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AN EXAMPLE OF A QUASI-COMMUTATIVE INVERSE SEMIGROUP

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Abstract. Constructing concrete examples of certain semigroups could help in implementing algorithms optimized for the users. We give concrete examples of certain finitely presented semigroups, namely $S_{p,n}$. Both computational and theoretical approaches are used for studying their structural properties to show that they are quasi-commutative and inverse semigroups.

1. Introduction

In [9], samples of finitely generated quasi-commutative semigroups that are not regular are given. Since the inverse semigroups are the main platform in solving distinct questions [6,8] and playing important rule in many branches of mathematics, we intend to present actual examples of semigroup presentations which are quasi-commutative and inverse semigroups. This helps the programmers to implement algorithms optimized for the users.

The notion of inverse semigroups was introduced by V. V. Wagner [10] through investigating generalized groups. A semigroup S is called an *inverse semigroup* if for each element $a \in S$ there exists a unique element $b \in S$ such that: $a = aba$ and $b = bab$. Equivalently, a semigroup S is an inverse semigroup if (1) it is regular, that is, for every element $a \in S$ there is an element b satisfying $a = aba$, $b = bab$ and (2) the set of all idempotents $E(S)$ is commutative subsemigroup of S. Usually, b is denoted by a^{-1}.

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Another special associative groupoid, namely a *quasi-commutative semigroup*, was introduced in the early of 1970s [2,5]. In such a semigroup, for all $a, b \in S$ there is an integer $r \geqslant 1$ such that $ab = b^r a$. Since every commutative semigroup is a candidate for a quasi-commutative semigroup, in this short article, we prefer to explore a non-commutative and non-group quasi-commutative semigroup.

The first section provides general information on the semigroup $S_{p,n}$ and then in the second section, we study some remarkable properties which make it special. The main results are given in the third section.

There is a constructive result being a consequence of [3, Theorem 4.11]:

Theorem 1.1. *[5, Theorem 2] A regular quasi-commutative semigroup is the disjoint union of groups.*

In the forth section we shall show that how the semigroup $S_{p,n}$ is partitioned into the union of groups.

Let $n \in \mathbb{N}$ and let $p \in \mathbb{N}$ be a prime such that $p \pmod 4 = 3$. We denote the semigroup defined by the presentation:

$$\langle a, b : a^{5^n} = a,\ aba = b,\ ab = b^p a\rangle,$$

by $S_{p,n}$. Some general preliminaries on semigroup presentations can be found in [1,7]. And for additional terminology and notation, one may consult the articles [4,5].

2. Preliminary properties

Considering the presentation of $S_{p,n}$, we first study the preliminary properties of it starting with a key lemma. Clearly for $a, b \in S_{p,n}$, the elements $a^0 b$ and ab^0 should be interpreted as b and a respectively.

Lemma 2.1. *The following identities containing $a, b \in S_{p,n}$ hold where k is a positive integer.*

(a1): $a^k b a^k = b$.

(a2): b^{2k} is central.

(a3): $ab^k a = b^k$ for odd values of k.

(a4): a^{2k} is central.

(a5): $b^j = a^k b^j a^k = a^{2k} b^j$ (j is odd and k is even).

(a6): $a^{(5^n-1)+i} = a^i$ where $1 \leqslant i < 5^n - 1$.

(a7): $a^2 b^{p+(i-1)} = b^i$ where $1 \leqslant i \leqslant 5^n - 1$.

Proof. Part (a1) is an immediate result of the relation $aba = b$ of $S_{p,n}$. Since

$$b^2 a = b(ba) = (aba)(ba) = (ab)(aba) = (ab)b = ab^2,$$

so part (a2) holds. Let $k = 2m + 1$ be an odd positive integer. Therefore with the use of (a2) we have $ab^{2m+1}a = b^{2m}(aba) = b^{2m+1}$ and so (a3) is proved. For part (a4) it is enough to show that a^2 and b commute. Obviously, $a^2 b = a(b^p a) = b^p$ (1). Moreover other parts of the lemma help to give $b = (a^2 b)a^2 \overset{(1)}{=} b^p a^2$ (2) and then $ba^2 = b^p a^4 = (ba^4)b^{p-1} = (ba^3)(b^p a)b^{p-2} = (ba^2)(ab^p a)b^{p-2} = b^{2p-1}a^2$. On the other hand (2) gives $b^p = b^{2p-1}a^2$ and so we finally show $ba^2 = b^{2p-1}a^2 = b^p \overset{(1)}{=} a^2 b$. Part (a5) is

