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## AN EXAMPLE OF A QUASI-COMMUTATIVE INVERSE SEMIGROUP

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**ABSTRACT.** Constructing concrete examples of certain semigroups could help in implementing algorithms optimized for the users. We give concrete examples of certain finitely presented semigroups, namely  $S_{p,n}$ . Both computational and theoretical approaches are used for studying their structural properties to show that they are quasi-commutative and inverse semigroups.

### 1. Introduction

In [9], samples of finitely generated quasi-commutative semigroups that are not regular are given. Since the inverse semigroups are the main platform in solving distinct questions [6, 8] and playing important role in many branches of mathematics, we intend to present actual examples of semigroup presentations which are quasi-commutative and inverse semigroups. This helps the programmers to implement algorithms optimized for the users.

The notion of inverse semigroups was introduced by V. V. Wagner [10] through investigating generalized groups. A semigroup  $S$  is called an *inverse semigroup* if for each element  $a \in S$  there exists a unique element  $b \in S$  such that:  $a = aba$  and  $b = bab$ . Equivalently, a semigroup  $S$  is an inverse semigroup if (1) it is regular, that is, for every element  $a \in S$  there is an element  $b$  satisfying  $a = aba$ ,  $b = bab$  and (2) the set of all idempotents  $E(S)$  is commutative subsemigroup of  $S$ . Usually,  $b$  is denoted by  $a^{-1}$ .

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Another special associative groupoid, namely a *quasi-commutative semigroup*, was introduced in the early of 1970s [2, 5]. In such a semigroup, for all  $a, b \in S$  there is an integer  $r \geq 1$  such that  $ab = b^r a$ . Since every commutative semigroup is a candidate for a quasi-commutative semigroup, in this short article, we prefer to explore a non-commutative and non-group quasi-commutative semigroup.

The first section provides general information on the semigroup  $S_{p,n}$  and then in the second section, we study some remarkable properties which make it special. The main results are given in the third section.

There is a constructive result being a consequence of [3, Theorem 4.11]:

**Theorem 1.1.** [5, Theorem 2] *A regular quasi-commutative semigroup is the disjoint union of groups.*

In the fourth section we shall show that how the semigroup  $S_{p,n}$  is partitioned into the union of groups.

Let  $n \in \mathbb{N}$  and let  $p \in \mathbb{N}$  be a prime such that  $p \pmod{4} = 3$ . We denote the semigroup defined by the presentation:

$$\langle a, b : a^{5^n} = a, aba = b, ab = b^p a \rangle,$$

by  $S_{p,n}$ . Some general preliminaries on semigroup presentations can be found in [1, 7]. And for additional terminology and notation, one may consult the articles [4, 5].

## 2. Preliminary properties

Considering the presentation of  $S_{p,n}$ , we first study the preliminary properties of it starting with a key lemma. Clearly for  $a, b \in S_{p,n}$ , the elements  $a^0 b$  and  $ab^0$  should be interpreted as  $b$  and  $a$  respectively.

**Lemma 2.1.** *The following identities containing  $a, b \in S_{p,n}$  hold where  $k$  is a positive integer.*

$$(a1): a^k b a^k = b.$$

$$(a2): b^{2k} \text{ is central.}$$

$$(a3): ab^k a = b^k \text{ for odd values of } k.$$

$$(a4): a^{2k} \text{ is central.}$$

$$(a5): b^j = a^k b^j a^k = a^{2k} b^j \text{ ( } j \text{ is odd and } k \text{ is even).}$$

$$(a6): a^{(5^n-1)+i} = a^i \text{ where } 1 \leq i < 5^n - 1.$$

$$(a7): a^2 b^{p+(i-1)} = b^i \text{ where } 1 \leq i \leq 5^n - 1.$$

*Proof.* Part (a1) is an immediate result of the relation  $aba = b$  of  $S_{p,n}$ . Since

$$b^2 a = b(ba) = (aba)(ba) = (ab)(aba) = (ab)b = ab^2,$$

so part (a2) holds. Let  $k = 2m + 1$  be an odd positive integer. Therefore with the use of (a2) we have  $ab^{2m+1}a = b^{2m}(aba) = b^{2m+1}$  and so (a3) is proved. For part (a4) it is enough to show that  $a^2$  and  $b$  commute. Obviously,  $a^2 b = a(b^p a) = b^p$  (1). Moreover other parts of the lemma help to give  $b = (a^2 b)a^2 \stackrel{(1)}{=} b^p a^2$  (2) and then  $ba^2 = b^p a^4 = (ba^4)b^{p-1} = (ba^3)(b^p a)b^{p-2} = (ba^2)(ab^p a)b^{p-2} = b^{2p-1}a^2$ .

