ and G has a presentation

$$\langle x_1, y_1, \dots, x_n, y_n, c | [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1 \text{ for } i \neq j,$$
$$[x_i, c] = [y_i, c] = 1,\ [x_i, y_i] = c,\ x_i^p = y_i^p = c^p = 1 \text{ for } 1 \leq i \leq n\rangle. \tag{2.1}$$

Let F be a free group on $\{x_1, y_1, \dots, x_n, y_n, c\}$. Then M is considered as an F-module in a natural way. Now we define a map $\delta : \{x_1, y_1, \dots, x_n, y_n, c\} \to M$. Let $1 \neq z_0 \in M^G$. For $i \in \{1, \dots, n\}$, let $\sigma_i : M \to (M^G)^{2n-2}$ be given by

$$\sigma_i(a) = \big((a^{-1})^{x_1}a, (a^{-1})^{y_1}a, \dots, \widehat{(a^{-1})^{x_i}}a, \widehat{(a^{-1})^{y_i}}a, \dots, (a^{-1})^{x_n}a, (a^{-1})^{y_n}a\big).$$

http://dx.doi.org/10.22108/ijgt.2023.135843.1816

For $x \in F$ and $a, b \in M$, as M is abelian, it follows that $((ab)^{-1})^x ab = (a^{-1})^x a(b^{-1})^x b$. Thus $a \mapsto (a^{-1})^x a$ defines a homomorphism $M \to M^G$, and we get that σ_i is a homomorphism. Noting that $|\,im(\sigma_i)| \leq p^{2n-2}$, $d(M) \geq 2n$ yields that $|\,ker(\sigma_i)| = \frac{|M|}{|\,im(\sigma_i)|} \geq p^2$. Since $|M^G| = p \lneq |\,ker(\sigma_i)|$, there exists $a_i \in ker(\sigma_i) \setminus M^G$. Note that either $(a_i^{-1})^{x_i} a_i \neq 1$ or $(a_i^{-1})^{y_i} a_i \neq 1$. If $(a_i^{-1})^{x_i} a_i \neq 1$, then we have $M^G = \langle (a_i^{-1})^{x_i} a_i \rangle$. Thus there exists $k \in \{1, \ldots, p-1\}$ such that $z_0 = ((a_i^{-1})^{x_i} a_i)^k = ((a_i^k)^{-1})^{x_i} a_i^k$. We replace a_i with a_i^k so that $(a_i^{-1})^{x_i} a_i = z_0$, and define $\delta(x_i) = 1$, $\delta(y_i) = a_i$. Similarly, if $(a_i^{-1})^{x_i} a_i = 1$ and $(a_i^{-1})^{y_i} a_i \neq 1$, we replace a_i with a power of a_i so that $a_i^{-1} a_i^{y_i} = z_0$, and define $\delta(x_i) = a_i$, $\delta(y_i) = 1$. Also define $\delta(c) = z_0$. By Lemma 2.1 (i), δ extends to a derivation $F \to M$. We proceed to check that δ preserves the relations in (2.1). First we see that $\gamma_2(F)$ acts trivially on M. For $a \in M$ and $x, y \in F$, $a^{xy} = (aa^{-1}a^x)^y = a^y a^{-1} a^x$, in which the last equality holds since $a^{-1}a^x \in M^G$. Similarly, $a^{yx} = a^x a^{-1} a^y = a^{xy}$, and hence $a^{[x,y]} = a$. Now we get an expression for $\delta([x, y])$, $x, y \in F$. Applying δ to the identity $xy = yx[x, y]$, we obtain $\delta(x)^y \delta(y) = \delta(y)^{x[x,y]} \delta(x)^{[x,y]} \delta([x, y])$. As the action of $[x, y]$ is trivial, we get that $\delta([x, y]) = \delta(x)^{-1}\delta(x)^y(\delta(y)^{-1})^x\delta(y)$. Thus, for all $i \in \{1, \ldots, n\}$,

$$\delta([x_i, y_i]) = \delta(x_i)^{-1}\delta(x_i)^{y_i}(\delta(y_i)^{-1})^{x_i}\delta(y_i) = z_0 = \delta(c).$$

Moreover, for all $x \in \{x_i, y_i\}$, $\delta([x, c]) = \delta(x)^{-1}\delta(x)^c(z_0^{-1})^x z_0$. Since $\gamma_2(F)$ acts trivially, and the action of relations in (2.1) is trivial, c acts trivially on M. Also since $z_0 \in M^G$, $(z_0^{-1})^x z_0 = 1$, and we obtain $\delta([x, c]) = 1$. Furthermore, as $a_i \in ker(\sigma_i)$, x_j, y_j acts trivially on a_i for $i \neq j$. Hence we deduce that $\delta([x, y]) = 1$ for all $x \in \{x_i, y_i\}$, $y \in \{x_j, y_j | i \neq j\}$. Now it remains to show $\delta(x^p) = 1$ for all $x \in \{x_i, y_i, c\}$. Let $\delta(x) = a$. Then $\delta(x^p) = aa^x \cdots a^{x^{p-1}}$. First we show that $a^{x^i} = a(a^{-1}a^x)^i$ for all $i \geq 1$, which is trivially true when $i = 1$. Now let $i \geq 1$. By induction hypothesis, $a^{x^{i+1}} = (a(a^{-1}a^x)^i)^x$. Since $a^{-1}a^x \in M^G$, $a^{x^{i+1}} = a^x(a^{-1}a^x)^i = aa^{-1}a^x(a^{-1}a^x)^i$, and the aim holds for $i + 1$. Therefore, $\delta(x^p) = a(aa^{-1}a^x)(a(a^{-1}a^x)^2) \cdots (a(a^{-1}a^x)^{p-1}) = a^p(a^{-1}a^x)^{\binom{p}{2}}$. We obtain $\delta(x^p) = 1$ since $p \mid \binom{p}{2}$ as $p \geq 3$ and M is elementary abelian. Hence, by Lemma 2.1 (ii), δ induces a derivation on G which we again denote with δ, and we have $\delta(c) = z_0 \neq 1$ as required. □

We prove the following lemma before proving the structure theorem mentioned in the introduction.

Lemma 2.3. *Let p be an odd prime and let G be a finite p-group with $|\gamma_2(G)| = \exp(G) = p$. Then either G is an extra-special group or there exists $U, V \leq G$ such that $G = U \times V$, where $U \leq Z(G)$ is elementary abelian, and V is an extra-special group. Furthermore, G has a minimal generating set $\{x_1, y_1, \ldots, x_n, y_n, x_{2n+1}, \ldots, x_{d(G)}\}$, $d(G) \geq 2n$, such that $V = \langle x_1, y_1, \ldots, x_n, y_n \rangle$ and $U = \langle x_{2n+1} \rangle \times \cdots \times \langle x_{d(G)} \rangle$.*

Proof. Since $|\gamma_2(G)| = p$ and $\exp(G) = p$, applying [6, Lemma 4.2] we obtain

$$G = VZ(G), \tag{2.2}$$

where V is an extra-special group. Note that $Z(V) \leq Z(G)$ by (2.2). If $|Z(G)| = p$, then we get $Z(G) = Z(V)$. Thus $G = VZ(G) = V$, and G is extra-special. Now let $|Z(G)| > p$. As $Z(G)$ is elementary abelian, we can write $Z(G) = Z(V) \times U$. By (2.2) it now follows that $G = UV$. Furthermore, $U \cap V \leq U \cap Z(V) = 1$, where $U \cap V \leq Z(V)$ holds since $U \leq Z(G)$. Thus $G = U \times V$. Moreover, $|V| = p^{2n+1}$, and for an odd prime p there is only one isomorphism class for extra-special groups of order p^{2n+1} and exponent p (cf. [16, p. 34]). Hence $d(V) = 2n$ and let $V = \langle x_1, y_1, \ldots, x_n, y_n \rangle$. Let $U = \langle x_{2n+1} \rangle \times \cdots \times \langle x_{2n+d(U)} \rangle$. We have that $G = \langle x_1, y_1, \ldots, x_n, y_n, x_{2n+1}, \ldots, x_{2n+d(U)} \rangle$. Since $\exp(G) = p$, $\Phi(G) = \gamma_2(G)$, and so $|G| = p^{d(G)}|\gamma_2(G)| = p^{d(G)}p$. On the other hand, $|G| = |U||V| = p^{d(U)}p^{2n+1}$, and hence we get $d(G) = 2n + d(U)$. This completes the proof. □

We recall the following fact about finite p-groups.

