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STRUCTURE OF FINITE GROUPS WITH TRAIT OF NON-NORMAL SUBGROUPS II

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Abstract. A finite non-Dedekind group G is called an $\mathcal{NAC}$-group if all non-normal abelian subgroups are cyclic. In this paper, all finite $\mathcal{NAC}$-groups will be characterized. Also, it will be shown that the center of non-nilpotent $\mathcal{NAC}$- groups is cyclic. If $\mathcal{NAC}$-group G has a non-abelian non-normal Sylow subgroup of odd order, then other Sylow subgroups of G are cyclic or of quaternion type.

1. Introduction

A finite non-Dedekind group G is called an $\mathcal{NAC}$-group ($\mathcal{NNC}$-group) if all of whose non-normal abelian (nilpotent) subgroups are cyclic. Any $\mathcal{NNC}$- group is $\mathcal{NAC}$-group, but the converse does not happen. The author and G. Tiemouri in [5], provided the complete characterization of finite non-nilpotent $\mathcal{NNC}$-groups. They showed that any $\mathcal{NNC}$-group is solvable [5, Theorem 2.2], but there exist a non-solvable $\mathcal{NAC}$-group.

Also, it can be seen in the introduction of [5] that if all the Sylow subgroups of G are cyclic, then G is an $\mathcal{NNC}$-group. Therefore G is an $\mathcal{NAC}$-group. But the converse does not hold. If all non-normal nilpotent (in particular abelian) subgroups of G are cyclic, not necessarily, its Sylow subgroups are cyclic.

In [9], L. Zhang and J. Zhang gave the classification of $\mathcal{NAC}$-p-groups. The purpose of this paper is to investigate the structure of finite non-Dedekindian $\mathcal{NAC}$-groups that containing at least a non-cyclic Sylow subgroup.

Our method in this article is to assume that the non-cyclic Sylow subgroup of G is abelian or not, in the non-abelian case, we consider this Sylow subgroup is normal or not.

Keywords: $\mathcal{NNC}$-group; $\mathcal{NAC}$-group; Non-nilpotent groups.

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In Theorems 4.1 and 4.2 we assume that G contains a non-cyclic abelian Sylow subgroup, in Theorems 5.1 and 5.2 we consider G has a non-abelian Sylow subgroup of odd order and in Theorems 5.4 and 5.6, G has a non-abelian Sylow subgroup of even order and all odd-order Sylow subgroups of G are cyclic.

In this paper, we use Q_{2^n}, D_{2^n} and $\mathbb{Z}_{p^n}$ to denote the generalized quaternion group of order 2^n, the dihedral group of order 2^n and the cyclic group of order p^n, respectively. Our notations are standard and can be found in [3].

2. Preliminaries

Throughout this paper, we used the following notations for the minimal non-abelian p-groups which are not isomorphic to Q_8.

$$M_p(m,n) = \langle a, b \,|\, a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle,$$

where $m \geq 2$.

$$M_p(m,n,1) = \langle a, b, c \,|\, a^{p^m} = b^{p^n} = c^p = 1, [a,b] = c, [c,a] = [c,b] = 1 \rangle,$$

where $m \geq n$, and if $p = 2$, then $m + n \geq 3$.

In the following theorem L. Zhang and J. Zhang gave the structure of non-abelian $\mathcal{NAC}$-p-group of odd order.

Theorem 2.1. [9, Theorem 3.3] *Assume G is a finite non-Dedekindian p-group and p is an odd prime. Then all non-normal abelian subgroups of G are cyclic if and only if G is one of the following groups.*

(i) $M_p(m,n)$, *where* $m \geq 2$.
(ii) $M_p(1,1,1) * C_{p^n}$.
(iii) $P_{81} = \langle a, b \,|\, a^9 = c^3 = 1, b^3 = a^3, [a,b] = c, [c,a] = a^3, [c,b] = 1 \rangle$.

The group P_{81} is a 3-group of maximal class of order 81.

Lemma 2.2. [5, Lemma 2.1] *Let G be a finite solvable group and P be a non-normal Sylow subgroup of G. If $\mathcal{N}_G(P)'$ is a Hall subgroup of $\mathcal{N}_G(P)$ and $\mathcal{N}_G(P)$ is contained in a maximal subgroup of prime index, then G is p-nilpotent.*

In the [5] the authors stated the above lemma with the assumption " $\mathcal{N}_G(P)'$ is a Hall subgroup of G". Without any change in the proof, Lemma 2.2 satisfies by the assumption "$\mathcal{N}_G(P)'$ is a Hall subgroup of $\mathcal{N}_G(P)$".

Theorem 2.3. [6, Main Theorem] *Let G be a finite group with quaternion (ordinary or generalized) Sylow 2-subgroup. Then G is 2-nilpotent if:*

(i) *either* $3 \nmid |G|$;
(ii) *or G is solvable with Sylow 2-subgroup of order greater that* 16.

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3. $\mathcal{NAC}$-groups with a subgroup of type (p,p)

In this section, we prove that the center of an $\mathcal{NAC}$-group is cyclic. Also, we find some properties of an $\mathcal{NAC}$-group with a subgroup of type (p,p).