On the other hand (2) gives  $b^p = b^{2p-1}a^2$  and so we finally show  $ba^2 = b^{2p-1}a^2 = b^p \stackrel{(1)}{=} a^2b$ . Part (a5) is a result of parts (a3) and (a4). For (a6), it is true for  $i = 1$ . Suppose that the result is true for  $i = m$  so  $a^{5^n+m} = a^{5^n+(m-1)+1} = a^{m+1}$  and induction gives the proof. Part (a7), by the help of (1), is valid for  $i = 1$ . Let  $i \geq 2$ . Then  $a^2b^p \cdot b^{i-1} = b^i$  and so (a7) is proved.  $\square$

**Lemma 2.2.** For a non-negative integer  $k$  we have  $ba = a^{4k+3}b$ . In particular  $ba = a^p b$ .

*Proof.* The result is true for  $k = 0$  since  $ba = (aba)a = a(ba^2) \stackrel{2.1(a4)}{=} a(a^2b) = a^3b$ . Let the claim be valid for  $k = m$ . Therefore  $a^{4(m+1)+3}b = a^4 \cdot (a^{4m+3}b) = a^3(aba) = a^3b = ba$ . and hence the result follows by induction.  $\square$

**Lemma 2.3.** Due to the presentation of  $S_{p,n}$ , the identities below are valid for  $a$  and  $b$  in  $S_{p,n}$ .

- (a8): If  $k$  is a positive integer then  $ab^k = b^{kp}a$ .
- (a9): If  $k$  is an odd positive integer then  $ba^k = (a^k)^3b$ .

*Proof.* For (a8), the result is true for  $k = 1$ . Suppose that the identity holds for  $k = m$  then  $ab^{m+1} = (ab^m) \cdot b = (b^{mp}a) \cdot b = b^{mp}(ab) = b^{mp}(b^p a) = b^{p(m+1)}a$ , and the result holds by induction. For part (a9) let  $k = 2m + 1$  be an odd positive integer. So

$$ba^{2m+1} \stackrel{2.1(a4)}{=} a^{2m}(ba) \stackrel{2.2}{=} a^{2m}(a^{4m+3}b) = (a^{2m+1})^3b.$$

Hence part (a9) holds for  $a$  and  $b$ .  $\square$

**Lemma 2.4.** If  $j$  is an odd positive integer then:

$$(ab^j)^k a = \begin{cases} b^{kj} & k \text{ is odd} \\ b^{kj} a & \text{otherwise} \end{cases}.$$

*Proof.* Because of  $(ab^j)a \stackrel{2.1(a3)}{=} b^j$ , so the result is true for  $k = 1$ . Assume that it is true for  $k = 2m + 1$ . Then  $(ab^j)^{2m+3}a = (ab^j)^{2m+1}(ab^j)^2a = (ab^j)^{2m+1}a(b^j ab^j a) \stackrel{2.1(a3)}{=} b^{j(2m+1)} \cdot b^{2j} = b^{j(2m+3)}$  and the result follows by induction. Now let  $k = 2$ . So  $(ab^j)^2a = ab^j ab^j a \stackrel{2.1(c,b)}{=} b^j \cdot (b^j a) = b^{2j}a$ . as desired. Assume it is valid for  $k = 2m$ . So by using induction we have  $(ab^j)^{2m+2}a = (ab^j)^{2m} \cdot (ab^j ab^j a) = (ab^j)^{2m} a \cdot (b^j ab^j a) \stackrel{2.1(a3)}{=} (b^{2mj} a) \cdot b^{2j} = b^{(2m+2)j}a$ .  $\square$

**Lemma 2.5.** If  $k$  is a positive integer then  $b^k = b^{2p+(k-2)}$ .

*Proof.* We use induction on  $k$ . If  $k = 1$  then  $b = aba = (b^p a) a \stackrel{2.1(a3)}{=} ab^p a^3 = (ab)(b^{p-1}a^3) = (b^p a)(b^{p-1}a^3) \stackrel{2.1(a2)}{=} b^{2p-1}a^4 = a^2 b^{2p-1} a^2 \stackrel{2.1(a3)}{=} b^{2p-1}$ . Note that  $2p - 1$  is an odd positive integer. Let the result be true for  $k = m$ . Then we have  $b^{2p+(m+1-2)} = b^{2p+(m-2)} \cdot b = b^{m+1}$  and the result holds by induction.  $\square$