Lemma 2.4. *Let G be a finite p-group and let N, L be normal subgroups of G. If $N \leq L[N, G]$, then $N \leq L$.*

Theorem 2.5. *Let p be an odd prime and let G be a finite p-group. If G is not powerful, then a normal subgroup N of G exists such that either G/N is an extra-special group of exponent p or $G/N = U \times V$ where $U \leq Z(G/N)$ is elementary abelian and V is an extra-special group of exponent p.*

Proof. Since G is not powerful, $G^p\gamma_3(G) \lneq \Phi(G)$. Otherwise, we have $\gamma_2(G) \leq G^p\gamma_3(G)$, and this yields $\gamma_2(G) \leq G^p$ by Lemma 2.4. Now there exists $G^p\gamma_3(G) \leq N \leq \Phi(G)$ such that $[\Phi(G) : N] = p$. Since $N \leq \Phi(G)$, and $[\Phi(G), G] \leq G^p\gamma_3(G) \leq N$, $N \trianglelefteq G$. Clearly $\exp(G/N) = p$. Furthermore, as $G^p \leq N$ and $N \lneq \Phi(G)$, $\gamma_2(G) \nleq N$, and hence $\Phi(G) = \gamma_2(G)N$. Thus, $\gamma_2(G/N) = \Phi(G)/N$ and has order p. Now the conclusion of the theorem follows by Lemma 2.3. □

3. Useful results

All commutators used in this paper are left-normed and $[g, h] = g^{-1}h^{-1}gh = g^{-1}g^h$. We often use the following lemma by Mann [18].

Lemma 3.1. *(Mann) Let G be a p-group of class less than or equal to p, and let $x, y \in G$. Then $[x, y^p] = 1$ is equivalent to $[x, y]^p = 1$ and, similarly, it is equivalent to $[x^p, y] = 1$.*

Corollary 3.2. *Let G be a finite p-group and let $t \in Z_p(G)$. Then the following are equivalent.*

(i) $t \in C_G(G^p)$.

(ii) $t^p \in Z(G)$.

(iii) $[g, t]^p = 1$ *for all* $g \in G$.

Let N be a normal subgroup of G, then $Z(N)$ can be regarded as a G/N-module via conjugation in G. Let $C_{\mathrm{Aut}(G)}(G/N, N)$ denote the subgroup of $\mathrm{Aut}(G)$ consisting of all automorphisms α such that $x^\alpha = x$ for all $x \in N$ and $g^{-1}g^\alpha \in N$ for all $g \in G$. We use the following well-known fact (cf. [14, Satz I.4.4]):

http://dx.doi.org/10.22108/ijgt.2023.135843.1816

Proposition 3.3. *Let N be a normal subgroup of a group G, then there is a natural isomorphism $\varphi : Z^1(G/N, Z(N)) \to C_{\mathrm{Aut}(G)}(G/N, N)$ given by $g^{\varphi(f)} = gf(gN)$ for all $g \in G$, $f \in Z^1(G/N, Z(N))$. The image of $B^1(G/N, Z(N))$ under φ is the group of inner automorphisms of G induced by elements of $Z(N)$.*

In addition to the above proposition, suppose that $f \in Z^1(G/N, Z(N) \cap Z_i(G))$ and $\varphi(f) = i_u$ is an inner automorphism induced by u, then it follows that

$$f(gN) = g^{-1}g^{\varphi(f)} = [g, u] \tag{3.1}$$

for all $g \in G$. Also $u \in C_G(N) \cap Z_{i+1}(G)$.

The following corollary is given for $N = \Phi(G)$ in [4, Lemma 3.3], but for an arbitrary N, the proof follows along the same lines, and we include a proof for the benefit of the reader.

Corollary 3.4. *Let G be a finite p-group and let N be a normal subgroup of G such that $C_G(N) = Z(N)$. Set $A = \Omega_1(Z(N))$ and $A^* = \{a \in Z(N) | a^p \in Z(G)\}$. Suppose that all the automorphisms of G of order p fixing N elementwise are inner, then $Z^1(G/N, A \cap Z_i(G)) \cong \frac{A^* \cap Z_{i+1}(G)}{Z(G)}$ for all $i \in \mathbb{N}$. In particular, if $G^p \leq N$, then*

$$Z^1(G/N, A \cap Z_i(G)) \cong \frac{Z(N) \cap Z_{i+1}(G)}{Z(G)} \tag{3.2}$$

for all $i \in \{1, \dots, p-1\}$.

Proof. Since $C_G(N) = Z(N)$, and all the automorphisms of G of order p fixing N elementwise are inner, by Proposition 3.3, we have $Z^1(G/N, A) = B^1(G/N, A^*)$. Moreover, $Z^1(G/N, A \cap Z_i(G)) = B^1(G/N, A^* \cap Z_{i+1}(G))$ for all $i \in \mathbb{N}$. Note that $Z(G) \leq C_G(N) = Z(N)$. Hence $Z(G) \leq A^*$ holds trivially. Thus $(A^* \cap Z_{i+1}(G))^{G/N} = Z(G)$, and by the fact that $B^1(H, M) \cong \frac{M}{M^H}$ for a H-module M, we get $B^1(G/N, A^* \cap Z_{i+1}(G)) \cong \frac{A^* \cap Z_{i+1}(G)}{Z(G)}$. In addition, if $G^p \leq N$, then $Z(N) \leq C_G(G^p)$, and hence $(Z(N) \cap Z_p(G))^p \leq Z(G)$ by Corollary *3.2*, and so $Z(N) \cap Z_p(G) \leq A^*$. Therefore, $A^* \cap Z_{i+1}(G) = Z(N) \cap Z_{i+1}(G)$ for all $i \in \{1, \dots, p-1\}$. □

Recall that for a maximal subgroup $\mathfrak{m}$ of a finite p-group G, we either have $Z(\mathfrak{m}) \leq Z(G)$ or $C_G(Z(\mathfrak{m})) = \mathfrak{m}$.

We now collect some facts, which gives a reduction to the conjecture.

Lemma 3.5. *Let G be a finite nonabelian p-group. Then G admits a non-inner automorphism of order p fixing $\Phi(G)$ elementwise, if one of the following occurs:*

- *(i) The nilpotency class of G is 2 or 3 and $p \geq 3$ (*[17, Theorem 1], [4, Theorem 4.4]*).*
- *(ii) $G/Z(G)$ is powerful (*[3, Theorem 2.6]*).*
- *(iii) $C_G(Z(\Phi(G))) \neq \Phi(G)$* [8].
- *(iv) G is regular* [22, 8].
- *(v) $Z(\mathfrak{m}) \leq Z(G)$ for a maximal subgroup $\mathfrak{m}$ of G (*[20, Lemma 9.108]*).*
- *(vi) $d\big(Z_2(G)/Z(G)\big) \neq d\big(Z(G)\big)d(G)$ (*[3, Corollary 2.3]*).*

(*vii*) $d\big(\Omega_1(Z_2(G))\big) < d\big(Z(G)\big)d(G)$ *(*[2, Remark 2]*).*

Lemma 3.6. *Let $1 \neq N$ be a proper normal subgroup of a group G. If $C_G(Z(N)) = N$, then $Z(G) \lneq Z(N) = C_G(N)$.*

Proof. As $Z(N) \leq N$, $C_G(N) \leq C_G(Z(N)) = N$, and hence $C_G(N) = Z(N)$. Furthermore, $Z(G) \leq C_G(N) = Z(N)$. Since $C_G(Z(N)) = N \lneq G$, we have $Z(N) \nleq Z(G)$. □

Remark 3.7. *Let p be a prime and let G be a finite nonabelian p-group. Let $Z_2^*(G) = \{a \in Z_2(G) | a^p \in Z(G)\}$. Then $Z_2^*(G) \leq C_G(G^p)$ by Corollary 3.2, and $[Z_2^*(G), \gamma_2(G)] = 1$ holds trivially. Thus $Z_2^*(G) \leq C_G(\Phi(G))$.*

Remark 3.8. *Let p be an odd prime and let G be a finite nonabelian p-group. Suppose that all the automorphisms of G of order p fixing $\Phi(G)$ elementwise are inner. Then $\Omega_1(Z_2(G)) \leq C_G(\Phi(G)) = Z(\Phi(G))$. Furthermore, setting $G = G/\Phi(G)$, $A = C = \Omega_1(Z_2(G))$, and $D = \Omega_1(Z(G))$ in* [4, Lemma 2.3] *yields that*

$$d\big(Z^1(G/\Phi(G), \Omega_1(Z_2(G)))\big) \geq d\big(\Omega_1(Z_2(G))\big)d(G) - d\big(\Omega_1(Z(G))\big)\binom{d(G)}{2}.$$

By (3.2), $Z^1(G/\Phi(G), \Omega_1(Z_2(G))) \cong \frac{Z(\Phi(G))\cap Z_3(G)}{Z(G)}$, and hence the above gives a lower bound for $d\left(\frac{Z(\Phi(G))\cap Z_3(G)}{Z(G)}\right)$.