Lemma 3.1. *Let H be a subgroup of $\mathcal{NAC}$-group G with non-cyclic center, then $H \trianglelefteq G$.*

Proof. Let $L \leqslant H$ be a central subgroup of type (p,p), for some prime number p. Then for any $h \in H$, $L\langle h\rangle$ is non-cyclic abelian subgroup of G so by assumption is normal in G. Now $h^g \in L\langle h\rangle \leqslant H$ for any $g \in G$. Hence $H \trianglelefteq G$. □

Theorem 3.2. *The center of any non-nilpotent $\mathcal{NAC}$-group is cyclic.*

Proof. Assume that $Z(G)$ is non-cyclic and $L \leqslant Z(G)$ is of type (q,q). According to previous Lemma 3.1, Sylow q-subgroup is normal in G. Now since $[L,\langle y\rangle] = 1$ for every q' element $y \in G$, so $L\langle y\rangle$ is non-cyclic abelian, hence $L\langle y\rangle \trianglelefteq G$, thus $\langle y\rangle \trianglelefteq G$. Therefore all Sylow p-subgroups are normal and Dedekindian for every $p \neq q$. So G is nilpotent which is a contradiction. Hence $Z(G)$ is cyclic. □

Lemma 3.3. *Let $G \cong N \rtimes H$ be an $\mathcal{NAC}$-group such that $(|N|,|H|) = 1$ and $K \leqslant H$.*

- (i) *If $K \ntrianglelefteq G$, then each abelian subgroup of $\mathcal{C}_N(K)$ is cyclic. Therefore all Sylow subgroups of odd order in $\mathcal{C}_N(K)$ are cyclic and the Sylow 2-subgroup is cyclic or of quaternion type.*
- (ii) *If K contain a non-cyclic abelian subgroup, then $\mathcal{C}_N(K)$ is Dedekindian.*

Proof. (i) Let L be an abelian subgroup of $\mathcal{C}_N(K)$. If L is not cyclic, then subgroup $\langle L,a\rangle$ is non-cyclic abelian for any $a \in K$, so it is normal in G. This implies that $\langle a\rangle \trianglelefteq G$ and so $K \trianglelefteq G$, a contradiction. Hence L is cyclic. It follows that $\mathcal{C}_N(K)$ does not contain any subgroup of type (p,p). Therefore any Sylow subgroup of odd order of $\mathcal{C}_N(K)$ is cyclic and Sylow 2-subgroup is cyclic or quaternion (ordinary or generalized).

(ii) Assume that L is a non-cyclic abelian subgroup of K. Then $L\langle y\rangle$ is non-cyclic abelian for any $y \in \mathcal{C}_N(K)$, therefore $\langle y\rangle \trianglelefteq G$. So $\mathcal{C}_N(K)$ is Dedekindian □

We will use the following lemma repeatedly to prove the converse of theorems.

Lemma 3.4. *Let $G \cong N \rtimes H$, where N is nilpotent Hall subgroup of G such that any non-cyclic Sylow subgroup of N is $\mathcal{NAC}$-group. Then G is an $\mathcal{NAC}$-group if:*

- (i) *any Sylow subgroup of H is cyclic or of quaternion type;*
- (ii) *for any non-normal subgroup K of H, each abelian subgroup of $\mathcal{C}_N(K)$ is cyclic;*
- (iii) *any non-cyclic abelian subgroup of N is H-invariant.*

Proof. Assume that S is an abelian non-normal subgroup of G, then $S = LK$ where $L \leqslant N$ and $K \leqslant H$. Obviously K is cyclic, by (i). If $K \ntrianglelefteq H$, then $L \leqslant \mathcal{C}_N(K)$ is cyclic, by (ii). Therefore S is cyclic. Assume that $K \trianglelefteq H$, then either $L = 1$ and so S is cyclic, or $L \neq 1$ and $L \ntrianglelefteq G$ (oterewise

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$S = LK \trianglelefteq G$). Hence N contains a non-cyclic Sylow subgroup P, such that $L \cap P \ntrianglelefteq G$. If $L \cap P$ is not cyclic, by (iii), is H-invariant, thus we get $L \cap P \ntrianglelefteq P$. By assumption P is an $\mathcal{NAC}$-geoup, thus $L \cap P$ is cyclic, a contradiction. Therefore L is cyclic, since any Sylow subgroup of L is cyclic, and so S is cyclic too. □

Remark 3.5. *By [9, Lemma 3.1-(2)], if $\mathcal{NAC}$-p-group G has a subgroup of type (p, p, p), then is Dedekindian. The same result is true for an $\mathcal{NAC}$-group G when $p = 2$.*

Let $L = \langle a, b, c\rangle \leqslant G$ be of type (p, p, p). As $\langle a, b\rangle$ and $\langle a, c\rangle$ are normal in G, then $\langle a\rangle \trianglelefteq G$. Similarly $\langle b\rangle, \langle c\rangle \trianglelefteq G$. Now for every element $t \in P$, there exists two elements, say $x, y \in L$, such that $\langle t\rangle \cap \langle x, y\rangle = 1$. Then $\langle t, x\rangle$ and $\langle t, y\rangle$ are normal is G. Therefore $\langle t\rangle \trianglelefteq G$. If $p = 2$, then $a, b, c \in Z(G)$. Hence for any element $t \in G$, $\langle t\rangle \trianglelefteq G$ and so G is a Dedekind group.

In the following we consider non-Dedekivdian $\mathcal{NAC}$-group G with a subgroup of type (p, p), for some prime factor p of $|G|$.