**Lemma 2.6.** *If  $j$  is a positive integer then:*

$$b^{pj} = \begin{cases} a^2b^j & j \text{ is odd} \\ b^ja & \text{otherwise} \end{cases}.$$

*Proof.* The result is trivially true for  $j = 1$  since according to (1) of 2.1 we have  $a^2b = b^p$ . Assume the result is true for  $j = 2m + 1$  so  $a^2b^{2m+3} = (a^2b^{2m+1})b^2 \stackrel{2.5}{=} b^{p(2m+1)} \cdot b^{2p} = b^{p(2m+3)}$ . Now let  $j = 2$ . Therefore  $b^p \cdot b^p = (a^2b)^2$ . By the previous part and part (a1) of lemma 2.1 we have the result, i.e.  $b^{2p} = b^2$ . If we assume the claim is true for  $j = 2m$  then  $b^{(2m+2)p} = b^{2m}(b^p)^2 = b^{2m}(a^2b)^2 = b^{2m+2}$  and induction completes the proof. □

**Lemma 2.7.** *If  $i, j$  are odd positive integers where  $i \geq 3$  then  $a^ib^j = b^ja^{i-2}$ .*

*Proof.* Let  $j = 1$  and  $i = 2k + 1$ . Then  $a^ib = a^{2k+1}b \stackrel{2.1(a4)}{=} (aba)a^{2k-1} = ba^{2k-1} = ba^{i-2}$ . Assume that the result is true for  $j = 2m + 1$ , i.e.  $a^ib^{2m+1} = b^{2m+1}a^{i-2}$  then by induction we have  $a^ib^{2m+3} = a^ib^{2m+1} \cdot b^2 = b^{2m+1}a^{i-2} \cdot b^2 \stackrel{2.1(a2)}{=} b^{2m+3}a^{i-2}$  and this proves the lemma. □

**Lemma 2.8.** *If  $i, j$  and  $k$  are positive integers where  $i, j$  are odd then  $(a^2b^j)^ka^i = a^ib^{kj}$ .*

*Proof.* Case 1 ( $i = 1$ ): We use an induction on  $k$  to show that  $(a^2b^j)^ka = ab^{kj}$ . Let  $k = 1$ , so  $(a^2b^j)a^i = a(ab^ja) \stackrel{2.1(a4)}{=} ab^j$  which shows that the identity holds. Now, assume that it is true for  $k = m$ , i.e.  $(a^2b^j)^ma = ab^{mj}$ . Then we have  $(a^2b^j)^{m+1}a = ((a^2b^j)^m \cdot a)(ab^ja) \stackrel{2.1(a4)}{=} (a^ib^{kj})b^j = a^ib^{j(k+1)}$ . Case 2 ( $i = 2m + 1, i \geq 3$ ): By an induction on  $k$  the identity is true for  $k = 1$ . In fact,  $(a^2b^j)a^i = (a^2b^j)a^{2m+1} \stackrel{2.1(a5)}{=} a^{2(1+m)}b^ja = a^{2m+1}(ab^ja) = a^ib^j$ . Let the identity be true for  $k = s$  so by induction we get  $(a^2b^j)^{s+1}a^i = (a^2b^j)(a^2b^j)^sa^i = (a^2b^j)(a^ib^{sj}) = (a^2b^ja^2)(a^{i-2}b^{sj}) = (b^ja^{i-2})b^{sj} \stackrel{2.7}{=} a^jb^{j(s+1)}$ . Hence the proof is complete. □

### 3. Main results

In this section, we show that for a positive integer  $n$  and a prime number  $p = 4k + 3$ , the semigroup  $S_{p,n}$  is a quasi-commutative and inverse semigroup.

**Theorem 3.1.** *A confluent and noetherian rewriting system (with respect to the short-lex order induced by  $a < b$ ) for  $S_{p,n}$  is:*

$$\langle a, b : aba \rightarrow b, ba^2 \rightarrow a^2b, b^2a \rightarrow ab^2, a^3b \rightarrow ba, b^p \rightarrow a^2b, bab^{p-1} \rightarrow ab, a^{5^n} \rightarrow a \rangle$$

*The normal forms of this rewriting system are:*

$$a^i \quad (i = 1, \dots, 5^n - 1), \quad a^2b^j, ab^j, bab^{j-1}, b^j \quad (j = 1, \dots, p - 1),$$

and so  $|S_{p,n}| = 5n + 4p - 5$ .