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a result of parts (a3) and (a4). For (a6), it is true for $i = 1$. Suppose that the result is true for $i = m$ so $a^{5^n+m} = a^{5^n+(m-1)+1} = a^{m+1}$ and induction gives the proof. Part (a7) , by the help of (1), is valid for $i = 1$. Let $i \geqslant 2$. Then $a^2b^p \cdot b^{i-1} = b^i$ and so (a7) is proved. □

Lemma 2.2. *For a non-negative integer k we have $ba = a^{4k+3}b$. In particular $ba = a^pb$.*

Proof. The result is true for $k = 0$ since $ba = (aba)a = a(ba^2) \overset{2.1(\text{a4})}{=} a(a^2b) = a^3b$. Let the claim be valid for $k = m$. Therefore $a^{4(m+1)+3}b = a^4 \cdot (a^{4m+3}b) = a^3(aba) = a^3b = ba$. and hence the result follows by induction. □

Lemma 2.3. *Due to the presentation of $S_{p,n}$, the identities below are valid for a and b in $S_{p,n}$.*

(a8): If k is a positive integer then $ab^k = b^{kp}a$.

(a9): If k is an odd positive integer then $ba^k = (a^k)^3b$.

Proof. For (a8), the result is true for $k = 1$. Suppose that the identity holds for $k = m$ then $ab^{m+1} = (ab^m) \cdot b = (b^{mp}a) \cdot b = b^{mp}(ab) = b^{mp}(b^pa) = b^{p(m+1)}a$, and the result holds by induction. For part (a9) let $k = 2m + 1$ be an odd positive integer. So

$$ba^{2m+1} \overset{2.1(\text{a4})}{=} a^{2m}(ba) \overset{2.2}{=} a^{2m}(a^{4m+3}b) = (a^{2m+1})^3b.$$

Hence part (a9) holds for a and b. □

Lemma 2.4. *If j is an odd positive integer then:*

$$(ab^j)^ka = \begin{cases} b^{kj} & k \text{ is odd} \\ b^{kj}a & \text{otherwise} \end{cases}.$$

Proof. Because of $(ab^j)a \overset{2.1(\text{a3})}{=} b^j$, so the result is true for $k = 1$. Assume that it is true for $k = 2m + 1$. Then $(ab^j)^{2m+3}a = (ab^j)^{2m+1}(ab^j)^2a = (ab^j)^{2m+1}a(b^jab^ja) \overset{2.1(\text{a3})}{=} b^{j(2m+1)} \cdot b^{2j} = b^{j(2m+3)}$ and the result follows by induction. Now let $k = 2$. So $(ab^j)^2a = ab^jab^ja \overset{2.1(\text{c,b})}{=} b^j \cdot (b^ja) = b^{2j}a$. as desired. Assume it is valid for $k = 2m$. So by using induction we have $(ab^j)^{2m+2}a = (ab^j)^{2m} \cdot (ab^jab^ja) = (ab^j)^{2m}a \cdot (b^jab^ja) \overset{2.1(\text{a3})}{=} (b^{2mj}a) \cdot b^{2j} = b^{(2m+2)j}a$. □

Lemma 2.5. *If k is a positive integer then $b^k = b^{2p+(k-2)}$.*

Proof. We use induction on k. If $k = 1$ then $b = aba = (b^pa)a \overset{2.1(\text{a3})}{=} ab^pa^3 = (ab)(b^{p-1}a^3) = (b^pa)(b^{p-1}a^3) \overset{2.1(\text{a2})}{=} b^{2p-1}a^4 = a^2b^{2p-1}a^2 \overset{2.1(\text{a3})}{=} b^{2p-1}$. Note that $2p - 1$ is an odd positive integer. Let the result be true for $k = m$. Then we have $b^{2p+(m+1-2)} = b^{2p+(m-2)} \cdot b = b^{m+1}$ and the result holds by induction. □