Proposition 3.6. *Let non-Dedekindian $\mathcal{NAC}$-group G contain a subgroup L of type (p, p), where p is prime. Assume that $Q \in \mathcal{S}y\ell_q(G)$, where $q \neq p$.*

- (i) *If $q > \max\{p, 3\}$, then for any q-element x, $\langle x\rangle \trianglelefteq G$. So Q is abelian normal subgroup of G.*
- (ii) *If $2 < q < p$ and Q is non-cyclic, then for any q-element x, $\langle x\rangle \trianglelefteq G$, so Q is abelian normal subgroup of G.*
- (iii) *If Sylow p-subgroup of G has a non-normal subgroup in G (especially if P is non-abelian), then for any odd prime $q \neq p$, Sylow q-subgroup is cyclic.*
- (iv) *If p is smallest prime factor of G, then for any $q \neq p$, Sylow q-subgroup of G is cyclic.*
- (v) *If p is odd prime, then Sylow 2-subgroup is cyclic or of quaternion type. Also if $q = 2$ and $Q \trianglelefteq G$, then for any $x \in Q$, $\langle x\rangle \trianglelefteq G$, so Q is cyclic or $Q \cong Q_8$. In additional Q is direct factor of G.*

Proof. (i) Now for $q > \max\{p, 3\}$, since q not divide $(p-1)$ and $(p+1)$, so $[L, x] = 1$, for any q-element x, if $p = 2$, then $\mathrm{Aut}(L) \cong S_3$, so for any $q > 3$, $[L, \langle x\rangle] = 1$. Hence $L\langle x\rangle \trianglelefteq G$, which implies that $\langle x\rangle \trianglelefteq G$. Therefore Q is Dedekindian normal subgroup of G, where $Q \in \mathcal{S}y\ell_q(G)$. So Q is abelian.

(ii) We show that the center of Q is non-cyclic. Assume that $y \in Z(Q)$ is of order q. Now for an element $y' \neq y$ of order q, $K = \langle y, y'\rangle$ is non-cyclic abelian subgroup so is normal in G. Hence $[K, L] = 1$, then $L\langle y'\rangle \trianglelefteq G$. Therefore $\langle y'\rangle \trianglelefteq Q$ and so $y' \in Z(Q)$.

Now for any $x \in Q$, there is a central element y of order q, such that $\langle y\rangle \cap \langle x\rangle = 1$. As $\langle x, y\rangle$ is non-cyclic abelian and necessarily is normal in G, then $[L, \langle x, y\rangle] = 1$ and so $L\langle x\rangle \trianglelefteq G$. Hence $\langle x\rangle \trianglelefteq G$. Therefore $\langle x\rangle \trianglelefteq G$ for any q-element x, which implies that Q is abelian normal subgroup of G.

(iii) By (i), for any $2 < q < p$, Q is cyclic, for $P \ntrianglelefteq G$. Similarly by (ii), for any $q > p$, Q is cyclic.

(iv) If G contains a subgroup K of type (q, q), where $q \neq p$ is prime number, since $K \trianglelefteq G$, $[L, K] = 1$. Hence for any $x \in L$, $\langle x\rangle K$ is non-cyclic abelian subgroup of G. Therefore $\langle x\rangle K \trianglelefteq G$ and so $\langle x\rangle \trianglelefteq G$.

Thus $L \leqslant Z(G)$. Now for any r-element x, when $r \neq p$, $\langle x\rangle \trianglelefteq G$, since $\langle L, x\rangle \trianglelefteq G$. Assume that $r = p$, again $\langle L, x\rangle \trianglelefteq G$, hence $[x, K] = 1$ and so $\langle x, K\rangle \trianglelefteq G$, which implies that $\langle x\rangle \trianglelefteq G$. Therefore G is Dedekind group, a contradiction.

(v) If G contains a subgroup of type $(2, 2)$, by (iv), L must be cyclic, a contradiction. So Sylow 2-subgroup of G is cyclic or of quaternion type. If $q = 2$ and $Q \trianglelefteq G$, then for any $x \in Q$, $\langle L, x\rangle \trianglelefteq G$, so $\langle x\rangle \trianglelefteq G$. Therefore Q is cyclic or $Q \cong Q_8$. Since, for any generator x of Q, $\mathrm{Aut}(\langle x\rangle)$ is 2-group, therefore Q is direct factors of G, for Q has a normal complement. □

Proposition 3.7. *Let non-Dedekindian $\mathcal{NAC}$-group G contain a 2-subgroup $L = \langle x, y\rangle$ of type $(2, 2)$, where $x \in Z(Q)$ and $Q \in \mathcal{S}y\ell_2(G)$. Then*

(i) *any Sylow subgroup of G of odd order is cyclic;*
(ii) *for any p-element a, $\langle a\rangle \trianglelefteq G$, where $p > 3$ is prime.*
Now assume that $|Q| > 4$, then
(iii) *$x \in Z(G)$ and so $2 \mid |Z(G)|$;*
(iv) *G is 2-nilpotent with cyclic 2-complement, furthermore Q is non-abelian and $\mathcal{C}_Q(L)$ is maximal of Q;*
In additional if G non-nilpotent then
(v) *Z(Q) is cyclic, $\langle y\rangle$ and $\langle xy\rangle$ are maximal cyclic subgroup of Q and $\Omega_1(Q)$ is unique subgroup of type $(2, 2)$;*
(vi) *$\mathcal{C}_Q(L)$ either is abelian of type $(2^\ell, 2)$, where $|Q| = 2^{\ell+2}$ and $\ell \geq 2$ or $\mathcal{C}_Q(L) \cong Q_8 \times \mathbb{Z}_2$.*

Proof. By Proposition 3.6, (i) and (ii) is obvious.

(iii) Suppose that $Q \setminus L$ contains an element c such that $x \notin \langle c\rangle$. Then $\langle x, c\rangle \trianglelefteq G$ and so $\langle x\rangle = L \cap \langle x, c\rangle \trianglelefteq G$, thus $x \in Z(G)$. Therefore we can assume that any element of $Q \setminus L$ is of order grater than 2 and $x \in \langle c\rangle$. Without loss of generality, we let $|c| = 4$. Since $(cy)^2 = x[y, c]$ and $[y, c] \in L$, then $|cy| = 4$ and so $x = (cy)^2 = x[c, y]$ thus $[c, y] = 1$. Therefore $\langle c^2\rangle = \Phi(\langle c, y\rangle) \trianglelefteq G$, again $x \in Z(G)$.

