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GOW-TAMBURINI TYPE GENERATION OF THE SPECIAL LINEAR GROUP FOR SOME SPECIAL RINGS

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Abstract. Let R be a commutative ring with unity and let $n \geq 3$ be an integer. Let $\mathrm{SL}_n(R)$ and $\mathrm{E}_n(R)$ denote respectively the special linear group and elementary subgroup of the general linear group $\mathrm{GL}_n(R)$. A result of Hurwitz says that the special linear group of size atleast three over the ring of integers of an algebraic number field is finitely generated. A celebrated theorem in group theory states that finite simple groups are two-generated. Since the special linear group of size atleast three over the ring of integers is not a finite simple group, we expect that it has more than two generators. In the special case, where R is the ring of integers of an algebraic number field which is not totally imaginary, we provide for $\mathrm{E}_n(R)$ (and hence $\mathrm{SL}_n(R)$) a set of Gow-Tamburini matrix generators, depending on the minimal number of generators of R as a $\mathbb{Z}$-module.

1. Introduction and summary of known results

We begin by recalling some basic definitions and well-known results regarding the elementary subgroup of the general linear group. Let R be a commutative ring with unity and $n \geq 3$ be an integer. Let $\mathrm{GL}_n(R)$ denote the general linear group consisting of all n by n invertible matrices over R.

Let $\mathrm{SL}_n(R)$ denote the special linear group consisting of all n by n matrices over R having determinant one i.e., $\mathrm{SL}_n(R)$ is the subgroup of $\mathrm{GL}_n(R)$ consisting of those matrices in $\mathrm{GL}_n(R)$ which have

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determinant one. We also recall for a group G, the definition of a commutator and use it to recall facts about perfect groups.

Definition 1.1. *(Commutator and commutator subgroup) Let G be a group and let $g, h \in G$. The commutator of g and h denoted by $[g, h]$ is defined as $[g, h] = g^{-1}h^{-1}gh$. The subgroup of G generated by commutators is called the commutator subgroup of G and is denoted by $[G,\ G]$. A group such that $G = [G,\ G]$ is called a perfect group.*

Definition 1.2. *(Elementary Matrices) Let R be a commutative ring with unity and let $n \geq 3$. Let i and j be integers with $1 \leq i \neq j \leq n$. Let $\mathrm{E}_{ij}(r)$ denote the $n \times n$ matrix with all ones along the diagonal, r in the (i, j)-th position and zeroes everywhere else. Clearly $\mathrm{E}_{ij}(r) \in \mathrm{GL}_n(R)$. Note that $\mathrm{E}_{ij}(-r)$ is the inverse of $\mathrm{E}_{ij}(r)$. These $\mathrm{E}_{ij}(r)$ are called elementary matrices. The subgroup of $\mathrm{GL}_n(R)$ generated by $\mathrm{E}_{ij}(r)$, $r \in R$ is called the elementary subgroup of $\mathrm{GL}_n(R)$ and is denoted by $\mathrm{E}_n(R)$.*

We now state a very useful splitting property of an elementary generator.

Remark 1.3. *Let R be a commutative ring with unity and let $n \geq 3$.*
(a) Let i and j be integers with $1 \leq i \neq j \leq n$. Then $\mathrm{E}_{ij}(r + s) = \mathrm{E}_{ij}(r)\mathrm{E}_{ij}(s)$.
(b) For $r, s \in R$ and three distinct indices i, j, k with $1 \leq i, j, k \leq n$, We have

$$\mathrm{E}_{ik}(rs) = [\mathrm{E}_{ij}(r), \mathrm{E}_{jk}(s)]. \tag{1.1}$$

Clearly, $\mathrm{E}_n(R) \subseteq \mathrm{SL}_n(R)$ and it is interesting to know when there is an equality here i.e., when is $\mathrm{E}_n(R) = \mathrm{SL}_n(R)$? A well-known fact in this direction is:

Theorem 1.4. *Let R be the ring of integers of an algebraic number field which is not totally imaginary. Then, for $n \geq 3$, we have $\mathrm{E}_n(R) = \mathrm{SL}_n(R)$.*

Proof. See [4, Theorem 3.6]. □

Theorem 1.5. *If $n \geq 3$, then $[\mathrm{E}_n(R), \mathrm{E}_n(R)] = \mathrm{E}_n(R)$. In particular, if R is the ring of integers of an algebraic number field which is not totally imaginary then $\mathrm{SL}_n(R)$, being equal to $\mathrm{E}_n(R)$ is perfect for $n \geq 3$ i.e., $[\mathrm{SL}_n(R), \mathrm{SL}_n(R)] = \mathrm{SL}_n(R)$, for $n \geq 3$.*

We now recall the motivating result for this paper.