Any of the conditions of Lemma 3.5 yields the existence of a non-inner automorphism of order p fixing $G^p\gamma_3(G) \leq \Phi(G)$ elementwise. In addition, we have the following lemma when $d(G) = 2$.

Lemma 3.9. *(*[2, Theorem 1]*) Let p be an odd prime and let G be a 2-generator finite p-group. If G fails to satisfy the condition $Z(\Phi(G)) \lneq Z(G^p\gamma_3(G)) = C_G(G^p\gamma_3(G))$ or if $d\left(\frac{Z(G^p\gamma_3(G))\cap Z_3(G)}{Z(G)}\right) < 2d\big(\Omega_1(Z_2(G))\big)$, then G admits a non-inner automorphism of order p fixing $G^p\gamma_3(G)$ elementwise.*

Remark 3.10. *Let p be an odd prime and let G be a finite nonabelian p-group such that $Z(G) \leq Z(G^p\gamma_3(G))$. Since $[Z(G^p\gamma_3(G)), G^p] = 1$, $(Z(G^p\gamma_3(G)) \cap Z_3(G))^p \leq Z(G)$ by Corollary 3.2. Moreover, $|\Omega_1(A)| = \left|\frac{A}{A^p}\right|$ for every finite abelian p-group A, so we obtain $\big|\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))\big| \geq \left|\frac{Z(G^p\gamma_3(G))\cap Z_3(G)}{Z(G)}\right|$. Both $\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))$ and $\frac{Z(G^p\gamma_3(G))\cap Z_3(G)}{Z(G)}$ are elementary abelian, and thus $d\big(\Omega_1(Z_3(G^p\gamma_3(G)) \cap Z_3(G))\big) \geq d\left(\frac{Z(G^p\gamma_3(G))\cap Z_3(G)}{Z(G)}\right)$.*

We rewrite these reductions as a hypothesis.

Definition 3.11. *We say that a finite nonabelian p-group G of odd order satisfies Hypothesis A, if the following holds true for G.*

(*i*) *The nilpotency class of G is at least 4.*

(*ii*) *$G/Z(G)$ is not powerful.*

(iii) $C_G(Z(\Phi(G)) = \Phi(G)$ *and* $Z(G) \lneq Z(\Phi(G)) = C_G(\Phi(G))$.

(iv) G *is not regular.*

(v) $C_G(Z(\mathfrak{m})) = \mathfrak{m}$ *and* $Z(G) \lneq Z(\mathfrak{m}) = C_G(\mathfrak{m})$ *for every maximal subgroup* $\mathfrak{m}$ *of* G.

(vi) $\Omega_1(Z_2(G)) \leq Z_2^*(G) \leq Z(\Phi(G))$.

(vii) G *satisfies* (3.3)–(3.5).

$$d\big(Z_2(G)/Z(G)\big) = d\big(Z(G)\big)d(G). \tag{3.3}$$

$$d\big(\Omega_1(Z_2(G))\big) \geq d\big(Z(G)\big)d(G). \tag{3.4}$$

$$d\left(\frac{Z(\Phi(G)) \cap Z_3(G)}{Z(G)}\right) \geq d\big(\Omega_1(Z_2(G))\big)d(G) - d\big(\Omega_1(Z(G))\big)\binom{d(G)}{2}. \tag{3.5}$$

(viii) *Either* $d(G) \geq 3$ *or* (3.6)–(3.8) *holds for* G*:*

$$Z(\Phi(G)) \lneq Z(G^p\gamma_3(G)) = C_G(G^p\gamma_3(G)). \tag{3.6}$$

$$d\left(\frac{Z(G^p\gamma_3(G)) \cap Z_3(G)}{Z(G)}\right) \geq 2d\big(\Omega_1(Z_2(G))\big). \tag{3.7}$$

$$d\big(\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))\big) \geq 2d\big(\Omega_1(Z_2(G))\big). \tag{3.8}$$

Remark 3.12. *Let* p *be an odd prime and let* G *be a finite nonabelian* p*-group. It follows from the discussion above that if* $d(G) \geq 3$ *and* G *does not have a non-inner automorphism of order* p *fixing* $\Phi(G)$ *elementwise or if* $d(G) = 2$ *and* G *does not have a non-inner automorphism of order* p *fixing* $G^p\gamma_3(G)$ *elementwise, then* G *satisfies Hypothesis* A.

4. Finite p-groups having cyclic center

This section proves the conjecture for every finite nonabelian p-group G with cyclic center satisfying $C_G(G^p\gamma_3(G)) \cap Z_3(G) \leq Z(\Phi(G))$. We begin by proving the following lemma.

Lemma 4.1. *Let* p *be an odd prime and let* G *be a finite nonabelian* p*-group with cyclic center. Then* $\Omega_1(Z_2(G)) \leq Z(G^p\gamma_3(G))$*, if all the automorphisms of* G *of order* p *fixing* $\Phi(G)$ *elementwise are inner.*

Proof. We assume Hypothesis A for G. Thus $\Omega_1(Z_2(G)) \leq Z(\Phi(G))$, and so $[G^p\gamma_3(G), \Omega_1(Z_2(G))] = 1$. Now we proceed to show that $\Omega_1(Z_2(G)) \leq G^p\gamma_3(G)$. Let $a \in \Omega_1(Z_2(G))$. For all $g \in G$, $a^g = a[a,g]$, and since $a^p = 1$, we get $[a,g]^p = 1$ by Corollary 3.2, and so $[a,g] \in \Omega_1(Z(G))$. Since $|\Omega_1(Z(G))| = p$, we obtain that the number of conjugates of a in G is at most p. Thus, either $a \in \Omega_1(Z(G))$, or $[G : C_G(a)] = p$, and $a \in Z(C_G(a))$ in the latter case. Furthermore, if $a \in \Omega_1(Z(G))$, then $a \in Z(\mathfrak{m})$ for every maximal subgroup $\mathfrak{m}$ of G by Hypothesis A. Thus in either case, $a \in Z(\mathfrak{m})$ for a maximal subgroup $\mathfrak{m}$ of G. Let $x \in G \setminus \mathfrak{m}$. Since $a \in \Omega_1(Z_2(G))$, $aa^x \cdots a^{x^{p-1}} = a^p[a,x]^{\binom{p}{2}} = 1$, and so a derivation $\delta : G/\mathfrak{m} \to Z(\mathfrak{m})$ exists with $\delta(x\,\mathfrak{m}) = a$. Note that the order of δ is p. Let $\alpha = \varphi(\delta) \in C_{\mathrm{Aut}(G)}(G/\mathfrak{m}, \mathfrak{m})$ given by $\alpha(g) = g\delta(g\,\mathfrak{m})$ for all $g \in G$. Then α has order p, and fixes $\Phi(G) \leq \mathfrak{m}$ elementwise. Hence $\alpha = i_u$ is an inner automorphism of G. It now follows that $u \in C_G(\Phi(G)) = Z(\Phi(G))$, and $a = [g,u]$ by (3.1). Thus $a \in [G, \Phi(G)] \leq G^p\gamma_3(G)$. □

http://dx.doi.org/10.22108/ijgt.2023.135843.1816

Below we recall a couple of well-known commutator identities that are often used in this paper. For x, y, z, elements of a group G, we have

$$[xy, z] = [x, z]^y\,[y, z] = [x, z][x, z, y][y, z] \text{ and} \tag{4.1}$$

$$[z, xy] = [z, y][z, x]^y = [z, y][z, x][z, x, y]. \tag{4.2}$$

Abdollahi and Ghoraishi [2, Theorem 1] proved that if a 2-generator finite p-group G of odd order does not have a non-inner automorphism of order p fixing $G^p\gamma_3(G)$ elementwise, then $Z(\Phi(G)) \lneq C_G(G^p\gamma_3(G)) = Z(G^p\gamma_3(G))$. Abdollahi [3, Theorem 2.5] proved that if a finite nonabelian p-group G of coclass r does not have a non-inner automorphism of order p fixing $\Phi(G)$ elementwise, then $d\big(Z(G)\big)\big(d(G)+1\big) \leq r+1$. In the next theorem, we obtain similar reductions to the conjecture when $Z(G)$ is cyclic.