(iv) By (i) and (ii), it is sufficient we show that for any 3-element a, $\langle a\rangle \trianglelefteq G$. By (iii), $x \in Z(G)$, so $[x, a] = 1$, now according to Maschke's Theorem, we can assume that $[y, a] = 1$, thus $[L, a] = 1$. Hence $\langle L, a\rangle \trianglelefteq G$ and so $\langle a\rangle \trianglelefteq G$. Therefore G has cyclic normal 2-complement. If Q is abelian, by Lemma 3.1, $Q \trianglelefteq G$ and so G must be abelian, a contradiction. Thus Q is non-abelian and $|Q : \mathcal{C}_Q(L)| = 2$, by N/C-Theorem.

(v) By (iv) G is 2-nilpotent, as G is non-nilpotent, $Q \ntrianglelefteq G$. So by Lemma 3.1, $Z(Q)$ is cyclic. Assume that for some $c \in Q \setminus L$, $c^2 = y$. Since $\langle x, c\rangle \trianglelefteq G$, so $\langle y\rangle = \langle x, c\rangle^2 \trianglelefteq G$. Therefore we have contradiction $y \in Z(G)$. Similarly $\langle xy\rangle$ is maximal cyclic. Assume that $c \in Q \setminus L$ such that $x \notin \langle c\rangle$. If $c \notin \mathcal{C}_Q(L)$, since $\langle x, c\rangle \trianglelefteq G$, we have contradiction $Q = \langle x, c\rangle\mathcal{C}_Q(L) \trianglelefteq G$. Hence $c \in \mathcal{C}_Q(L)$. Now $\langle c, y\rangle \trianglelefteq G$ thus $\langle y\rangle = \langle c, y\rangle \cap L \trianglelefteq G$, then $y \in Z(G)$, a contradiction. Then for any $c \in Q \setminus L$, $x \in \langle c\rangle$ and so $|c| > 2$. Therefore $\Omega_1(Q) = L$.

(vi) By (v), for any element $c \in \mathcal{C}_Q(L) \setminus L$, $x \in \langle c \rangle$, so $\mathcal{C}_Q(L)/\langle y \rangle$ just contains one subgroup of order 2, hence either is cyclic or of quaternion type. If $\mathcal{C}_Q(L)/\langle y \rangle$ is cyclic, then $\mathcal{C}_Q(L)$ is abelian of type $(2^\ell, 2)$, where $|Q| = 2^{\ell+2}$ and $\ell \geq 2$. Suppose that $\mathcal{C}_Q(L)/\langle y \rangle$ is of quaternion type. Since $\langle y \rangle$ is central maximal cyclic subgroup of $\mathcal{C}_Q(L)$ and $\mathcal{C}_Q(L)$ does not contain any non-normal subgroup of type $(2^2, 2)$, so $\mathcal{C}_Q(L) \cong Q_8 \times \mathbb{Z}_2$. □

In the part (vi) of above Theorem, when $\mathcal{C}_Q(L) \cong Q_8 \times \mathbb{Z}_2$, $|Q| = 32$. By using GAP we see $Q \cong \mathrm{SmallGroup}(32, 8)$. For a presentation of Q see part (i-1) of Theorem 5.4.

Theorem 3.8. *Let G be a non-Dedekindian nilpotent group. Then G is $\mathcal{NAC}$-group if and only if G is isomorphic to one of the following groups.*

(i) *$Q \times C$, where $Q \ncong Q_8$ is non-abelian $\mathcal{NAC}$-2-group.*

(ii) *$Q \times P \times C$, where P is non-abelian $\mathcal{NAC}$-p-group of odd order and Q is cyclic or $Q \cong Q_8$.*

Here C is cyclic Hall subgroup of odd order.

Proof. If G has no any subgroup of type (p, p), where $p \geq 2$. Then the $2'$-Hall subgroup of G is cyclic, so $2 \mid |G|$ and Sylow 2-subgroup of G is generalized quaternion of order at least 16. So G is isomorphic to the group (i).

Assume that G contains a subgroup H of type (p, p). If $p = 2$, then by Proposition 3.7 (iv), the $2'$-Hall subgroup of G is cyclic. So Q (the Sylow 2-subgroup of G) is one of the $\mathcal{NAC}$-2-group listed in [9, Theorems 3.5 and 3.6]. Hence G is isomorphic to the group (i).

Now assume that p is odd prime number and $P \in \mathcal{S}y\ell_p(G)$. Since for any p'-element y, $H\langle y \rangle$ is abelian non-cyclic subgroups of G, so $H\langle y \rangle \trianglelefteq G$. Which implies that $\langle y \rangle \trianglelefteq G$. Therefore P has a Dedekindian complement. Since P is non-abelian, for G is non-Dedekindian, so G is isomorphic to the group (ii), by Proposition 3.6 (iii) and (v).

By Lemma 3.4 and assumption $H = C$ or $H = Q \times C$ in the case (ii), the converse is true. □

4. $\mathcal{NAC}$-groups with a non-cyclic abelian Sylow subgroup

According to Theorem 3.8, in this section we characterized the structure of the non-nilpotent $\mathcal{NAC}$-group with an abelian Sylow subgroup.