Theorem 1.6. *(Gow-Tamburini theorem)*

(a) *For $n \geq 2, n \neq 4$, $\mathrm{SL}_n(\mathbb{Z})$ is two-generated with generators the Jordan matrix $x_n(1)$ and its transpose $x_n(1)^t$, where*
$$x_n(1) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & 1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix} = \prod_{i=1}^{n-1} \mathrm{E}_{i+1,i}(1).$$

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(b) *Further, for* $n = 4$, *the subgroup of* $\mathrm{SL}_4(\mathbb{Z})$ *generated by* $x_4(1)$ *and* $x_4(1)^t$ *has index* 8 *in* $\mathrm{SL}_4(\mathbb{Z})$.

Proof. See [3, Theorem 1, Theorem 2 and Theorem 3]. □

Motivated from the above result, we introduce the notion of a Gow-Tamburini matrix.

Definition 1.7 (Gow-Tamburini type matrix)**.** *Let* R *be the ring of integers of an algebraic number field and let* $\alpha \in R$. *Then a matrix of the form:*

$$x_n(\alpha) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ \alpha & 1 & 0 & \dots & 0 & 0 \\ 0 & \alpha & 1 & \dots & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & \alpha & 1 & 0 \\ 0 & 0 & 0 & \dots & \alpha & 1 \end{pmatrix} = \prod_{i=1}^{n-1} \mathrm{E}_{i+1,i}(\alpha) \text{ or its transpose } x_n(\alpha)^t = \prod_{i=n-1}^{1} \mathrm{E}_{i,i+1}(\alpha) \text{ is called as}$$

a Gow-Tamburini type matrix.

With the above notation, the result of Gow-Tamburini may be restated as: $\mathrm{SL}_n(\mathbb{Z})$, $n \neq 4$ is generated by the Gow-Tamburini matrices $x_n(1)$ and $x_n(1)^t$. We now state our main result.

Theorem 1.8. (a) *If* $n = 5$ *and further if* R *is the ring of integers of a quadratic number field, which is not totally imaginary and which is finitely generated as a* $\mathbb{Z}$*-module by* 1 *and* α, *then* $\mathrm{E}_5(R)$ *(and hence* $\mathrm{SL}_5(R)$*) is generated by the three Gow-Tamburini matrices* $x_5(1), x_5(1)^t$ *and* $x_5(\alpha)^t$. *(In fact, it is generated by* $x_5(1), x_5(1)^t$ *and* $x_5(\alpha)$ *as well.)*

(b) *If* $n = 5$ *and* R *is the ring of integers of an algebraic number field which is not totally imaginary and which is a finitely generated* $\mathbb{Z}$*-module with a minimal set of generators* $\{1, \alpha_1, \alpha_2, \dots, \alpha_n\}$, *then* $\mathrm{E}_5(R)$ *(and hence* $\mathrm{SL}_5(R)$*) is generated by the* $n+2$ *Gow-Tamburini matrices* $x_5(1), x_5(1)^t, x_5(\alpha_1)^t, \dots, x_5(\alpha_n)^t$. *(In fact, it is generated by the Gow-Tamburini matrices* $x_5(1), x_5(1)^t$, $x_5(\alpha_1), x_5(\alpha_2), \dots, x_5(\alpha_n)$ *as well.)*

(c) *If* $n \geq 6$ *and* R *is the ring of integers of a quadratic number field which is not totally imaginary and which is finitely generated as a* $\mathbb{Z}$*-module by* 1 *and* α, *then* $\mathrm{E}_n(R)$ *(and hence* $\mathrm{SL}_n(R)$*) is generated by the three Gow-Tamburini matrices* $x_n(1), x_n(1)^t$ *and* $x_n(\alpha)$.

(d) *If* $n \geq 6$ *and* R *is the ring of integers of an algebraic number field which is not totally imaginary and which is finitely generated as a* $\mathbb{Z}$*-module with a minimal set of generators* $\{1, \alpha_1, \alpha_2, \dots, \alpha_n\}$, *then* $\mathrm{E}_n(R)$ *(and hence* $\mathrm{SL}_n(R)$*) is generated by the* $n+2$ *Gow-Tamburini matrices* $x_n(1), x_n(1)^t, x_n(\alpha_1), \dots, x_n(\alpha_n)$.

In the second section, we prove parts (a) and (b) of the above result. In the last section we give the proofs of part (c) and (d).