*Proof.* The rewriting system in the statement is the output of the Knuth-Bendix algorithm with respect to the stated reduction ordering. The normal forms are then just all words in  $\{a, b\}^+$  not containing the left hand side of one of the rewrite rules in the given systems. It is straightforward to show that the stated normal forms are precisely these words.  $\square$

**Lemma 3.2.** *For the semigroup  $S_{p,n}$  we have  $|E(S_{p,n})| \geq 2$ .*

*Proof.* With the help of lemma 2.1, we have  $(a^2b^{p-1})^2 = (a^2b^pa^2)b^{p-2} = (ab^pa)b^{p-2} = a(b^pa)b^{p-2} = a(ab)b^{p-2} = a^2b^{p-1}$ . On the other hand, since  $s = 5^n - 1$  where  $1 \leq s < 5^n$  is the only positive integer which is divisible by  $5^n - 1$  so  $a^{5^n-1} = \text{id}_{\langle a \rangle}$  belongs to  $E(S_{p,n})$ .  $\square$

**Theorem 3.3.** *The semigroup  $S_{p,n}$  is a quasi-commutative semigroup.*

*Proof.* To prove the quasi-commutativity of  $S_{p,n}$ , we consider the different cases for  $x, y \in S_{p,n}$  and then we find a positive integer  $r$  satisfying

$$xy = y^r x \quad (1).$$

In what follows, we assume  $1 \leq i \leq 5^n - 1$  and  $1 \leq j, s, t \leq p - 1$ . Trivially, there are three cases in which  $r = 1$ , i.e.  $x = y$ ,  $x, y \in \{a^i\}$  and  $x, y \in \{b^j\}$ . Moreover, parts (e) and (c) of lemma 2.1 yield that in the following cases the value  $r = 1$  satisfies (1):

- $x, y \in \{a^2b^t\}$  and  $x, y \in \{b^j, a^2b^s\}$ .
- $x, y \in \{a^i, b^j\}$ ;  $x, y \in \{a^i, ab^j\}$ ;  $x, y \in \{a^i, a^2b^j\}$  ( $i$  is even and for all values of  $j$ ).
- $x, y \in \{a^i, b^j\}$ ;  $x, y \in \{a^i, ab^j\}$ ;  $x, y \in \{a^i, a^2b^j\}$  ( $j$  is even and for all values of  $i$ ).
- $x, y \in \{b^j, ab^s\}$  ( $j$  is even and for all values of  $s$ ).
- $x, y \in \{b^j, ab^s\}$  ( $s$  is even and for all values of  $j$ ).

The essential cases for  $x$  and  $y$ , that lead to finding appropriate integers  $r, n$  satisfying  $xy = y^r x$  and  $yx = x^r y$  are as follows. We assume that  $i$  and  $j$  are both odd.

**Case 1:**

- $x = a^i, y = b^j$

If  $i = 1$  then  $xy \stackrel{2.3}{=} y^p x$ . If  $i = (2m + 1) \geq 3$  then by lemma 2.1 part (e) we have  $xy = a^{2m+1}b^j = ab^j a^{2m} \stackrel{2.3}{=} y^p x$ .

- $x = b^j, y = a^i$

If  $i = 1, j = 1$  then  $xy \stackrel{2.2}{=} a^3 b = y^3 x$ . If  $i = 1$  and  $j = 2m + 1$  then  $xy = b^{2m+1} a \stackrel{2.1(c)}{=} (ba)b^{2m} = (a^3 b)b^{2m} = y^3 x$ . If  $i = 2m + 1$  and  $j = 1$  then  $xy = ba^{2m+1} = (ba)a^{2m} = a^{2m}(ba) \stackrel{2.2}{=} a^{2m}(a^{4k+3}b) = y^3 b$  where  $k$  is a non-negative integer. And if  $i = 2m + 1$  and  $j = 2l + 1$  for some positive integers  $m, l$  then  $xy \stackrel{2.1(c,e)}{=} a^{2m}(ba)b^{2l} \stackrel{2.2}{=} a^{2m}(a^{4m+3}b)b^{2l} = y^3 x$ .