Theorem 4.2. *Let p be an odd prime and let G be a finite nonabelian p-group with cyclic center. Suppose that all the automorphisms of G of order p fixing $G^p\gamma_3(G)$ elementwise are inner, then the following holds:*

- *(i) $C_G(G^p\gamma_3(G)) \cap Z_3(G) \nleq Z(\Phi(G))$.*
- *(ii) $\binom{d(G)+1}{2} \leq r$, where r is the coclass of G.*

Proof. We assume that G satisfies Hypothesis A. We prove (i) by showing the existence of an automorphism of order p that fixes $G^p\gamma_3(G)$ and $G/\Omega_1(Z_2(G))$ elementwise, but not $\Phi(G)$, and we prove (ii) by using the conditions of Hypothesis A and by (i).

(i) By Hypothesis A, G is not powerful. Hence, by Theorem 2.5, G has a normal subgroup $G^p\gamma_3(G) \leq N \lneq \Phi(G)$ such that $G/N = U/N \times V/N$, where V/N is an extra-special group and $U/N \leq Z(G/N)$ is elementary abelian. Furthermore, we assume $\{x_1, y_1, \ldots, x_n, y_n, x_{2n+1}, \ldots, x_{d(G)}\}$, $d(G) \geq 2n$, is a minimal generating set for G such that $V/N = \langle \bar{x_1}, \ldots, \bar{y_n}\rangle$ and $U/N = \langle \overline{x_{2n+1}}\rangle \times \cdots \times \langle \overline{x_{d(G)}}\rangle$. Since $N \leq \Phi(G)$ and $\Omega_1(Z_2(G)) \leq Z(\Phi(G))$, we obtain $[\Omega_1(Z_2(G)), N] = 1$, and $\Omega_1(Z_2(G)) \leq G^p\gamma_3(G) \leq N$ by Lemma 4.1. Thus $\Omega_1(Z_2(G)) \leq Z(N)$ which is a G/N-module. Set $M = C_{\Omega_1(Z_2(G))}(U) \leq Z(N)$. Since $U/N \leq Z(G/N)$, $U/N \trianglelefteq G/N$, and $U \trianglelefteq G$. Thus $C_G(U) \trianglelefteq G$, and hence $M = C_G(U) \cap \Omega_1(Z_2(G)) \trianglelefteq G$. Therefore, M is a G/N-module. Next we check that the conditions of Lemma 2.2 holds when M is considered as a V/N-module. Since U/N acts trivially on M, we obtain $M^{V/N} = M^{G/N} = \Omega_1(Z(G))$. In particular, $|M^{V/N}| = p$. Now we look for a comparison of $d(M)$ and $d(V/N)$. If $G/N = V/N$, then $M = \Omega_1(Z_2(G))$, and we deduce that $d\big(\Omega_1(Z_2(G))\big) \geq d(G) = d(G/N)$ by (3.4). Now let $G/N \gneq V/N$. Consider the map $\sigma : \Omega_1(Z_2(G)) \to \Omega_1(Z(G))^{d(G)-2n}$ given by

$$\sigma(a) = \big([x_{2n+1}, a], \ldots, [x_{d(G)}, a]\big).$$

Let $x \in G$. By Corollary 3.2, $[x, a]^p = 1$ for all $a \in \Omega_1(Z_2(G))$, and expanding $[x, ab]$ using (4.2), we obtain that $a \mapsto [x, a]$ defines a homomorphism $\Omega_1(Z_2(G)) \to \Omega_1(Z(G))$. Thus σ is a homomorphism and $im(\sigma) \leq \Omega_1(Z(G))^{d(G)-2n}$. Note that $ker(\sigma) = M$. Hence $|\Omega_1(Z_2(G))| = |M||\,im(\sigma)| \leq |M|p^{d(G)-2n}$, and $p^{d(G)} \leq |\Omega_1(Z_2(G))|$ by (3.4), so that $p^{2n} \leq |M|$. Thus $2n = d(V/N) \leq d(M)$, because M is elementary abelian. Moreover, $2n \leq d(M)$ implies that $M \not\leq \Omega_1(Z(G))$, so we get $[M, V/N] = \Omega_1(Z(G)) = M^{V/N}$. Therefore, applying Lemma 2.2 with $G = V/N$, we obtain a derivation $\delta \in Z^1(V/N, M)$ with $\delta([x_1, y_1]) \neq 1$, and δ has order p. If $G/N = V/N$, then $\delta \in Z^1(G/N, M)$, otherwise the extension $\delta' \in Z^1(G/N, M)$ of δ that corresponds to $(1_{U/N}, \delta) \in \mathrm{Hom}(U/N, M) \times Z^1(V/N, M)$ (See [7, Lemma 1.2]) will have order p, and satisfies $\delta'([x_1, y_1]) \neq 1$, and we denote δ' with δ in the latter case. Let $\alpha = \varphi(\delta) \in C_{\mathrm{Aut}(G)}(G/N, N)$ given by $\alpha(g) = g\delta(gN)$ for all $g \in G$. Then α has order p, and fixes $G^p\gamma_3(G) \leq N$ elementwise. Hence $\alpha = i_u$ is an inner automorphism of G. It now follows that $u \in C_G(G^p\gamma_3(G)) \cap Z_3(G)$, and $[x_1, y_1, u] = \delta([x_1, y_1]) \neq 1$ by (3.1), so that $u \notin C_G(\Phi(G)) = Z(\Phi(G))$.

(*ii*) Let $|G| = p^n$. By Hypothesis *A*, $n - r \geq 4$. First note that $\frac{G}{Z_{n-r-1}(G)}$ is not cyclic, and so $\left|\frac{G}{Z_{n-r-1}(G)}\right| \geq p^2$, and $\left|\frac{Z_{n-r-1}(G)}{Z_3(G)}\right| \geq p^{n-r-4}$. Now we obtain a lower bound for $|Z_3(G)/Z(G)|$. Since $Z(G)$ is cyclic, we get $d\big(\Omega_1(Z_2(G))\big) \geq d(G)$ by (3.4), and using this in (3.5) yields

$$d\left(\frac{Z(\Phi(G)) \cap Z_3(G)}{Z(G)}\right) \geq d(G)^2 - \binom{d(G)}{2} = \binom{d(G)+1}{2}. \tag{4.3}$$

Furthermore, $Z_3(G) \not\leq Z(\Phi(G))$ by (*i*) so that $\left|\frac{Z_3(G)}{Z(\Phi(G))\cap Z_3(G)}\right| \geq p$. Hence $\left|\frac{Z_3(G)}{Z(G)}\right| \geq p^{\binom{d(G)+1}{2}+1}$ by (4.3). Thus

$$\begin{aligned} p^n =& \big|Z(G)\big| \left|\frac{Z_3(G)}{Z(G)}\right| \left|\frac{Z_{n-r-1}(G)}{Z_3(G)}\right| \left|\frac{G}{Z_{n-r-1}(G)}\right| \\ \geq& p\; p^{\binom{d(G)+1}{2}+1}\; p^{n-r-4}\; p^2, \end{aligned}$$

yielding $\binom{d(G)+1}{2} \leq r$.

□

Let G be a finite nonabelian p-group, $p \geq 3$. Suppose that $Z(G)$ is cyclic and all the automorphisms of G of order p leaving $G^p\gamma_3(G)$ elementwise fixed are inner, then

$$C_G(G^p\gamma_3(G)) \cap Z_3(G) \not\leq Z(\Phi(G)). \tag{4.4}$$

In next definition we write this as a hypothesis.