Lemma 2.6. *If j is a positive integer then:*

$$b^{pj} = \begin{cases} a^2b^j & j \text{ is odd} \\ b^ja & \text{otherwise} \end{cases}.$$

Proof. The result is trivially true for $j = 1$ since according to (1) of 2.1 we have $a^2b = b^p$. Assume the result is true for $j = 2m + 1$ so $a^2b^{2m+3} = (a^2b^{2m+1})b^2 \overset{2.5}{=} b^{p(2m+1)} \cdot b^{2p} = b^{p(2m+3)}$. Now let $j = 2$. Therefore $b^p \cdot b^p = (a^2b)^2$. By the previous part and part (a1) of lemma 2.1 we have the result, i.e. $b^{2p} = b^2$. If we assume the claim is true for $j = 2m$ then $b^{(2m+2)p} = b^{2m}(b^p)^2 = b^{2m}(a^2b)^2 = b^{2m+2}$ and induction completes the proof. □

Lemma 2.7. *If i, j are odd positive integers where $i \geqslant 3$ then $a^ib^j = b^ja^{i-2}$.*

Proof. Let $j = 1$ and $i = 2k+1$. Then $a^ib = a^{2k+1}b \overset{2.1(\mathrm{a4})}{=} (aba)a^{2k-1} = ba^{2k-1} = ba^{i-2}$. Assume that the result is true for $j = 2m+1$, i.e. $a^ib^{2m+1} = b^{2m+1}a^{i-2}$ then by induction we have $a^ib^{2m+3} = a^ib^{2m+1} \cdot b^2 = b^{2m+1}a^{i-2} \cdot b^2 \overset{2.1(\mathrm{a2})}{=} b^{2m+3}a^{i-2}$ and this proves the lemma. □

Lemma 2.8. *If i, j and k are positive integers where i, j are odd then $(a^2b^j)^ka^i = a^ib^{kj}$.*

Proof. Case 1($i = 1$): We use an induction on k to show that $(a^2b^j)^ka = ab^{kj}$. Let $k = 1$, so $(a^2b^j)a^i = a(ab^ja) \overset{2.1(\mathrm{a4})}{=} ab^j$ which shows that the identity holds. Now, assume that it is true for $k = m$, i.e. $(a^2b^j)^ma = ab^{mj}$. Then we have $(a^2b^j)^{m+1}a = \big((a^2b^j)^m \cdot a\big)(ab^ja) \overset{2.1(\mathrm{a4})}{=} (a^ib^{kj})b^j = a^ib^{j(k+1)}$. Case 2 ($i = 2m+1, i \geqslant 3$): By an induction on k the identity is true for $k = 1$. In fact, $(a^2b^j)a^i = (a^2b^j)a^{2m+1} \overset{2.1(\mathrm{a5})}{=} a^{2(1+m)}b^ja = a^{2m+1}(ab^ja) = a^ib^j$. Let the identity be true for $k = s$ so by induction we get $(a^2b^j)^{s+1}a^i = (a^2b^j)(a^2b^j)^sa^i = (a^2b^j)(a^ib^{sj}) = (a^2b^ja^2)(a^{i-2}b^{sj}) = (b^ja^{i-2})b^{sj} \overset{2.7}{=} a^jb^{j(s+1)}$. Hence the proof is complete. □

3. Main results

In this section, we show that for a positive integer n and a prime number $p = 4k + 3$, the semigroup $S_{p,n}$ is a quasi-commutative and inverse semigroup.

Theorem 3.1. *A confluent and noetherian rewriting system (with respect to the short-lex order induced by $a < b$) for $S_{p,n}$ is:*

$$\langle a, b : aba \longrightarrow b, ba^2 \longrightarrow a^2b, b^2a \longrightarrow ab^2, a^3b \longrightarrow ba, b^p \longrightarrow a^2b, bab^{p-1} \longrightarrow ab, a^{5^n} \longrightarrow a\rangle$$

The normal forms of this rewriting system are:

$$a^i \quad (i = 1, \ldots, 5^n - 1), \quad a^2b^j,\ ab^j,\ bab^{j-1},\ b^j \quad (j = 1, \ldots, p-1),$$

and so $|S_{p,n}| = 5n + 4p - 5$.