Theorem 4.1. *Let G be a non-nilpotent group with a non-cyclic abelian Sylow 2-subgroup. Then G is an $\mathcal{NAC}$-group if and only if $G \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times C) \rtimes \mathbb{Z}_{3^m}$ where C is cyclic $\{2, 3\}'$-Hall subgroup.*

Proof. Assume that $Q \in \mathcal{S}y\ell_2(G)$ and $L \leqslant Q$ is of type $(2, 2)$. By Proposition 3.7 (i) and (ii), G contains cyclic normal $\{2, 3\}'$-Hall subgroup C. If $|Q| > 4$, by Proposition 3.7 (iv), we get contradiction Q is non-abelian, so $Q = L$. Hence $3 \mid |G|$ and Sylow 3-subgroup is cyclic non-normal in G (Proposition 3.7 (i)-(ii)) and acts irreducibly on Q. So $G \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times C) \rtimes \mathbb{Z}_{3^m}$.

The converse by Lemma 3.4 is true. □

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We observed that in Theorem 4.1, if Q, the Sylow 2-subgroup of G is non-cyclic abelian, then it is of type $(2,2)$. Because the center of $\mathcal{NAC}$-group is cyclic, so $Q \cap Z(G) = 1$, by Maschke's theorem. Therefore no subgroup of Q is normal in G. We now extend this problem to the abelian Sylow p-subgroups of odd order.

Theorem 4.2. *Let G be a non-nilpotent group with a non-cyclic abelian Sylow subgroup P of odd order. Then G is $\mathcal{NAC}$-group if and only if G is isomorphic to one of the following groups.*

- (i) *If P has a subgroup which is non-normal in G, then G has one of the following structures.*
  - (i-1) $G \cong (P \times C) \rtimes H$.
  - (i-2) $G \cong Q \times (P \times C) \rtimes H$, *where $Q \in \mathcal{S}y\ell_2(G)$ is cyclic or $Q \cong Q_8$.*

  In all cases C is cyclic normal Hall subgroup of odd order, any Sylow subgroup of H is cyclic or of quaternion type, $P \cong \mathbb{Z}_p \times \mathbb{Z}_p$ is the only non-cyclic abelian Sylow subgroup of G and H acts irreducibly on P.
- (ii) *If any subgroup of P is normal in G, then $G \cong N \rtimes H$, where N is Dedekindian Hall subgroup of G and any Sylow subgroup of H is cyclic or of quaternion type. We can assume that p is the smallest prime factor of $|G|$ such that G has a subgroup of type (p,p). Also any prime factor of $|H|$ is a divisor of $p-1$.*

In any case, each abelian subgroup of $\mathcal{C}_N(K)$ is cyclic, where $K \leqslant H$ is non-normal in G and N is the normal complement of H.

Proof. (i) By Remark 3.5, $\Omega_1(P)$ is of type (p,p). Without loss of generality, we can assume that P has a subgroup $\langle x \rangle$, which is non-normal in G. We can choose an element z of order p such that $z \notin \langle x \rangle$, then $L = \langle x, z \rangle \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_p \trianglelefteq G$, where $|x| = p^n$. Since G non-nilpotent, G contains a non-normal Sylow subgroup, so there exist an element g of order r coprime to p such that $g \notin \mathcal{N}_G(\langle x \rangle)$. If g fixed some element of $\Omega_1(L) = \Omega_1(P)$, then $L = [L,g] \times \mathcal{C}_L(g)$, where $\mathcal{C}_L(g) \neq 1$. Hence $\langle x \rangle = [L,g]$ and we have contradiction $g \in \mathcal{N}_G(\langle x \rangle)$. Now we assume that any subgroup of $\Omega_1(L)$ is g-invariant. Then $|x| > p$ and for distance subgroups $\langle z_1 \rangle$, $\langle z_2 \rangle$ and $\langle z_1 z_2 \rangle$, by assumption $z_1^g = z_1^j$, $z_2^g = z_2^j$ and $(z_1 z_2)^g = (z_1 z_2)^\ell$, where $1 < i, j, \ell \leq p-1$, we have

$$(z_1 z_2)^\ell = (z_1 z_2)^g = z_1^g z_2^g = z_1^i z_2^j.$$

Therefore $z_1^{\ell - i} = z_2^{j-\ell}$ hence $i = \ell = j$. Since $L/\langle z \rangle$ is g-invariant, we can assume that$x^g = x^t z$, where $(t,p) = 1$ so $(x^{p^{n-1}})^g = (x^{p^{n-1}})^t$, thus $t = kp + \ell$. By simple computation we see $x = x^{g^r} = x^{t^r} z^{r\ell^{r-1}}$, so $z^{r\ell^{r-1}}$. Therefore $p \mid r\ell^{r-1}$ a contradiction. Hence $\Omega_1(P)$ contains a subgroup which is maximal cyclic non-normal in G, since for any element $y \in P$ of order greater that p, there exist an element $z \in \Omega_1(P)$, such that $z \notin \langle y \rangle$, then $\langle y, z \rangle \trianglelefteq G$. Thus $\langle y^p \rangle \trianglelefteq G$. Therefore $P \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_p$.

According to Proposition 3.6 (iii), any Sylow r-subgroup of odd order is cyclic, where $r \neq p$ and Sylow 2-subgroup is cyclic or of quaternion type.

http://dx.doi.org/10.22108/ijgt.2023.135678.1814

Now assume that $Q \ntrianglelefteq G$. Then G has the structure $(i-1)$. If $Q \trianglelefteq G$, by Proposition 3.6 (v), G has structure $(i-2)$.

(ii) Suppose that any subgroup of P is normal in G and q is the smallest factor of $|G|$ such that G has a subgroup of type (q,q). Since P is non-cyclic, hence every subgroup of Sylow q-subgroup of G is normal in G, so we can assume that $q = p$. Now according to Proposition 3.6 (i) and (ii), R is abelian normal subgroup of G, for every $r > p$ and R is cyclic for every $2 < r < p$, by choice p, where $R \in \mathcal{S}y\ell_r(G)$. Thus $G \cong N \rtimes H$ where N is the product of all normal Sylow subgroups and H complement of N in G. Therefore any Sylow subgroup of H is cyclic or of quaternion type and any subgroup of N is normal in G, by Proposition 3.6 (i), (ii) and (v), so N is Dedekindian.