Definition 4.3. *A finite nonabelian p-group G of odd order satisfies Hypothesis B, if*

(*i*) *Hypothesis A holds for G and*

(*ii*) *either $Z(G)$ is not cyclic, or (4.4) holds for G.*

The homomorphisms like σ considered in the proof of Theorem 4.2 appear several times in the next section. To avoid the repetition of arguments, we will record the following lemma.

Lemma 4.4. *Let G be a finite p-group, $N \trianglelefteq G$. Let H be a group and $\chi : H \to G/N$ be a homomorphism. Let $M \leq Z(N) \cap Z_n(G)$ be a normal abelian subgroup of G considered as a H-module via χ. Let $x'_1, \ldots, \ldots, x'_l \in H$, then the map $\lambda : M \to Z(N) \cap Z_{n-w+1}(G)$ given by $\lambda(a) = c(x'_1, \ldots, x'_l, a)$ is a homomorphism, where c is a commutator of weight $w \geq 2$ in $\{x'_1, \ldots, x'_l, a\}$ and of weight 1 in a. Furthermore, if $n \leq p$, then the image of M^* under λ is contained in $\Omega_1(Z(N) \cap Z_{n-w+1}(G))$, where $M^* = \{a \in M | a^p \in Z(G)\}$.*

Proof. We have that $c(x'_1, \ldots, x'_l, a) = c(x_1, \ldots, x_l, a)$, where $x_k \in G$ satisfy $\chi(x'_k) = x_k N$ for all $k \in \{1, \ldots, l\}$. Thus $im(\lambda) \leq Z(N) \cap Z_{n-w+1}(G)$, and expanding $c(x'_1, \ldots, x'_l, ab)$ for all $a, b \in M$, we deduce that λ is a homomorphism. Furthermore, if $n \leq p$, then $M \leq Z_p(G)$ so that $\lambda(M^*) \leq \Omega_1(Z(N))$ by Corollary 3.2. Thus we obtain $\lambda(M^*) \leq \Omega_1(Z(N) \cap Z_{n-w+1}(G))$. □

The following technical lemma will be useful in Section 5.

Lemma 4.5. *Let G and H be two groups, and let $N \trianglelefteq G$. Let $\chi : H \to G/N$ be a homomorphism. Let $M \leq Z(N) \cap Z_4(G)$ be a normal abelian subgroup of G considered as a H-module via χ and let $\delta : H \to M$ be a derivation. Let $x', y', z', w' \in H$ and let $x, y, z, w \in G$ such that $\chi(x') = xN, \chi(y') = yN, \chi(z') = zN, \chi(w') = wN$. If $\delta(x') = a_1$, $\delta(y') = a_2$, and $\delta(z') = a_3$, then we have the following.*

(*i*) $\delta([y', x']) = [a_2, x][y, a_1][y, x, a_2][y, x, a_1][y, x, [a_1, y]]$.
(*ii*) $\delta([y', x', z']) = [a_2, x, z][y, a_1, z][y, x, a_2, z][y, x, a_1, z][y, x, a_3][y, x, z, a_3]$.
(*iii*) *if $M \leq C_G(\gamma_3(G))$, then* $\delta([y', x', z', w']) = [\delta([y', x', z']), w]$.
(*iv*) *if G is a p-group, $p \geq 5$, and $a_1^p = 1$, then* $\delta((x')^p) = 1$.
(*v*) *if G is a 3-group, $a_1 \in Z_3(G)$ such that $a_1^3 = 1$, and $[a_1, x, x] = 1$, then* $\delta((x')^3) = 1$.

Proof. For $a \in M$ and $h \in H$ with $\chi(h) = gN$, we have $a^h = g^{-1}ag$. Thus applying δ to the identity $y'x' = x'y'[y', x']$, we obtain

$$a_2^x a_1 = a_1^{y[y,x]} a_2^{[y,x]} \delta([y', x']).$$

Writing $a_1^{y[y,x]} = a_1[a_1, y[y, x]]$, and expanding $[a_1, y[y, x]]$ by (4.2), we get

$$a_2[a_2, x]a_1 = a_1[a_1, [y, x]][a_1, y][a_1, y, [y, x]]a_2[a_2, [y, x]]\delta([y', x']),$$

which yields (*i*). Next to prove (*ii*), we apply (*i*) with $x' = z'$ and $y' = [y', x']$. Since $M \leq Z_4(G)$, we get $[y, x, z, [a_3, [y, x]]] = 1$, and since $\delta([y', x']) \in Z_3(G)$ by (*i*), we get $[y, x, z, \delta([y', x'])] = 1$. Thus

$$\delta([y', x', z']) = [\delta([y', x']), z][y, x, a_3][y, x, z, a_3].$$

Using (*i*) in $[\delta([y', x']), z]$ yields $[[a_2, x][y, a_1][y, x, a_2][y, x, a_1][y, x, [a_1, y]], z]$, and expanding this by the repeated use of (4.1) gives (*ii*). Similarly to prove (*iii*), we apply (*i*) with $x' = w'$ and $y' = [y', x', z']$. Since $M \leq Z_4(G)$, we obtain $\delta([y', x', z', w']) = [\delta([y', x', z']), w][y, x, z, \delta(w')]$, in which $[y, x, z, \delta(w')] = 1$ since $[\gamma_3(G), M] = 1$ by the assumption in (*iii*). To prove (*iv*) and (*v*), let us first express $\delta((x')^p)$ as

$$\delta((x')^p) = a_1 a_1^x \cdots a_1^{x^{p-1}} = a_1^p [a_1, x]^{\binom{p}{2}} [a_1, x, x]^{\binom{p}{3}} [a_1, x, x, x]^{\binom{p}{4}}. \tag{4.5}$$

When $p \geq 5$, since $a_1^p = 1$ and $a_1 \in Z_4(G) \leq Z_p(G)$, we obtain $[a_1, x]^p = [a_1, x, x]^p = [a_1, x, x, x]^p = 1$ by Corollary 3.2. Moreover, p divides $\binom{p}{i}$, $i = 2, 3, 4$, hence we get $\delta((x')^p) = 1$ by (4.5). In order to prove (v), we deduce that $\delta((x')^3) = a_1^3[a_1, x]^3$ by (4.5). Since $a_1^3 = 1$ and $a_1 \in Z_3(G)$, we get $[a_1, x]^3 = 1$ by Corollary 3.2. This yields $\delta((x')^3) = 1$. □

5. Existence of a non-inner automorphism of order p in finite p-groups of coclass 4 and 5

In this section we prove the conjecture for finite nonabelian p-groups of coclass 4 and 5, $p \geq 5$. Suppose that G is a finite nonabelian p-group of order p^n, class c, and coclass r. Since $\left|\frac{G}{Z_{c-1}(G)}\right| \geq p^2$, and $\left|\frac{Z_{c-1}(G)}{Z_i(G)}\right| \geq p^{c-1-i}$, we have $\left|\frac{G}{Z_i(G)}\right| \geq p^{c+1-i}$, and thus $|Z_i(G)| \leq p^{i+r-1}$ for all $i \in \{1, \ldots, c-1\}$. In particular, for finite p-groups of coclass 4 and 5, we have $|Z_i(G)| \leq p^{i+4}$ for all $i \in \{1, \ldots, c-1\}$.

Theorem 5.1. *Let p be an odd prime and let G be a finite nonabelian p-group of class c such that $|Z_i(G)| \leq p^{i+4}$ for all $i \in \{1, \ldots, c-1\}$. Then G admits a non-inner automorphism of order p fixing $G^p\gamma_3(G)$ elementwise, if one of the following occurs:*

(i) $Z(G)$ *is not cyclic.*

(ii) $d(G) \geq 3$.

(iii) $|\Omega_1(Z_2(G))| \geq p^3$.