**Case 2:**

- $x = a^i, y = ab^j$

If  $i = 1$  then  $xy = a^2 b^j \stackrel{2.6}{=} b^{pj} y^p x \stackrel{2.4}{=} (ab^j)^p a = y^p x$ . If  $i = (2m + 1) \geq 3$  then by lemma 2.6 we have

$xy = a^{2m+1}(ab^j) = a^{2m}b^ja^2$  so  $xy \stackrel{2.4}{=} a^{2m}(ab^j)^pa^2 \stackrel{2.1(e)}{=} (ab^j)^pa^i = y^px$ .

- $x = ab^j, y = a^i$

If  $i = 1$  then  $xy = b^j = a^2b^ja^2 \stackrel{2.1(a8)}{=} a^3 \cdot ab^j = y^3x$ . Assume it is true for  $i = 2m + 1$  so for  $i = 2m + 3$  we have  $xy \stackrel{2.1(e)}{=} a^2 \cdot (ab^j)a^{2m+1} = a^{2+3(2m+1)}(ab^j) \stackrel{2.1(a8)}{=} a^{2+3(2m+1)} \cdot a(a^4b^j) = y^3x$ .

**Case 3:**

- $x = a^i, y = a^2b^j$

Considering Lemmas 2.6 and 2.8, we have  $xy = a^i(a^2b^j) = a^ib^{2j} = (a^2b^j)^pa^i = y^px$ .

- $x = a^2b^j, y = a^i$

If  $i = 1$  then  $xy = a^2(ab^ja)a \stackrel{2.3}{=} a^3(a^2b^j) = y^3x$ . Assume it is true for  $i = 2m + 1$  then by induction on  $i$  we have  $xy = (a^2b^j)a^{2m+3} = a^2(a^{3(2m+1)} \cdot a^2b^j) \stackrel{2.1(a8)}{=} a^{3(2m+1)+6} \cdot a^2b^j = y^3x$ .

**Case 4:**

- $x = b^j, y = ab^s$

By using Lemmas 2.1 part (d), 2.4 and 2.6, we have  $xy = b^ja(ab^sa) = (a^2b^s)(b^ja) = b^{2j}(b^ja) = (ab^s)^pa \cdot (b^ja) = (ab^s)^pb^j = y^px$ .

- $x = ab^s, y = b^j$

Lemmas 2.1 part (d), 2.4 and 2.6 help us to have  $xy = a(ab^sa) \cdot (ab^ja) = (a^2b^ja)(ab^sa) = (a^2b^j)(ab^s) = b^{2j}(ab^s) = y^px$ .

**Case 5:**

- $x = ab^j, y = a^2b^s$

Assume  $s$  is even. So  $xy \stackrel{2.1(c,d)}{=} yx$ . If  $s$  is odd so by Lemmas 2.1 part (e), 2.8 and 2.6 we have  $xy = a^3b^{s+j} = ab^{ps+j} = (a^2b^s)^pa \cdot b^j = y^px$ .

- $x = a^2b^s, y = ab^j$

If  $s$  is even, the proof is similar to the previous subcase so let it be odd. We have  $xy = a(ab^sa)b^j \stackrel{2.1(d), 2.3}{=} b^{2j}ab^s \stackrel{2.4}{=} (ab^j)^pa \cdot ab^s = y^px$ . □

**Theorem 3.4.** *Semigroup  $S_{p,n}$  is an inverse semigroup.*

*Proof.* Since the semigroup  $S_{p,n}$  is quasi-commutative so its idempotents commute [4] so we only show that  $S_{p,n}$  is regular, i.e. for all  $a \in S_{p,n}$ , the associated set  $V(a) = \{x \in S_{p,n} : axa = a \text{ and } xax = x\}$  is singleton. For the sake of showing the regularity of  $S_{p,n}$ , one may use the original definition of a regular semigroup i.e. for all  $x \in S_{p,n}$  there is  $y \in S_{p,n}$  such that  $xyx = x$  and we check this for elements of  $S_{p,n}$  by considering Theorem 3.1. The different cases are as follows:

**Case 1** ( $x = a^i$ ):

From 2.1 part (a6) we have

$$x \cdot a^{5^n - (1+i)} \cdot x = x,$$

where  $1 \leq i < 5^n - 1$ . And if  $i = 5^n - 1$  then  $x \cdot a^i \cdot x = (a^{5^n})^2 \cdot a^{5^n - 3} = x$ .