Proof. The rewriting system in the statement is the output of the Knuth-Bendix algorithm with respect to the stated reduction ordering. The normal forms are then just all words in $\{a, b\}^+$ not containing the left hand side of one of the rewrite rules in the given systems. it is straightforward to show that the stated normal forms are precisely these words. □

Lemma 3.2. *For the semigroup $S_{p,n}$ we have $|E(S_{p,n})| \geqslant 2$.*

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Proof. With the help of lemma 2.1, we have $(a^2b^{p-1})^2 = (a^2b^pa^2)b^{p-2} = (ab^pa)b^{p-2} = a(b^pa)b^{p-2} = a(ab)b^{p-2} = a^2b^{p-1}$. On the other hand, since $s = 5^n - 1$ where $1 \leqslant s < 5^n$ is the only positive integer which is divisible by $5^n - 1$ so $a^{5^n-1} = \mathrm{id}_{\langle a\rangle}$ belongs to $E(S_{p,n})$. □

Theorem 3.3. *The semigroup $S_{p,n}$ is a quasi-commutative semigroup.*

Proof. To prove the quasi-commutativity of $S_{p,n}$, we consider the different cases for $x, y \in S_{p,n}$ and then we find a positive integer r satisfying

$$xy = y^r x \quad (1).$$

In what follows, we assume $1 \leqslant i \leqslant 5^n - 1$ and $1 \leqslant j, s, t \leqslant p - 1$. Trivially, there are three cases in which $r = 1$, i.e. $x = y$, $x, y \in \{a^i\}$ and $x, y \in \{b^j\}$. Moreover, parts (e) and (c) of lemma 2.1 yield that in the following cases the value $r = 1$ satisfies (1):

- $x, y \in \{a^2b^t\}$ and $x, y \in \{b^j, a^2b^s\}$.
- $x, y \in \{a^i, b^j\}$; $x, y \in \{a^i, ab^j\}$; $x, y \in \{a^i, a^2b^j\}$ (i is even and for all values of j).
- $x, y \in \{a^i, b^j\}; x, y \in \{a^i, ab^j\}; x, y \in \{a^i, a^2b^j\}$ (j is even and for all values of i).
- $x, y \in \{b^j, ab^s\}$ (j is even and for all values of s).
- $x, y \in \{b^j, ab^s\}$ (s is even and for all values of j).

The essential cases for x and y, that lead to finding appropriate integers r, n satisfying $xy = y^r x$ and $yx = x^n y$ are as follows. We assume that i and j are both odd.

Case 1:

- $x = a^i, y = b^j$

If $i = 1$ then $xy \overset{2.3}{=} y^p x$. If $i = (2m + 1) \geqslant 3$ then by lemma 2.1 part (e) we have $xy = a^{2m+1}b^j = ab^ja^{2m} \overset{2.3}{=} y^p x$.

- $x = b^j, y = a^i$

If $i = 1, j = 1$ then $xy \overset{2.2}{=} a^3b = y^3x$. If $i = 1$ and $j = 2m + 1$ then $xy = b^{2m+1}a \overset{2.1(c)}{=} (ba)b^{2m} = (a^3b)b^{2m} = y^3x$. If $i = 2m + 1$ and $j = 1$ then $xy = ba^{2m+1} = (ba)a^{2m} = a^{2m}(ba) \overset{2.2}{=} a^{2m}(a^{4k+3}b) = y^3b$ where k is a non-negative integer. And if $i = 2m + 1$ and $j = 2l + 1$ for some positive integers m, l then $xy \overset{2.1(c,e)}{=} a^{2m}(ba)b^{2l} \overset{2.2}{=} a^{2m}(a^{4m+3}b)b^{2l} = y^3x$.