Proof. Suppose that G does not satisfy the conclusion of the theorem, then we can assume Hypothesis B for G. Furthermore, since $|Z_2(G)| \leq p^6$, we get $\left|\frac{Z_2^*(G)}{Z(G)}\right| \leq p^5$, and hence we obtain (5.1) by (3.3):

$$d(G)d\big(Z(G)\big) \leq 5. \tag{5.1}$$

(i) Since G is nonabelian, $d(G) \geq 2$. Thus, if $Z(G)$ is not cyclic, then we get $d(G) = d\big(Z(G)\big) = 2$ by (5.1). Using $d(G) = d\big(Z(G)\big) = 2$ in (3.4) yields $d\big(\Omega_1(Z_2(G))\big) \geq 4$, and using this in (3.5) gives that $d\left(\frac{Z(\Phi(G)) \cap Z_3(G)}{Z(G)}\right) \geq 6$. Thus $|Z(\Phi(G)) \cap Z_3(G)| \geq p^6|Z(G)| \geq p^8$, where we get the second inequality since $d\big(Z(G)\big) = 2$. This is a contradiction to $|Z_3(G)| \leq p^7$, whence the proof.

(ii) We now assume $Z(G)$ is cyclic by (i). Then using that $\left|\frac{Z_3(G)}{Z(G)}\right| \geq p^{\binom{d(G)+1}{2}+1}$, which we obtained in the proof of Theorem 4.2 (ii), we get $\left|\frac{Z_3(G)}{Z(G)}\right| \geq p^7$. This yields $|Z_3(G)| \geq p^8$, a contradiction to $|Z_3(G)| \leq p^7$.

(iii) We now assume that $Z(G)$ is cyclic and $d(G) = 2$ by (i) and (ii). Since $\Omega_1(Z_2(G))$ is elementary abelian, $|\Omega_1(Z_2(G))| \geq p^3$ implies that $d\big(\Omega_1(Z_2(G))\big) \geq 3$, and thus we obtain (5.2) by (3.8):

$$d\big(\Omega_1(Z(G^p\gamma_3(G)\big) \cap Z_3(G))) \geq 6. \tag{5.2}$$

In the next few lines, we aim to find the isomorphism class of $G/G^p\gamma_3(G)$. First note that $G/G^p\gamma_3(G)$ is a nonabelian group. Otherwise, we have $\gamma_2(G) \leq G^p\gamma_3(G)$, which yields that G is powerful by Lemma 2.4. Now let $H = \langle h_1, h_2\rangle$ be a finite p-group of class 2 and of exponent p. Then it follows that $\gamma_2(H) = \langle [h_1, h_2] \rangle$, and since $\exp(H) = p$, we get $|\gamma_2(H)| = p$. Furthermore, since the class of H is 2, we have $\gamma_2(H) \leq Z(H)$, and since $\exp(H) = p$, we have $\gamma_2(H) = \Phi(H)$, and so we get $[H : Z(H)] \leq [H : \gamma_2(H)] = p^2$. Note that $H/Z(H)$ is not cyclic. Thus $[H : Z(H)] = p^2$, and we get $\gamma_2(H) = Z(H)$. Therefore, H is an extra-special group of order p^3 and of exponent p. Taking $H = G/G^p\gamma_3(G)$ in the above discussion, we obtain the below presentation for $G/G^p\gamma_3(G)$:

$$\langle x, y \mid x^3, y^3, [y, x, x], [y, x, y] \rangle. \tag{5.3}$$

We now proceed to give a family of derivations from $G/G^p\gamma_3(G) \to \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))$. Let F be a free group on $\{x, y\}$. Then $\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))$ is an F-module. By taking $H = F$ and $N = G^p\gamma_3(G)$ in Lemma 4.4, we see that the map

$$\begin{aligned} \tau : \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))^2 &\to \Omega_1(Z(G))^4 \\ (a, b) \mapsto &\big([b, x, x][y, a, x][y, x, a],\ [b, x, y][y, a, y][y, x, b],\ [a, x, x],\ [b, y, y]\big) \end{aligned}$$

is a homomorphism. Let $(a, b) \in ker(\tau)$. By Lemma 2.1 (i), the assignment $x \mapsto a$, $y \mapsto b$ extends to a derivation $\delta_{a,b}$ of F. We now check that $\delta_{a,b}$ preserves the relations in (5.3). By Lemma 4.5 (ii), we have

$$\begin{aligned} \delta_{a,b}([y, x, x]) =& [b, x, x][y, a, x][y, x, a] \text{ and} \\ \delta_{a,b}([y, x, y]) =& [b, x, y][y, a, y][y, x, b]. \end{aligned}$$

Since $(a, b) \in ker(\tau)$, we get $\delta_{a,b}([y, x, x]) = \delta_{a,b}([y, x, y]) = 1$. Similarly we get $\delta_{a,b}(x^p) = \delta_{a,b}(y^p) = 1$ by Lemma 4.5 (iv) and (v). Hence, by Lemma 2.1 (ii), $\delta_{a,b}$ induces a unique derivation from $G/G^p\gamma_3(G) \to \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))$, and so

$$|\, ker(\tau)| \leq |Z^1(G/G^p\gamma_3(G), \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G)))|.$$

Since $d(G) = 2$, we have $C_G(G^p\gamma_3(G)) = Z(G^p\gamma_3(G))$ by (3.6). Therefore, applying Corollary 3.4 with $N = G^p\gamma_3(G)$, we obtain

$$|\, ker(\tau)| \leq \left| \frac{A^* \cap Z_4(G)}{Z(G)} \right|, \tag{5.4}$$

where $A^* = \{a \in Z(G^p\gamma_3(G)) | a^p \in Z(G)\}$. Now we will find a lower bound for $|\, ker(\tau)|$. Since $|\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))| \geq p^6$ by (5.2), and $|\, im(\tau)| \leq p^4$, we have

$$|\, ker(\tau)| = \frac{|\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))|^2}{|\, im(\tau)|} \geq p^8.$$

This yields $|A^* \cap Z_4(G)| \geq p^8 |Z(G)| \geq p^9$ by (5.4), a contradiction to $|Z_4(G)| \leq p^8$, whence the proof.

□

Theorem 5.2. *Let $p \geq 5$ and let G be a finite nonabelian p-group of class c such that $|Z_i(G)| \leq p^{i+4}$ for all $i \in \{1, \ldots, c-1\}$. Then G admits a non-inner automorphism of order p fixing $G^p\gamma_3(G)$ elementwise, if either of the following occurs:*

(*i*) $\left|\frac{Z(G^p\gamma_3(G))\cap Z_4(G)}{Z(G)}\right| \leq p^5$.

(*ii*) $\left|\frac{Z(G^p\gamma_3(G))\cap Z_5(G)}{Z(G)}\right| \leq p^7$.

Proof. Suppose that all the automorphisms of G of order p fixing $G^p\gamma_3(G)$ elementwise are inner. Then we assume that Hypothesis B holds for G. Furthermore, we assume that $Z(G)$ is cyclic, $d(G) = 2$, and $|\Omega_1(Z_2(G))| \leq p^2$ by Theorem 5.1. Thus we get $|\Omega_1(Z_2(G))| = p^2$ by (3.4). Moreover, $G/G^p\gamma_3(G)$ is an extra-special group of order p^3, exponent p, and has presentation (5.3). Let F be a free group on $\{x, y\}$. It follows that $Z(G^p\gamma_3(G))$ is an F-module.

(*i*) We now give a family of derivations from $G/G^p\gamma_3(G) \to \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))$. First note that the map

$$\tau_1 : \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))^2 \to \Omega_1(Z(G))^2$$
$$(a, b) \mapsto \big([b, x, x][y, a, x][y, x, a],\ [b, x, y][y, a, y][y, x, b]\big)$$

is a homomorphism by Lemma 4.4, and let $(a, b) \in ker(\tau_1)$. Since F is a free group, the map $x \mapsto a, y \mapsto b$ extends to a derivation $\delta_{1a,b}$ of F, and by Lemma 4.5 (*ii*) and (*iv*), we check that $\delta_{1a,b}$ preserves the relations in (5.3). This implies that $\delta_{1a,b}$ induces a unique derivation from $G/G^p\gamma_3(G) \to \Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))$ by Lemma 2.1 (*ii*). Furthermore, since $d(G) = 2$, we have $C_G(G^p\gamma_3(G)) = Z(G^p\gamma_3(G))$ by (3.6), and thus applying (3.2) with $N = G^p\gamma_3(G)$ yields that

(5.5) $$\left|\frac{Z(G^p\gamma_3(G)) \cap Z_4(G)}{Z(G)}\right| \geq |\, ker(\tau_1)|.$$

On the other hand, since $|\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))| \geq p^4$ by (3.8), and $|\, im(\tau_1)| \leq p^2$, we obtain

$$|\ker(\tau_1)| = \frac{|\Omega_1(Z(G^p\gamma_3(G)) \cap Z_3(G))|^2}{|\, im(\tau_1)|} \geq p^6.$$

Now using (5.5), we obtain that G admits a non-inner automorphism of order p fixing $G^p\gamma_3(G)$ elementwise whenever $\left|\frac{Z(G^p\gamma_3(G))\cap Z_4(G)}{Z(G)}\right| \leq p^5$.