2. Gow-Tamburini generation for $\mathrm{E}_5(R)$

We begin this section by the following observation:

Lemma 2.1. *Let R be the ring of integers of a quadratic number field which is finitely generated as a $\mathbb{Z}$-module by 1 and α. Let $n \geq 3$ be an integer. Then, $\mathrm{E}_n(R)$ is generated by $x_n(1), x_n(1)^t$ and $\mathrm{E}_{12}(\alpha)$.*

Proof. Let $\varepsilon \in \mathrm{E}_n(R)$. By using Remark 1.3, $\varepsilon = \prod_{(i,j)} \mathrm{E}_{ij}(a + b\alpha) = \prod_{(i,j)} \mathrm{E}_{ij}(a)\mathrm{E}_{ij}(b\alpha); a, b \in \mathbb{Z}$. Hence it is enough to prove that $\mathrm{E}_{ij}(\alpha); (i,j) \neq (1,2)$ is generated by $x_n(1), x_n(1)^t$ and $\mathrm{E}_{12}(\alpha)$. Consider the following equations:

$$\mathrm{E}_{i,i+1}(-1)\mathrm{E}_{i+1,i}(1)\mathrm{E}_{i,j}(r)\mathrm{E}_{i+1,i}(-1)\mathrm{E}_{i,i+1}(1) = \mathrm{E}_{i+1,j}(r). \tag{2.1}$$

$$\mathrm{E}_{j+1,j}(1)\mathrm{E}_{j,j+1}(-1)\mathrm{E}_{i,j}(r)\mathrm{E}_{j,j+1}(1)\mathrm{E}_{j+1,j}(-1) = \mathrm{E}_{i,j+1}(r). \tag{2.2}$$

Using Equation (2.2), we observe that all elementary generators of the type $\mathrm{E}_{1j}(r);\ 3 \leq j \leq n$ can be generated using $\mathrm{E}_{12}(r)$ and $x_n(1), x_n(1)^t$, since by Gow-Tamburini theorem, $\mathrm{E}_n(\mathbb{Z})$ is generated by $x_n(1)$ and $x_n^t(1)$. Using Equation (2.1), we observe that all elementary generators in any fixed column can be generated using $\mathrm{E}_{12}(\alpha)$ and $x_n(1), x_n(1)^t$. Thus $\mathrm{E}_n(R)$ is generated by $x_n(1), x_n(1)^t$ and $\mathrm{E}_{12}(\alpha)$. □

Corollary 2.2. *An analysis of the proof above gives that if R is the ring of integers of an algebraic number field which is a finitely generated $\mathbb{Z}$-module with a minimal set of generators $\{1, \alpha_1, \alpha_2, \ldots, \alpha_n\}$, then $\mathrm{E}_n(R)$ is generated by $x_n(1), x_n(1)^t, \mathrm{E}_{12}(\alpha_1), \ldots, \mathrm{E}_{12}(\alpha_n)$.*

Theorem 2.3. *Let R be the ring of integers of a quadratic number field which is a finitely generated $\mathbb{Z}$-module generated by 1 and α. Then $\mathrm{E}_5(R)$ is generated by the Gow-Tamburini matrices $x_5(1), x_5(1)^t$ and $x_5(\alpha)^t$.*

Proof. Let G be the subgroup of $\mathrm{E}_5(R)$ generated by $x_5(1), x_5(1)^t$ and $x_5(\alpha)^t$. By Gow-Tamburini theorem, we know that $\mathrm{E}_5(\mathbb{Z})$ is generated by $x_5(1)$ and $x_5(1)^t$. Consider

$$z_5(\alpha) = x_5(\alpha)^t x_5(1)^{-1} x_5(1)^t = x_5(\alpha)^t \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix} x_5(1)^t = \begin{pmatrix} 1-\alpha & 1 & \alpha & 0 & 0 \\ \alpha-1 & 0 & 1 & \alpha & 0 \\ 1-\alpha & 0 & 0 & 1 & \alpha \\ \alpha-1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $P_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} z_5(\alpha) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-\alpha & 1 & \alpha & 0 & 0 \\ \alpha-1 & 0 & 1 & \alpha & 0 \\ 1-\alpha & 0 & 0 & 1 & \alpha \\ \alpha-1 & 0 & 0 & 0 & 1 \end{pmatrix} \in G$. As $\mathrm{E}_{31}(1)P_5\mathrm{E}_{31}(-1) = \mathrm{E}_{21}(-\alpha)P_5$, it follows that $\mathrm{E}_{31}(1)P_5\mathrm{E}_{31}(-1)P_5^{-1} = \mathrm{E}_{21}(-\alpha) \in G$. Further by Lemma 2.1, G contains all the elementary matrices, so $G = \mathrm{E}_5(R)$. □