Assume that L is a subgroup of order (p,p) and S is a Sylow s-subgroup of H, for some prime factor s of $|H|$. As $S \ntrianglelefteq G$, thus S acts non-trivially on a proper subgroup of L. Therefore $s \mid p-1$.

Finally, assume that $K \leqslant H$ is non-normal in G. Then by Lemma 3.3, each abelian subgroup of $\mathcal{C}_N(K)$ is cyclic.

Again the converse is provided by Lemma 3.4. □

As an example of a group that satisfies Theorem 4.2 (i-1), it can be considered the group $(\mathbb{Z}_7 \times \mathbb{Z}_7) \rtimes \mathrm{SL}(2,3)$ that $\mathrm{SL}(2,3)$ acts irreducibly on $\mathbb{Z}_7 \times \mathbb{Z}_7$, (the SmallGroup$(1176, 215)$ of the GAP library [8]). Of course for some prime this type of group can be non-solvable. For example $(\mathbb{Z}_{11} \times \mathbb{Z}_{11}) \rtimes \mathrm{SL}(2,5)$ is a non-solvable $\mathcal{NAC}$-group, where $\mathrm{SL}(2,5)$ acts irreducibly on $\mathbb{Z}_{11} \times \mathbb{Z}_{11}$.

The group of type (ii) is solvable, because all subgroups of N are normal in G. If $L \cong L_1 \times L_2$ is of type (p,p), then $H/\mathcal{C}_H(L_i)$ is cyclic for $i = 1,2$. Hence $H' \leqslant \mathcal{C}_H(L_1) \cap \mathcal{C}_H(L_2)$ commute with L. Now either $Q \leqslant H'$, then Q is a direct factor of G and so H is cyclic, or $Q \cap H'$ is cyclic, so any Sylow subgroup of H is cyclic. Therefore in the any case G is solvable.

Lemma 4.3. *Let $G \cong N \rtimes H$ be $\mathcal{NAC}$-group such that N is a maximal nilpotent subgroup of G and any Sylow subgroup of H is cyclic. If any maximal subgroups of N are H-invariant, then H is cyclic.*

Proof. We show that $H' = 1$, so H is cyclic. Assume that $H' \neq 1$ then H' is a cyclic Hall subgroup of H with cyclic complement U.

Let R be a Sylow subgroup of H', then $R \trianglelefteq H$ and so $H \leqslant \mathcal{N}_G(R)$. If $\mathcal{N}_G(R)$ contains a subgroup P of type (p,p), then $R \trianglelefteq G$, for $[P,R] = 1$. Hence $[N,R] = 1$ and NR is nilpotent, a contradiction. Therefore all Sylow subgroups of $\mathcal{N}_G(R)$ are cyclic, so $\mathcal{N}_G(R)'$ is a Hall subgroup of $\mathcal{N}_G(R)$.

Let N_1 be a maximal subgroup of N which contains $\mathcal{N}_N(R)$. Since N is nilpotent, N_1 is of the prime index, so $M = N_1\mathcal{N}_G(R)$ is a maximal subgroup of G of the prime index, by assumption N is normal in G. Now according to Lemma 2.2, R has a normal complement N_R. Assume that S is an arbitrary Sylow subgroup of U, therefore $S \leqslant N_R$. By using Frattini's argument we will have $G = N_R\mathcal{N}_G(S)$. Hence $R \leqslant \mathcal{N}_G(S)$, therefore $[R,S] = 1$, so $[U,R] = 1$. If T is a complement of R in H', then $H/T \cong RU$ where we will have $H' \leqslant T$, a contradiction. Thus H is abelian and so is cyclic. □

Proposition 4.4. *Let G be a non-nilpotent $\mathcal{NAC}$-group such that all Sylow subgroups of G are abelian. Then G has one of the following structures.*

(i) *G is non-abelian meta-cyclic group such that G' is cyclic Hall-subgroup.*
(ii) *$G \cong ((\mathbb{Z}_2 \times \mathbb{Z}_2) \times C) \rtimes \mathbb{Z}_{3^m}$ where C is a cyclic $\{2,3\}'$-Hall subgroup.*
(iii) *$G \cong ((\mathbb{Z}_p \times \mathbb{Z}_p) \times C) \rtimes H$ where p is odd, C and H are cyclic Hall subgroups and H acts irreducibly on $\mathbb{Z}_p \times \mathbb{Z}_p$.*
(iv) *$G \cong (P \times C) \rtimes H$ where P is non-cyclic abelian Sylow p-subgroup of odd order, C is abelian Hall subgroup and H is cyclic Hall subgroup. Also, all subgroups of P and C are H-invariant.*

Proof. If all Sylow subgroups of G are cyclic, then G is an $\mathcal{NNC}$-group which is necessarily an $\mathcal{NAC}$-group and has structure (i).

Assume G has a subgroup of type (p,p). If $p=2$, then according to Theorem 4.1, G will have structure (ii). Let p be odd. According to Theorem 4.2, G will have the structure (iii) or (iv) corresponding to whether the Sylow p-subgroup of G has a non-normal or normal subgroup in G. By Lemma 4.3, H in (iv) is cyclic. □

5. $\mathcal{NAC}$-groups without non-cyclic abelian Sylow subgroup

In the previous section, we see that if $\mathcal{NAC}$-group contains a subgroup of type (p,p), then for any $2 < q \neq p$, Sylow q-subgroup will be abelian. Therefore, if an $\mathcal{NAC}$-group contains one non-abelian Sylow subgroup of odd order, then other Sylow subgroups are cyclic or quaternion (ordinary or generalized). Hence G can only contain one non-abelian odd-order Sylow subgroup.