Case 2:

- $x = a^i, y = ab^j$

If $i = 1$ then $xy = a^2b^j \overset{2.6}{==} b^{pj}y^px \overset{2.4}{=} (ab^j)^pa = y^px$. If $i = (2m + 1) \geqslant 3$ then by lemma 2.6 we have $xy = a^{2m+1}(ab^j) = a^{2m}b^{pj}$ so $xy \overset{2.4}{=} a^{2m}(ab^j)^pa \overset{2.1(e)}{=} (ab^j)^pa^i = y^px$.

- $x = ab^j, y = a^i$

If $i = 1$ then $xy = b^j = a^2b^ja^2 \overset{2.1\,(a8)}{=} a^3 \cdot ab^j = y^3x$. Assume it is true for $i = 2m + 1$ so for $i = 2m + 3$ we have $xy \overset{2.1(e)}{=} a^2 \cdot (ab^j)a^{2m+1} = a^{2+3(2m+1)}(ab^j) \overset{2.1\,(a8)}{=} a^{2+3(2m+1)} \cdot a(a^4b^j) = y^3x$.

Case 3:

- $x = a^i, y = a^2b^j$

Considering Lemmas 2.6 and 2.8, we have $xy = a^i(a^2b^j) = a^ib^{pj} = (a^2b^j)^pa^i = y^px$.

- $x = a^2b^j, y = a^i$

If $i = 1$ then $xy = a^2(ab^ja)a \overset{2.3}{=} a^3(a^2b^j) = y^3x$. Assume it is true for $i = 2m+1$ then by induction on i we have $xy = (a^2b^j)a^{2m+3} = a^2(a^{3(2m+1)} \cdot a^2b^j) \overset{2.1\,(a8)}{=} a^{3(2m+1)+6} \cdot a^2b^j = y^3x$.

Case 4:

- $x = b^j, y = ab^s$

By using Lemmas 2.1 part (d), 2.4 and 2.6, we have $xy = b^ja(ab^sa) = (a^2b^s)(b^ja) = b^{pj}(b^ja) = (ab^s)^pa \cdot (b^ja) = (ab^s)^pb^j = y^px$.

- $x = ab^s, y = b^j$

Lemmas 2.1 part (d), 2.4 and 2.6 help us to have $xy = a(ab^sa) \cdot (ab^ja) = (a^2b^ja)(ab^sa) = (a^2b^j)(ab^s) = b^{pj}(ab^s) = y^px$.

Case 5:

- $x = ab^j, y = a^2b^s$

Assume s is even. So $xy \overset{2.1(c,d)}{=} yx$. If s is odd so by Lemmas 2.1 part (e), 2.8 and 2.6 we have $xy = a^3b^{s+j} = ab^{ps+j} = (a^2b^s)^pa \cdot b^j = y^px$.

- $x = a^2b^s, y = ab^j$

If s is even, the proof is similar to the previous subcase so let it be odd. We have $xy = a(ab^sa)b^j \overset{2.1(d),\,2.3}{=} b^{pj}ab^s \overset{2.4}{=} (ab^j)^pa \cdot ab^s = y^px$. □

Theorem 3.4. *Semigroup $S_{p,n}$ is an inverse semigroup.*

Proof. Since the semigroup $S_{p,n}$ is quasi-commutative so its idempotents commute [4] so we only show that $S_{p,n}$ is regular, i.e. for all $a \in S_{p,n}$, the associated set $V(a) = \{x \in S_{p,n} : axa = a \text{ and } xax = x\}$ is singleton. For the sake of showing the regularity of $S_{p,n}$, one may use the original definition of a regular semigroup i.e. for all $x \in S_{p,n}$ there is $y \in S_{p,n}$ such that $xyx = x$ and we check this for elements of $S_{p,n}$ by considering Theorem 3.1. The different cases are as follows:

Case 1 ($x = a^i$):

From 2.1 part ($a6$) we have

$$x \cdot a^{5^n-(1+i)} \cdot x = x,$$

where $1 \leqslant i < 5^n - 1$. And if $i = 5^n - 1$ then $x \cdot a^i \cdot x = (a^{5^n})^2 \cdot a^{5^n-3} = x$.

Case 2 ($x = b^j$):

By ($a7$) we get

$$x \cdot (a^2b^{p-(j+1)}) \cdot x = a^2b^{p+j-1} = x,$$

where $1 \leqslant j \leqslant p - 1$.