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Corollary 2.4. *Let R be the ring of integers of a quadratic number field which is a finitely generated $\mathbb{Z}$-module generated by 1 and α. Then $\mathrm{E}_5(R)$ is generated by the Gow-Tamburini matrices $x_5(1), x_5(1)^t$ and $x_5(\alpha)$.*

Proof. The proof follows from Theorem 2.3 and the identity $Tx_5(\alpha)T^{-1} = x_5(\alpha)^t$,

where $T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathrm{E}_5(\mathbb{Z})$. Then the result follows from Gow-Tamburini theorem. □

Theorem 2.5. *Let R be a ring of integers of a quadratic number field which is not totally imaginary and which is a finitely generated $\mathbb{Z}$-module generated by 1 and α. Then $\mathrm{SL}_5(R)$ is generated by $x_5(1), x_5(1)^t$ and $x_5(\alpha)$.*

Proof. The proof follows from Corollary 2.4 and Theorem 1.4. □

The statements and proofs above also help us to deduce the following consequences:

Theorem 2.6. *Let R be the ring of integers of an algebraic number field which is not totally imaginary and which is a finitely generated $\mathbb{Z}$-module with a minimal set of generators $\{1, \alpha_1, \alpha_2, \ldots, \alpha_n\}$. Then, $\mathrm{E}_5(R)$ (and hence $\mathrm{SL}_5(R)$ too) is generated by $x_5(1), x_5(1)^t$ and $x_5(\alpha_1)^t, \ldots, x_5(\alpha_n)^t$ (and hence also by $x_5(1), x_5(1)^t$ and $x_5(\alpha_1), \ldots, x_5(\alpha_n)$).*

Proof. Follows using Remark 1.3(a) and proof similar to Theorem 2.3. □

3. Generation for $n \geq 6$

In this section, we first prove $\mathrm{E}_n(R)$ for $n \geq 6$ is generated by $x_n(1), x_n(1)^t$ and $x_n(\alpha)$, if R is the ring of integers of a quadratic number field which is a finitely generated $\mathbb{Z}$-module, generated by 1 and α. For this we recall first some necessary notations and results:

Definition 3.1. *Let $e_1 = (1, \ldots, 0) \in \mathbb{Z}[\alpha]^n$. Define $Stab_n(e_1, R) = \{\sigma \in \mathrm{SL}_n(R) | e_1\sigma = e_1\}$. Clearly $Stab_n(e_1, R)$ is a subgroup of $\mathrm{SL}_n(R)$ and*

$$Stab_n(e_1, R) = \left\{ \begin{pmatrix} 1 & 0 \\ u & \beta \end{pmatrix} | u \in R^{n-1}, \beta \in \mathrm{SL}_{n-1}(R) \right\}.$$

More notations and observations:

(a) There is a surjective homomorphism

$\nu_n : Stab_n(e_1, R) \longrightarrow \mathrm{SL}_{n-1}(R)$ given by $\nu_n \begin{pmatrix} 1 & 0 \\ u & \beta \end{pmatrix} = \beta$.

The kernel of this homomorphism is denoted as $\mathcal{N}_n$ and is equal to $\left\{ \begin{pmatrix} 1 & 0 \\ u & I_{n-1} \end{pmatrix} | u \in R^{n-1} \right\}$ and is a normal subgroup of $Stab_n(e_1, R)$.

(b) We embed $\mathrm{SL}_{n-1}(R)$ inside $SL_n(R)$ as the lower right diagonal entry i.e., $\sigma \in \mathrm{SL}_{n-1}(R)$ is lies in $\mathrm{SL}_n(R)$ via the homomorphism $\theta : \mathrm{SL}_{n-1}(R) \longrightarrow \mathrm{SL}_n(R)$ given by $\theta(\sigma) = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}$. We can observe that $S = Image(\theta) \cong \mathrm{SL}_{n-1}(R)$ is a subgroup of $Stab_n(e_1, R)$.

(c) Now any element $\begin{pmatrix} 1 & 0 \\ u & \beta \end{pmatrix} \in Stab_n(e_1, R)$ can be decomposed as $\begin{pmatrix} 1 & 0 \\ u & \beta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \in \mathcal{N}_n S$. Therefore $Stab_n(e_1, R) = \mathcal{N}_n S$. Also $\mathcal{N}_n \cap S = I_n$. Hence $Stab_n(e_1, R)$ is a semi-direct product of $\mathcal{N}_n$ and S i.e., $Stab_n(e_1, R) = \mathcal{N}_n \rtimes S$.