In this section, we characterized the non-nilpotent $\mathcal{NAC}$-group G with a non-abelian Sylow subgroup. By Theorems 4.1 and 4.2, in the following theorems, we can assume that G does not contain a non-cyclic abelian Sylow subgroup. First, we consider that a non-abelian Sylow subgroup is of odd order next, of even order such that odd-order Sylow subgroups are cyclic.

Theorem 5.1. *Assume that group G contains a non-abelian non-normal Sylow subgroup of odd order, P say, and $Q \in \mathcal{S}y\ell_2(G)$. Then G is $\mathcal{NAC}$-group if and only if $G \cong Q \times C \rtimes P$, where C is the normal cyclic $\{2,p\}'$-Hall subgroup of G, Q is either cyclic or $Q \cong Q_8$ and P is one of the following groups.*

(i) *$M_p(m,1) \cong \mathbb{Z}_{p^m} \rtimes \mathbb{Z}_p$, where $m \geq 2$.*
(ii) *$P_{81} = \langle a,b,c \,|\, a^9 = c^3 = 1, a^3 = b^3, [a,b] = c, [c,a] = a^3, [c,b] = 1\rangle$.*

Furthermore $\mathcal{C}_P(C) = T$ where $T = \langle a^p, b\rangle$ if $P \cong M_p(m,1)$ and $T = \langle b,c\rangle$ if $P \cong P_{81}$.

Proof. Since every non-normal abelian subgroup of P is cyclic, so P is one of the groups listed in Theorem 2.1. Assume that $P \cong M_p(1,1,1) * \mathbb{Z}_{p^n}$. Since $\langle a,c\rangle$ and $\langle b,c\rangle$ are non-cyclic abelian subgroups of $M_p(1,1,1)$ and commute with $\langle z\rangle$ the generator of $Z(P)$, thus $K = \langle a,z\rangle$ and $L = \langle b,z\rangle$ are non-cyclic abelian maximal subgroups in P, by assumption are normal in G. So P is normal in G a contradiction.

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Let $P \cong M_p(m,n)$. By Lemma 3.1, $Z(P) = \langle a^{p^{m-1}}, b^{p^{n-1}} \rangle$ is cyclic then $n = 1$, for $m \geq 2$. Thus $P \cong M_p(m,1)$.

We set $T = \langle a^p, b \rangle$ in group (i) and $T = \langle b, c \rangle$ in group (ii), then T is abelian maximal normal subgroup of P and $P = \langle a \rangle T$. Now we set $(u,v) = (a^{p^{m-1}}, b)$ in group (i) and $(u,v) = (a^3, c)$ in the group (ii) hence $Z(P) = \langle u \rangle$ and $\langle v \rangle \ntrianglelefteq P$. We know $\langle u \rangle$ is the only subgroup of $\langle u, v \rangle$ which is not maximal cyclic, so $\langle u \rangle \trianglelefteq G$ and for any $x \in Q$, $\langle u \rangle^x = \langle u \rangle$.

According to Proposition 3.6 (iii), (v), all Sylow subgroups of odd order distinct from P are cyclic and Sylow 2-subgroup of G is cyclic or of quaternion type. By [2], G is not simple, so G contains at least a normal cyclic Sylow subgroup of odd order. Assume that C is the cyclic normal Hall subgroup of G of odd order, (in group (ii), C is the $\{2,3\}'$-Hall subgroup of G, by Proposition 3.6 (i)). So $[C,T] = 1$ and $T = \mathcal{C}_P(C)$, for T is maximal subgroup of P.

For any element $y \in C$, $\langle y \rangle \trianglelefteq G$. So for any element $x \in Q$, $y^x \in \langle y \rangle$. Hence a and x as elements of $\mathrm{Aut}(\langle y \rangle)$ are commutes, so

$$y^{ax} = y^{xa} \Rightarrow y^{[a,x]} = y,$$

now we get $[a,x] \in T$.

Assume that P is group (i) and x is a p'-element of $\mathcal{N}_G(P)$ such that $[x, \langle u, v_1 \rangle] \neq 1$. The subgroups $\langle ab^i \rangle$ for $0 \leq i \leq p-1$ are cyclic so for some i there exist a subgroup $\langle ab^i \rangle$ which is x-invariant. Without loss of generality, we can assume that $\langle a \rangle$ is x-invariant, since for any i, $P = \langle ab^i, v_1 \rangle$ and $(ab^i)^{p^{m-1}} = u$. If $[a,x] = 1$, then $[x,u] = 1$ thus $v_1^x = v_1^i$ for some $i \not\equiv 1$ modulo p. Now

$$u = u^x = [a, v_1]^x = [a, v_1^i] = u^i,$$

because $[a,b] = [a, v_1]$. So we get contradiction $p \mid i-1$. Hence $[a,x] \neq 1$, as $(|a|,|x|) = 1$ we get contradiction $\langle a \rangle = \langle [a,x] \rangle \leqslant T$. Therefore $\langle x \rangle \trianglelefteq G$, for any p'-element x, in particular $Q \trianglelefteq G$ and C is $\{2,p\}'$-Hall subgroup of G.

In group (ii), assume that $Q \ntrianglelefteq G$, as Sylow 2-subgroup of $\mathrm{Aut}(\langle b, c \rangle)$ is of type $(2,2)$, therefore the action x on $\langle b, c \rangle$ is of order 2 and $x^2 \in \mathcal{C}_Q(\langle b, c \rangle)$. On the other hand there is a $x \in Q$ such that x acts non-trivially on c or b^3, because $Q \ntrianglelefteq G$.