Case 3 ($x = ab^s$):

Let s where $1 \leqslant s \leqslant p-1$ be odd so we have $x \cdot (a^3 b^{p-(s+1)}) \cdot x = (ab^s a)a(ab^{p-(s+1)}a)b^s = b^s ab^{p-1} \overset{2.7}{=} a^3 b^s \cdot b^{p-1} \overset{(1)}{=} ab^p \cdot b^{p+s-2} \overset{2.5}{=} x$.
Let s be even so we have $x \cdot (ab^{p-(s+1)}) \cdot x \overset{2.1(a2)}{=} a^2 b^{p+s-1} a$. It is equal to $(a^2 b^p)(b^{s-1}a) \overset{2.1(a4)}{=} (b^p a^2)(b^{s-1}a) = (b^p a)(ab^{s-1}a) = b^p ab^{s-1} = b^p(b^{p(s-1)}a) \overset{2.3}{=} x$.

Case 4 ($x = a^2 b^t$):

For this kind of element wherein $1 \leqslant t \leqslant p-1$ we consider $a4$ and 2.6 to have $x \cdot b^{p-(j+1)} \cdot x = a^2 b^{p-(j+1)} a^2 = a^2 b^p a^2 \cdot b^{j-1} = b^{p-(1-j)} = b^p b^{j-1} = (a^2 b) b^{j-1} = x$. □

4. Decomposition as the union of groups

Let the values of n and p be fixed. Using the following codes in GAP [11]:

```
gap> T1:=Range(IsomorphismTransformationSemigroup([a^2b^2]));;
     T2:=AsGroup(MagmaByMultiplicationTable(MultiplicationTable(AsList(T1))));;
     StructureDescription(T2);
```

shows that the subsemigroup

$$H = \langle a^2 b^2 \rangle = \{a^2 b^2, b^4, a^2 b^6, b^8, \ldots, a^2 b^{4k-2}, b^{4k}, a^2 b^{4k+2}\},$$

has a group structure being isomorphic to $\mathbb{Z}_{\frac{p-1}{2}}$. Also,

$$K = \overbrace{\langle\, ab^{\frac{p-1}{2}}, b^{\frac{p-1}{2}} \,\rangle}^{\langle\, ab^{2k+1}, b^{2k+1} \,\rangle} = \{b^{2k+1}, ab^{2k+1}, bab^{2k}, a^2 b^{2k+1}, b^{4k+2}, ab^{4k+2}, bab^{4k+2}, a^2 b^{4k+2}\},$$

is a subgroup of $S_{p,n}$, isomorphic to Q_8 and $H \cap K = a^2 b^{4k+2} = e_K = e_H$. Clearly $|H \times K| = (2k+1)8$ and from the presentation of $S_{p,n}$, if $w \in \{a,b\}^+$ contains the letter b then so does every word in $\{a,b\}^+$ that represents the same element of $S_{p,n}$. So, the set of all elements in $S_{p,n}$ which are represented by a word containing the letter b forms an ideal of $S_{p,n}$ and this makes $H \times K$ to be an ideal of $S_{p,n}$ as well. Furthermore, the monogenic semigroup

$$A = \langle a \rangle = \{a^i : i \in \{1, \ldots, 5^n - 1\}\},$$

is obviously isomorphic to $\mathbb{Z}_{5^n-1}$ and so we proved the following result:

Corollary 4.1. *The semigroup $S_{p,n}$ is decomposed as $S_{p,n} = A \cup (H \times K)$.*

Remark 4.2. *Since the $\mathcal{D}$-class of A is isomorphic to $\mathbb{Z}_{5^n-1}$ and $H \times K \cong \mathbb{Z}_{\frac{p-1}{2}} \times Q_8$ so the $\mathscr{D}$-class structure of $S_{p,n}$ can also be given in Figure 1.*

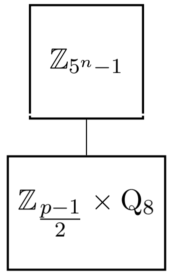

FIGURE 1. The $\mathscr{D}$-class structure of $S_{p,n}$

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