(d) Let $z_n(\alpha) = (x_n(1)^t)^{-1} x_n(1) x_n(1)^t x_n(1)^{-1} x_n(1)^t$. Then

$$z_n(\alpha) = \begin{pmatrix} 1 & 0 \\ c & x_{n-1}(1)^t \end{pmatrix},$$

where c is the vector $[-1, (-1)^2, \ldots, (-1)^{n-1}]^t$. Then $z_n(\alpha) \in Stab_n(e_1, R)$. Let $\mathcal{B}_n$ be the subgroup of $Stab_n(e_1, R)$ generated by $x_n(1), z_n(\alpha), x_n(\alpha)$ i.e., $\mathcal{B}_n = \langle x_n(1), z_n(\alpha), x_n(\alpha) \rangle \leq Stab_n(e_1, R)$.

(e) We also wish to consider another stabilizer inside $\mathrm{SL}_{n-1}(R)(\subseteq SL_n(R))$. Hence, let $Stab_{n-1}(e_1, R) = \left\{ \begin{pmatrix} 1 & 0 \\ v & \beta \end{pmatrix} | v \in R^{n-2}, \beta \in \mathrm{SL}_{n-2}(R) \right\}$, where $e_1 = (1, \ldots, 0) \in R^{n-1}$.

Lemma 3.2. *Let $n \geq 4$ and let R be the ring of integers of an algebraic number field which is not totally imaginary. Then, $\mathcal{N}_n[Stab_n(e_1, R), Stab_n(e_1, R)] = Stab_n(e_1, R)$.*

Proof. Since $\mathcal{N}_n \leq Stab_n(e_1, R)$, clearly $\mathcal{N}_n[Stab_n(e_1, R), Stab_n(e_1, R)] \leq Stab_n(e_1, R)$. We know $Stab_n(e_1, R) = \mathcal{N}_n \rtimes S$ and as $n - 1 \geq 3$, by Theorem 1.5, S is perfect and hence $S = [S, S] \leq [Stab_n(e_1, R), Stab_n(e_1, R)]$. Thus, $Stab_n(e_1, R) \leq \mathcal{N}_n[Stab_n(e_1, R), Stab_n(e_1, R)]$. This completes the proof. □

In fact, we now prove that $Stab_n(e_1, R)$ is perfect, for $n \geq 4$.

Lemma 3.3. *Let R be the ring of integers of an algebraic number field which is not totally imaginary. Then $Stab_n(e_1, R)$ is perfect for $n \geq 4$.*

Proof. It is enough to prove that $Stab_n(e_1, R) \leq [Stab_n(e_1, R), Stab_n(e_1, R)]$.
Let $\alpha = \begin{pmatrix} 1 & 0 \\ u & \beta \end{pmatrix} \in Stab_n(e_1, R)$. We know $\alpha = \begin{pmatrix} 1 & 0 \\ u & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$, where $\beta \in \mathrm{SL}_{n-1}(R) =$

$\mathrm{E}_{n-1}(R)$ by Theorem 1.4. As $\mathrm{E}_{n-1}(R)$ is perfect, we get $\beta \in [\mathrm{E}_{n-1}(R), \mathrm{E}_{n-1}(R)]$ and hence $\begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix} \in$ $[Stab_n(e_1, R), Stab_n(e_1, R)]$. It now remains to prove that $\begin{pmatrix} 1 & 0 \\ u & I_{n-1} \end{pmatrix} \in [Stab_n(e_1, R), Stab_n(e_1, R)]$. For this, observe that we can write $\begin{pmatrix} 1 & 0 \\ u & I_{n-1} \end{pmatrix} = \prod_{i=2}^{n} \mathrm{E}_{i,1}(u_i), u_i \in R$. For $2 \leq i \leq n-1$, $\mathrm{E}_{i,1}(u_i) =$ $[\mathrm{E}_{i,n}(u_i), \mathrm{E}_{n,1}(1)]$, with $\mathrm{E}_{i,n}(u_i), \mathrm{E}_{n,1}(1) \in Stab_n(e_1, R)$. Further, $\mathrm{E}_{n,1}(u_n) = [\mathrm{E}_{n,n-1}(u_n), \mathrm{E}_{n-1,1}(1)]$, with $\mathrm{E}_{n,n-1}(u_n), \mathrm{E}_{n-1,1}(1) \in Stab_n(e_1, R)$. Combining the above steps we get that $Stab_n(e_1, R) \leq$ $[Stab_n(e_1, R), Stab_n(e_1, R)]$. This completes the proof. □