(*ii*) We now assume that $\left|\frac{Z(G^p\gamma_3(G))\cap Z_4(G)}{Z(G)}\right| \geq p^6$ by (*i*), and since $Z_4(G) \leq Z_p(G)$, as explained in Remark 3.10, this implies that

(5.6) $$|\Omega_1(Z(G^p\gamma_3(G)) \cap Z_4(G))| \geq p^6.$$

In order to prove (ii), consider the map

$$\tau_2 : \Omega_1(Z(G^p\gamma_3(G)) \cap Z_4(G))^2 \to \Omega_1(Z_2(G))^2$$
$$(a, b) \mapsto \big([b, x, x][y, a, x][y, x, b, x][y, x, a, x][y, x, a],$$
$$[b, x, y][y, a, y][y, x, b, y][y, x, a, y][y, x, b]\big).$$

We have that τ_2 is a homomorphism by Lemma 4.4. As in the proof of (i), we obtain that every $(a, b) \in ker(\tau_2)$ determines a unique derivation from $G/G^p\gamma_3(G) \to \Omega_1(Z(G^p\gamma_3(G)) \cap Z_4(G))$. Furthermore, since $d(G) = 2$, we have $C_G(G^p\gamma_3(G)) = Z(G^p\gamma_3(G))$ by (3.6), and hence applying (3.2) with $N = G^p\gamma_3(G)$ yields that

$$\left|\frac{Z(G^p\gamma_3(G)) \cap Z_5(G)}{Z(G)}\right| \geq |\,ker(\tau_2)|. \tag{5.7}$$

On the other hand, since $|\,im(\tau_2)| \leq p^4$, using (5.6) we get

$$|\,ker(\tau_2)| = \frac{|\Omega_1(Z(G^p\gamma_3(G)) \cap Z_4(G))|^2}{|\,im(\tau_2)|} \geq p^8.$$

Thus (5.7) yields that G admits a non-inner automorphism of order p leaving $G^p\gamma_3(G)$ elementwise fixed, whenever $\left|\frac{Z(G^p\gamma_3(G)) \cap Z_5(G)}{Z(G)}\right| \leq p^7$.

□

The theorem below appears in [9].

Theorem 5.3. *(*[9, Theorem 2.4 and Theorem 2.5]*) Let G be a finite p-group and let N and M be normal subgroups of G. Then, for all $r, l \geq 0$, we have*

(i) $[N^{p^r}, M] \equiv [N, M]^{p^r} \left(\mod \prod_{k=1}^{r}[M, {}_{p^k} N]^{p^{r-k}} \right)$.

(ii) $[N^{p^r}, {}_l G] \equiv [N, {}_l G]^{p^r} \left(\mod \prod_{k=1}^{r}[N, {}_{p^k+l-1} G]^{p^{r-k}} \right)$.

In the next theorem, we prove that every finite nonabelian p-group of coclass 4 and 5 admits a non-inner automorphism of order p for $p \geq 5$. Let us recall an elementary fact that if G is a finite p-group of class c and of coclass r, and if $|Z_i(G)| = p^{i+r-1}$ for an $i \in \{1, \dots, c-1\}$, then $|Z_j(G)| = p^{j+r-1}$ and $G/Z_j(G)$ is a group of maximal class for all $j \in \{i, \dots, c-1\}$.

Theorem 5.4. *Let $p \geq 5$ and let G be a finite nonabelian p-group.*

(i) *If G is of coclass 4, then G admits a non-inner automorphism of order p fixing $G^p\gamma_3(G)$ elementwise.*

(ii) *If G is of coclass 5, then G admits a non-inner automorphism of order p fixing $G^p\gamma_4(G)$ elementwise.*

Proof. (i) Since the coclass of G is 4, we have $|Z_5(G)| \leq p^8$, and thus Theorem 5.2 (ii) yields (i).

(ii) As in the proof of Theorem 5.2, we assume that Hypothesis B holds for G, $Z(G)$ is cyclic, $d(G) = 2$, $|\Omega_1(Z_2(G))| = p^2$, and $G/G^p\gamma_3(G)$ is an extra-special group of order p^3. Furthermore, by Theorem 5.2 (ii), $\left|\frac{Z(G^p\gamma_3(G))\cap Z_5(G)}{Z(G)}\right| \geq p^8$. Since the coclass of G is 5, we have $|Z_5(G)| \leq p^9$, and this implies that $|Z_5(G)| = p^9$, $|Z(G)| = p$, and

$$Z_5(G) \leq Z(G^p\gamma_3(G)). \tag{5.8}$$

Moreover, $|Z_5(G)| = p^9$ implies that $G/Z_5(G)$ is of maximal class. Now we proceed to show the existence of $K \trianglelefteq G$ such that $Z_4(G) \leq K \leq G^p\gamma_3(G)$ and G/K is a group of maximal class and of order p^4. If $[G : Z_5(G)] \geq p^4$, then we have that $G/\gamma_4(G)Z_5(G)$ is a group of maximal class and of order p^4, and $Z_4(G) \leq \gamma_4(G)Z_5(G) \leq G^p\gamma_3(G)$ holds by (5.8). Now let $[G : Z_5(G)] \leq p^3$. In this case, since $[G : G^p\gamma_3(G)] = p^3$, using (5.8) we get $Z_5(G) = Z(G^p\gamma_3(G)) = G^p\gamma_3(G)$, and $G^p\gamma_3(G)$ is abelian. Furthermore, since $|Z_5(G)| = p^9$, we obtain that $|G| = p^{12}$ and the class of G is 7. Note that, as the class of G is 7, $Z_4(G) \lneq Z_5(G) = G^p\gamma_3(G)$. Hence, there exists $Z_4(G) \leq K \leq G^p\gamma_3(G)$ with $[G^p\gamma_3(G) : K] = p$. Now let us note that $G^p \leq Z_3(G) \leq K$. Since $\gamma_8(G) = 1$, by Theorem 5.3 (ii) we obtain $[G^p,\ _3\ G] = \gamma_4(G)^p$, and taking $N = G$, $M = \gamma_3(G)$ in Theorem 5.3 (i) yields that $[G^p, \gamma_3(G)] = \gamma_4(G)^p$. We have $[G^p, \gamma_3(G)] = 1$ as $G^p\gamma_3(G)$ is abelian, and hence we get $G^p \leq Z_3(G)$. Moreover, $\gamma_8(G) = 1$ implies that $\gamma_4(G) \leq Z_4(G) \leq K$. Thus $[G^p\gamma_3(G), G] \leq G^p\gamma_4(G) \leq K$ yielding $K \trianglelefteq G$, and $|G/K| = p^4$. Furthermore, $G^p \leq K$ implies that $\gamma_3(G) \nleq K$, and hence G/K is of maximal class. This proves the existence of K in either case as required. Noting that $Z_4(G) \leq Z(G^p\gamma_3(G))$ by (5.8), $Z_4(G) \leq K \leq G^p\gamma_3(G)$ implies $Z_4(G) \leq Z(K)$, and thus $Z_4(G)$ is a G/K-module. Now we proceed to give a family of derivations from $G/K \to \Omega_1(Z_4(G))$. The isomorphism class given by isomorphism type 12 in Huppert's classification of finite p-groups of order p^4 [14, Chapter 3 p. 346] is the only isomorphism class for finite p-groups of maximal class and order p^4, $p \geq 5$. Hence G/K has a presentation

$$\langle x, y \mid x^p, y^p, [y, x, y], [y, x, x, y], [y, x, x, x]\rangle. \tag{5.9}$$