**Case 2** ( $x = b^j$ ):

By (a7) we get

$$x \cdot (a^2b^{p-(j+1)}) \cdot x = a^2b^{p+j-1} = x,$$

where  $1 \leq j \leq p - 1$ .

**Case 3** ( $x = ab^s$ ):

Let  $s$  where  $1 \leq s \leq p - 1$  be odd so we have  $x \cdot (a^3b^{p-(s+1)}) \cdot x = (ab^s a)(ab^{p-(s+1)} a)b^s = b^s ab^{p-1} \stackrel{2.7}{=} a^3 b^s \cdot b^{p-1} \stackrel{(1)}{=} ab^p \cdot b^{p+s-2} \stackrel{2.5}{=} x$ .

Let  $s$  be even so we have  $x \cdot (ab^{p-(s+1)}) \cdot x \stackrel{2.1(a2)}{=} a^2 b^{p+s-1} a$ . It is equal to  $(a^2 b^p)(b^{s-1} a) \stackrel{2.1(a4)}{=} (b^p a^2)(b^{s-1} a) = (b^p a)(ab^{s-1} a) = b^p ab^{s-1} = b^p (b^{p(s-1)} a) \stackrel{2.3}{=} x$ .

**Case 4** ( $x = a^2 b^t$ ):

For this kind of element wherein  $1 \leq t \leq p - 1$  we consider a4 and 2.6 to have  $x \cdot b^{p-(j+1)} \cdot x = a^2 b^{p-(j+1)} a^2 = a^2 b^p a^2 \cdot b^{j-1} = b^{p-(1-j)} = b^p b^{j-1} = (a^2 b) b^{j-1} = x$ . □

### 4. Decomposition as the union of groups

Let the values of  $n$  and  $p$  be fixed. Using the following codes in GAP [11]:

```
gap> T1:=Range(IsomorphismTransformationSemigroup([a^2b^2]));;
      T2:=AsGroup(MagmaByMultiplicationTable(MultiplicationTable(AsList(T1))));;
      StructureDescription(T2);
```

shows that the subsemigroup

$$H = \langle a^2 b^2 \rangle = \{a^2 b^2, b^4, a^2 b^6, b^8, \dots, a^2 b^{4k-2}, b^{4k}, a^2 b^{4k+2}\},$$

has a group structure being isomorphic to  $\mathbb{Z}_{\frac{p-1}{2}}$ . Also,

$$K = \langle \overbrace{ab^{\frac{p-1}{2}}, b^{\frac{p-1}{2}}}^{ab^{2k+1}, b^{2k+1}} \rangle = \{b^{2k+1}, ab^{2k+1}, bab^{2k}, a^2 b^{2k+1}, b^{4k+2}, ab^{4k+2}, bab^{4k+2}, a^2 b^{4k+2}\},$$

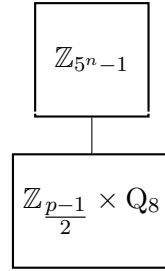
is a subgroup of  $S_{p,n}$ , isomorphic to  $Q_8$  and  $H \cap K = a^2 b^{4k+2} = e_K = e_H$ . Clearly  $|H \times K| = (2k + 1)8$  and from the presentation of  $S_{p,n}$ , if  $w \in \{a, b\}^+$  contains the letter  $b$  then so does every word in  $\{a, b\}^+$  that represents the same element of  $S_{p,n}$ . So, the set of all elements in  $S_{p,n}$  which are represented by a word containing the letter  $b$  forms an ideal of  $S_{p,n}$  and this makes  $H \times K$  to be an ideal of  $S_{p,n}$  as well. Furthermore, the monogenic semigroup

$$A = \langle a \rangle = \{a^i : i \in \{1, \dots, 5^n - 1\}\},$$

is obviously isomorphic to  $\mathbb{Z}_{5^n-1}$  and so we proved the following result:

**Corollary 4.1.** *The semigroup  $S_{p,n}$  is decomposed as  $S_{p,n} = A \cup (H \times K)$ .*

**Remark 4.2.** *Since the  $\mathcal{D}$ -class of  $A$  is isomorphic to  $\mathbb{Z}_{5^n-1}$  and  $H \times K \cong \mathbb{Z}_{\frac{p-1}{2}} \times Q_8$  so the  $\mathcal{D}$ -class structure of  $S_{p,n}$  can also be given in Figure 1.*

FIGURE 1. The  $\mathcal{D}$ -class structure of  $S_{p,n}$ 

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