Lemma 3.4. *If* $\mathrm{SL}_{n-1}(R)$ *is generated by* $\nu_n(x_n(1))(= x_{n-1}(1))$, $\nu_n(z_n((\alpha))(= x_{n-1}(1)^t)$, $\nu_n(x_n(\alpha))(=$ $x_{n-1}(\alpha))$, *then* $Stab_n(e_1, R) = \mathcal{N}_n\mathcal{B}_n$.

Proof. We know that $\mathcal{N}_n$ is a normal subgroup of $Stab_n(e_1, R)$ and $\mathcal{B}_n$ is a subgroup of $Stab_n(e_1, R)$. Hence, $\mathcal{N}_n\mathcal{B}_n$ is a subgroup of $Stab_n(e_1, R)$. Now let $\sigma = \begin{pmatrix} 1 & 0 \\ u & \beta \end{pmatrix} \in Stab_n(e_1, R)$ where $u \in$ $R^{n-1}, \beta \in \mathrm{SL}_{n-1}(R)$. By assumption, $\nu_n(\sigma) = \beta = \nu_n(\rho)$, where ρ is in the subgroup of $\mathrm{SL}_n(R)$ generated by $x_n(1), z_n(\alpha)$ and $x_n(\alpha)$ i.e., $\rho \in \mathcal{B}_n$. Hence, write $\rho = \begin{pmatrix} 1 & 0 \\ w & \beta \end{pmatrix}$ for some $w \in R^{n-1}$. Hence, $\sigma = \begin{pmatrix} 1 & 0 \\ u - w & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ w & \beta \end{pmatrix} \in \mathcal{N}_n\mathcal{B}_n$, proving that $Stab_n(e_1, R) \leq \mathcal{N}_n\mathcal{B}_n$ and hence completing the proof. □

We now introduce a notation which is useful in the further lemmas. Let $\nu_n^{-1}(Stab_{n-1}(R))$ be denoted by $\mathcal{K}_n$. It is easy to see that $\mathcal{K}_n \cong \left\{ \begin{pmatrix} 1 & 0 & \bar{0} \\ a & 1 & \bar{0} \\ P & N & A \end{pmatrix} | \bar{0}^t \in R^{n-2}, a \in R, P, N \in R^{n-2}, A \in \mathrm{SL}_{n-2}(R) \right\}$. Let $\mathcal{D}_n$ denote the preimage in $\mathcal{B}_n$ under ν_n of $Stab_{n-1}(e_1, R)$ i.e., $\mathcal{D}_n = \mathcal{K}_n \cap \mathcal{B}_n$.

Lemma 3.5. *If* $\mathrm{SL}_{n-1}(R)$ *is generated by* $\nu_n(x_n(1)), \nu_n(z_n(\alpha))$ *and* $\nu_n(x_n(\alpha))$, *then* $\mathcal{K}_n = \mathcal{N}_n\mathcal{D}_n$.

Proof. Clearly $\mathcal{N}_n\mathcal{D}_n \subseteq \mathcal{K}_n$. Conversely, let $\sigma \in \mathcal{K}_n$. Write

$$\sigma = \begin{pmatrix} 1 & 0 & \bar{0} \\ a & 1 & \bar{0} \\ P & N & A \end{pmatrix},$$

for some $\bar{0} \in R^{n-2}, a \in R, P, N \in R^{n-2}, A \in \mathrm{SL}_{n-2}(R)$. Let $\tau = \begin{pmatrix} 1 & \bar{0} \\ N & A \end{pmatrix} \in Stab_{n-1}(e_1, R) \leq$ $\mathrm{SL}_{n-1}(R)$. By the given assumption and the fact that ν_n is an homomorphism, we write $\tau = \nu_n(\lambda)$, where λ belongs to the subgroup of $\mathrm{SL}_n(R)$ generated by $x_n(1), z_n(\alpha)$ and $x_n(\alpha)$ i.e., $\lambda \in \mathcal{B}_n$. Hence