Let F be a free group on $\{x, y\}$. Then $\Omega_1(Z_4(G))$ is an F-module. By Lemma 4.4, the map

$$\tau_3 : \Omega_1(Z_4(G))^2 \to \Omega_1(Z_2(G)) \times \Omega_1(Z(G))^2$$
$$(a, b) \mapsto \big([b, x, y][y, a, y][y, x, b, y][y, x, a, y][y, x, b],\ [\nu(a, b), y],\ [\nu(a, b), x]\big)$$

is a homomorphism, where $\nu(a, b) = [b, x, x][y, a, x][y, x, b, x][y, x, a, x][y, x, a]$. Let $(a, b) \in ker(\tau_3)$. Since F is a free group, the map $x \mapsto a$, $y \mapsto b$ extends to a derivation $\delta_{3_{a,b}}$ of F. Now we proceed to check that $\delta_{3a,b}$ preserves the relations in (5.9). Using Lemma 4.5 (ii), we get $\delta_{3a,b}([y, x, y]) = [b, x, y][y, a, y][y, x, b, y][y, x, a, y][y, x, b] = 1$. Similarly $\delta_{3a,b}([y, x, x]) = \nu(a, b)$, and so, by Lemma 4.5 (iii), we obtain $\delta_{3a,b}([y, x, x, y]) = [\nu(a, b), y] = 1$ and $\delta_{3a,b}([y, x, x, x]) = [\nu(a, b), x] = 1$. By Lemma 4.5 (iv) we get $\delta_{3a,b}(x^p) = \delta_{3a,b}(y^p) = 1$. Hence $\delta_{3a,b}$ induces a derivation on G/K by

Lemma 2.1 (ii), which we again denote with $\delta_{3a,b}$. Let $\alpha_{3a,b} = \varphi(\delta_{3a,b}) \in C_{\mathrm{Aut}(G)}(G/K, K)$ given by $\alpha_{3a,b}(g) = g\delta_{3a,b}(gK)$ for all $g \in G$. It follows that $\alpha_{3a,b}$ has order p, and fixes K and $G/\Omega_1(Z_4(G))$ elementwise. Since the class of G/K is 3, G/K is a regular group, and since $x^p, y^p \in K$ by (5.9), we obtain $G^p \leq K$. Furthermore, $\gamma_4(G) \leq K$ as $|G/K| = p^4$, and thus $\alpha_{3a,b}$ fixes $G^p\gamma_4(G) \leq K$ elementwise. Suppose $\alpha_{3a,b} = i_{u_{3a,b}}$ is an inner automorphism of G, then $\alpha_{3a,b}|_{G/\Omega_1(Z_4(G))} = id$ implies that $u_{3a,b} \in Z_5(G)$. Hence if $\alpha_{3a,b}$ is an inner automorphism of G for all $(a,b) \in ker(\tau_3)$, then we obtain

$$|\,ker(\tau_3)| \leq \left|\frac{Z_5(G)}{Z(G)}\right| = p^8. \tag{5.10}$$

Now we conclude the proof by contradicting (5.10). By (5.8), $Z_4(G) \leq Z(G^p\gamma_3(G))$, and hence using Theorem 5.2 (i) we obtain $|\frac{Z_4(G)}{Z(G)}| \geq p^6$. As explained in the Remark 3.10, this implies that

$$|\Omega_1(Z_4(G))| \geq p^6. \tag{5.11}$$

Now we find an upper bound for $|\,im(\tau_3)|$. Let us consider the map

$$\mu : \Omega_1(Z_2(G)) \to \Omega_1(Z(G))^2$$
$$w \mapsto ([w,y],[w,x]).$$

We have that μ is a homomorphism by Lemma 4.4. Noting that $ker(\mu) = \Omega_1(Z(G))$, and since $|\Omega_1(Z_2(G))| = p^2$, we get $|\,im(\mu)| = p$. Since $\nu(a,b) \in \Omega_1(Z_2(G))$ for all $a, b \in \Omega_1(Z_4(G))$, we obtain that $im(\tau_3) \leq \Omega_1(Z_2(G)) \times im(\mu)$. Thus $|\,im(\tau_3)| \leq |\Omega_1(Z_2(G))||\,im(\mu)| = p^3$, and using (5.11) we get

$$|\,ker(\tau_3)| \geq \frac{|\Omega_1(Z_4(G))|^2}{|\,im(\tau_3)|} \geq p^9.$$

□

Acknowledgments

I would like to thank my advisor Viji Z Thomas for the careful reading of the manuscript and his valuable suggestions and constant support. I am also thankful to the referee for his helpful comments and suggestions in improving the paper.

References

[1] A. Abdollahi, Finite p-groups of class 2 have noninner automorphisms of order p, *J. Algebra*, **312** (2007) 876–879.

[2] A. Abdollahi, and S. Mohsen Ghoraishi, On noninner automorphisms of 2-generator finite p-groups, *Comm. Algebra*, **45** (2017) 3636–3642.

[3] A.Abdollahi, Powerful p-groups have non-inner automorphisms of order p and some cohomology, *J. Algebra*, **323** (2010) 779–789.

[4] A. Abdollahi, S. M. Ghoraishi and B. Wilkens, Finite p-groups of class 3 have noninner automorphisms of order p, *Beitr. Algebra Geom.*, **54** (2013) 363–381.

[5] A. Abdollahi, S. M. Ghoraishi, Y. Guerboussa, M. Reguiat, and B. Wilkens, Noninner automorphisms of order p for finite p-groups of coclass 2, *J. Group Theory*, **17** (2014) 267–272.

[6] Y. Berkovich, *Groups of prime power order*, **1**, De Gruyter Expositions in Mathematics, **46**, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[7] G. Cutolo, On a question about automorphisms of finite p-groups, *J. Group Theory*, **9** (2006) 231–250.

[8] M. Deaconescu and G. Silberberg, Noninner automorphisms of order p of finite p-groups, *J. Algebra*, **250** (2002) 283–287.

[9] G. A. Fernández-Alcober, J. González-Sánchez and A. Jaikin-Zapirain, Omega subgroups of pro-p groups, *Israel J. Math.*, **166** (2008) 393–412.

[10] W. Gaschütz, Nichtabelsche p-Gruppen besitzen äussere p-Automorphismen, *J. Algebra*, **4** (1966) 1–2.

[11] S. Mohsen Ghoraishi, On noninner automorphisms of finite nonabelian p-groups, *Bull. Aust. Math. Soc.*, **89** (2014) 202–209.

[12] S. Mohsen Ghoraishi, A note on automorphisms of finite p-groups, *Bull. Aust. Math. Soc.*, **87** (2013) 24–26.

[13] K. W. Gruenberg, *Cohomological topics in group theory*, Lecture Notes in Mathematics, **143**, Springer-Verlag, Berlin-New York, 1970.

[14] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der Mathematischen Wissenschaften, **134**, Springer-Verlag, Berlin-New York, 1967.

[15] A. R. Jamali and M. Viseh, On the existence of noninner automorphisms of order two in finite 2-groups, *Bull. Aust. Math. Soc.*, **87** (2013) 278–287.

[16] C. R. Leedham-Green and S. McKay, *The structure of groups of prime power order*, London Mathematical Society Monographs, New Series, **27**, Oxford Science Publications, Oxford University Press, Oxford, 2002.

[17] H. Liebeck, Outer automorphisms in nilpotent p-groups of class 2, *J. London Math. Soc.*, **40** (1965) 268–275.

[18] A. Mann, Groups with few class sizes and the centralizer equality subgroup, *Israel J. Math.*, **142** (2004) 367–380.

[19] V. D. Mazurov and E. I. Khukhro (eds.), *Unsolved problems in group theory, The Kourovka notebook*, augmented ed., Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, 1995.

[20] J. J. Rotman, *An introduction to homological algebra*, second ed., Universitext, Springer, New York, 2009.

[21] M. Ruscitti, L. Legarreta and M. K. Yadav, Non-inner automorphisms of order p in finite p-groups of coclass 3, *Monatsh. Math.*, **183** (2017) 679–697.

[22] P. Schmid, A cohomological property of regular p-groups, *Math. Z.*, **175** (1980) 1–3.

[23] M. Shabani-Attar, Existence of noninner automorphisms of order p in some finite p-groups, *Bull. Aust. Math. Soc.*, **87** (2013) 272–277.

Komma Patali

School of Mathematics, Indian Institute of Science Education and Research Thiruvananthapuram, 695551, Thiruvananthapuram, India

Email: `patalikommagmail.com`

http://dx.doi.org/10.22108/ijgt.2023.135843.1816