$\lambda \in \nu_n^{-1}(Stab_{n-1}(e_1, R)) \cap \mathcal{B}_n$ i.e., $\lambda \in \mathcal{D}_n$. Write $\lambda = \begin{pmatrix} 1 & 0 & \bar{0} \\ b & 1 & \bar{0} \\ Q & N & A \end{pmatrix}$, for some $b \in R, Q \in R^{n-2}$ and N, A as above. Hence,

$$\sigma = \begin{pmatrix} 1 & 0 & \bar{0} \\ a-b & 1 & \bar{0} \\ P-Q & 0 & I_{n-2} \end{pmatrix} \begin{pmatrix} 1 & 0 & \bar{0} \\ b & 1 & \bar{0} \\ Q & N & A \end{pmatrix} \in \mathcal{N}_n \mathcal{D}_n.$$

Hence, $\mathcal{K}_n \subseteq \mathcal{N}_n \mathcal{D}_n$, completing the proof. □

Let $\varphi_n : \mathcal{K}_n \longrightarrow R$ be the map which sends an element of $\mathcal{K}_n$ described above to the element a of R. It is easy to see that φ_n is a surjective ring homomorphism. We now prove some results on commutator of $\mathcal{D}_n$, which has been defined above the previous lemma.

Lemma 3.6. *Let $n \geq 5$. Then, $[\mathcal{D}_n, \mathcal{D}_n] \leq [\mathcal{K}_n, \mathcal{K}_n] \leq ker\ \varphi_n$.*

Proof. As $\mathcal{D}_n \leq \mathcal{K}_n$, clearly $[\mathcal{D}_n, \mathcal{D}_n] \leq [\mathcal{K}_n, \mathcal{K}_n]$. We now prove that $[\mathcal{K}_n, \mathcal{K}_n] \leq ker\ \varphi_n$ by checking that $\varphi_n([\beta_1, \beta_2]) = 0$, where $\beta_i \in \mathcal{K}_n$. This is enough as φ_n is a homomorphism. For this, write $\beta_i = \begin{pmatrix} 1 & 0 & \bar{0} \\ a_i & 1 & \bar{0} \\ P_i & N_i & A_i \end{pmatrix}$, where $\bar{0}^t \in R^{n-1}, a_i \in R, P_i, N_i \in R^{n-1}, A_i \in \mathrm{SL}_{n-2}(R)$. A straight forward calculation gives that $[\beta_1, \beta_2] = \begin{pmatrix} 1 & 0 & \bar{0} \\ 0 & 1 & \bar{0} \\ T & V & [A_1, A_2] \end{pmatrix}$, with $T \in R^{n-2}, V \in R^{n-2}$. Then $\varphi_n([\beta_1, \beta_2]) = 0$, proving that $[\mathcal{K}_n, \mathcal{K}_n] \leq ker\ \varphi_n$, as claimed. □

Lemma 3.7. *Let $n \geq 6$. Assume that $\mathrm{SL}_{n-1}(R)$ is generated by $\nu_n(x_n(1)), \nu_n(z_n(\alpha)), \nu_n(x_n(\alpha))$, then $\mathcal{K}_n = \mathcal{N}_n[\mathcal{D}_n, \mathcal{D}_n]$.*

Proof. By Lemma 3.5, it is enough to prove that $\mathcal{K}_n \subseteq \mathcal{N}_n[\mathcal{D}_n, \mathcal{D}_n]$. In order to prove this, we first prove that $\nu_n(\mathcal{D}_n) = Stab_{n-1}(e_1, R)$ and then use it to prove the required result. In order to prove $\nu_n(\mathcal{D}_n) = Stab_{n-1}(e_1, R)$, note that it is enough to prove $Stab_{n-1}(e_1, R) \subseteq \nu_n(\mathcal{D}_n)$. Let $\tau \in Stab_{n-1}(e_1, R) \leq \mathrm{SL}_{n-1}(R)$. The fact that ν_n is a homomorphism, along with the given assumption imply the existence of $\lambda \in \mathcal{B}_n$ such that $\nu_n(\lambda) = \tau$. Hence, $\lambda \in \nu_n^{-1}(Stab_{n-1}(e_1, R) \cap \mathcal{B}_n := \mathcal{D}_n$. Hence $\tau \in \nu_n(\mathcal{D}_n)$. We now prove $\mathcal{K}_n \subseteq \mathcal{N}_n[\mathcal{D}_n, \mathcal{D}_n]$. Given $\sigma \in \mathcal{K}_n$, write $\sigma = \begin{pmatrix} 1 & 0 \\ u & \tau \end{pmatrix}$, where $u \in R^{n-1}$ and $\tau = \begin{pmatrix} 1 & \bar{0} \\ N & A \end{pmatrix} \in \mathrm{SL}_{n-1}(R)$, where $N \in R^{n-2}$ and $A \in \mathrm{SL}_{n-2}(R)$. Then, $\tau \in Stab_{n-1}(e_1, R)$. By Lemma 3.3 and using ν_n is a homomorphism, we get $Stab_{n-1}(e_1, R) = [Stab_{n-1}(e_1, R), Stab_{n-1}(e_1, R)] = [\nu_n(\mathcal{D}_n), \nu_n(\mathcal{D}_n)] =$

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$\nu_n([\mathcal{D}_n, \mathcal{D}_n])$. Hence there exists $\beta \in [\mathcal{D}_n, \mathcal{D}_n]$ such that $\nu_n(\beta) = \tau$. Hence, write $\beta = \begin{pmatrix} 1 & 0 \\ w & \tau \end{pmatrix}$ for some $w \in R^{n-1}$. Hence, $\sigma = \begin{pmatrix} 1 & 0 \\ u-w & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ w & \tau \end{pmatrix} \in \mathcal{N}_n[\mathcal{D}_n, \mathcal{D}_n]$, completing the proof. □

We now prove the main theorem.

Theorem 3.8. *Let R be the ring of integers of a quadratic number field which is not totally imaginary and assume that R is finitely generated as a $\mathbb{Z}$-module by 1 and α. Then for all $n \geq 5$, $\mathrm{E}_n(R)$ (and hence $\mathrm{SL}_n(R)$) is generated by the Gow-Tamburini matrices $x_n(1), x_n(1)^t$ and $x_n(\alpha)$.*

Proof. The proof is by induction on n and we have already established it for $n = 5$. By induction hypothesis, $x_{n-1}(1), x_{n-1}(1)^t$ and $x_{n-1}(\alpha)$ generate $\mathrm{SL}_{n-1}(R).(= \mathrm{E}_n(R))$. Let G be the subgroup of $\mathrm{SL}_n(R)$ generated by $x_n(1), x_n(1)^t$ and $x_n(\alpha)$. As $x_n(\alpha) \in \mathcal{K}_n$, by Lemma 3.7 $x_n(\alpha) = g_n h_n, g_n \in \mathcal{N}_n$ and $h_n \in [\mathcal{D}_n, \mathcal{D}_n] \leq ker\ \varphi_n$ by Lemma 3.6. Clearly $x_n(\alpha) \in \mathcal{B}_n$ and $h_n \in [\mathcal{D}_n, \mathcal{D}_n] \leq \mathcal{D}_n \leq \mathcal{B}_n$, gives $g_n = x_n(\alpha)h_n^{-1} \in \mathcal{N}_n \cap \mathcal{B}_n$. Hence we have $\varphi_n(x_n(\alpha)) = \alpha = \varphi_n(g_n h_n) = \varphi_n(g_n) + \varphi_n(h_n) = \varphi_n(g_n)$ as $\varphi_n(h_n) = 0$. Hence, we get that $g_n \in \mathcal{N}_n \cap \mathcal{B}_n$ with $\varphi_n(g_n) = \alpha$. This implies $g_n = \begin{pmatrix} 1 & 0 \\ v & I_{n-1} \end{pmatrix} \in G$, where $v = (\alpha, a_3 + \alpha b_3, \ldots, a_n + \alpha b_n) \in R^{n-1}$, where $a_i, b_i \in \mathbb{Z}$. Let $N = \prod_{i=3}^{n} \mathrm{E}_{i1}(-a_i) \in G$ then $N g_n = \begin{pmatrix} 1 & 0 \\ w & I_{n-1} \end{pmatrix}$, where $w = (\alpha, \alpha b_3, \ldots, \alpha b_n) \in R^{n-1}$. Let $M = \prod_{i=3}^{n} \mathrm{E}_{i2}(-b_i) \in G$, then $MNg_n = \mathrm{E}_{21}(\alpha) \in G$. Hence $G = \mathrm{E}_n(R) = \mathrm{SL}_n(R)$. □

Theorem 3.9. *If $n \geq 6$ and R is the ring of integers of an algebraic number field which is not totally imaginary and which is finitely generated as a $\mathbb{Z}$-module with a minimal set of generators $\{1, \alpha_1, \alpha_2, \ldots, \alpha_n\}$, then $\mathrm{E}_n(R)$ (and hence $\mathrm{SL}_n(R)$) is generated by the $n+2$ Gow-Tamburini matrices $x_n(1), x_n(1)^t, x_n(\alpha_1), \ldots, x_n(\alpha_n)$.*

Proof. Follows similar to that of Theorem 3.8. □

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