If $(b^3)^x = b^3$, by assumption $c^x = c^i b^{3j}$ and according to $[c,a] = a^3$ $^{(\star)}$, we get

$$a^3 = (a^3)^x = [c,a]^x = [c^i b^{3j}, az] = [c,a]^i = a^{3i},$$

where $z = [a,x] \in T$. So we can assume that $i = 1$, therefore $3 \nmid j$. Now,

$$c = c^{x^2} = (cb^{3j})^x = c^x b^{3j} = cb^{6j}.$$

Hence $b^{6j} = 1$, which is a contradiction. Thus $(b^3)^x = b^{-3}$. By reusing the equality $(\star)$, we get

$$a^{-3} = [c^x, a^x] = [c^i b^{3j}, az] = [c,a]^i = a^{3i}.$$

http://dx.doi.org/10.22108/ijgt.2023.135678.1814

So we can assume that $i=-1$. Again using the equality $(\star)$, we get

$$c = c^{x^2} = (c^{-1}b^{3j})^x = (c^x)^{-1}b^{-3j} = cb^{-6j}.$$

Therefore $j=0$ and $c^x = c^{-1}$.

Now assume that $b^x = b^i c^j$, so $b^{-3} = (b^3)^x = b^{3i}$, therefore $i=-1$. By $[a,b]=c$ we get,

$$c^{-1} = c^x = [a^x, b^x] = [az, b^{-1}c^j] = [a, b^{-1}][a,c]^j = c^{-1}a^{3j}.$$

As a result, $j=0$ and $b^x = b^{-1}$. Assume that $z = b^r c^s$. Since

$$a^{-3} = (a^3)^x = (a^x)^3 = (az)^3 = z^3a^3 = b^{3r}a^3,$$

thus $r=1$ and $z = bc^s$. So x acts on the generators of P as follows:

$$a^x = ac^sb, \quad b^x = b^{-1}, \quad c^x = c^{-1}.$$

Also,

$$(ba)^3 = b^3a^3[a,b]^3[a,[a,b]] = b^3a^3c^3[a,c] = b^3a^3a^{-3} = b^3.$$

Hence $(ab)^3 = b^3$ and will have,

$$b^{-3} = (b^3)^x = (a^xb^x)^3 = (ac^sbb^{-1})^3 = (ac^s)^3 = a^3c^{3s}[c^3,a]^3 = a^3 = b^3.$$

Consequently we get the contradiction $b^6 = 1$. Therefore Sylow 2-subgroup is normal in G. By Proposition 3.6 (v), Q is direct factor of G also Q is cyclic or $Q \cong Q_8$.

Converse: Assume that $S = LK$ is abelian non-normal subgroup of G, where $L \leqslant Q \times C$ and $K \leqslant P$. Thus L is cyclic normal subgroup of G and $K \ntrianglelefteq G$. If $K \trianglelefteq P$, then $S \trianglelefteq G$, hence $K \ntrianglelefteq P$ and so K is cyclic, for P is $\mathcal{NAC}$-group. Therefore S is cyclic. □

Assume that $G \cong Q_8 \times \mathrm{SmallGroup}(105, 37)$, where $\mathrm{SmallGroup}(105, 37) \cong \mathbb{Z}_{13} \rtimes \mathrm{P}_{81}$. Then G satisfy of the above theorem.

By characterization of Theorem 4.2, in the following, we can assume that G does not contain any non-cyclic abelian Sylow subgroup.

Theorem 5.2. *Assume that the group G contains a non-abelian normal Sylow subgroup of odd order, P say, and $Q \in \mathcal{S}y\ell_2(G)$. Then G is $\mathcal{NAC}$-group if and only if G is one of the following groups.*

(i) *$G \cong Q \times (P \times C) \rtimes H$, where H is cyclic Hall subgroup and Q is cyclic or $Q \cong Q_8$.*

(ii) *$G \cong (P \times C) \rtimes H$, where any Sylow subgroup of H is either cyclic or of quaternion type.*

In both cases C is the cyclic normal Hall subgroup of G and P is one of the groups listed in Theorem 2.1. Also all non-cyclic abelian subgroups of P are H-invariant and $|H/\mathrm{core}_G(H)| \mid (p-1)$.

Furthermore, let $L \leqslant P$ be of type (p,p), then $\mathcal{C}_Q(L) = \mathrm{core}_G(Q) \trianglelefteq G$ is Dedekindian and $\mathcal{C}_H(L) = \mathrm{core}_G(H) \trianglelefteq G$ is cyclic. Also each abelian subgroup of $\mathcal{C}_N(K)$ is cyclic, where $K \leqslant H$ is non-normal in G and N is normal complement of H.

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Proof. By Proposition 3.6 (iii), for any odd prime number $r \neq p$, Sylow r-subgroup is cyclic and by Proposition 3.6 (v), Q is cyclic or of quaternion type. Assume that N is the product of all normal Sylow subgroups of G and H is complement N in G. Hence $N = PC$ is nilpotent where C is the cyclic normal Hall subgroup of G. Also any Sylow subgroup of H is cyclic or of quaternion type. Therefore, G is group (i) or (ii), respect to Q is normal or not. By Proposition 3.6 (v) in the case (i), Q is cyclic or $Q \cong Q_8$.

As P is non-abelian $\mathcal{NAC}$-group, P is group listed in Theorem 2.1. Let A be maximal abelian subgroup of P. Then A is characteristic subgroup of P, so $A \trianglelefteq G$. If A is not of type (p,p), then by Theorem 4.2, whose subgroups are H-invariant, for AH is $\mathcal{NAC}$-group. Therefore any non-cyclic abelian subgroup of P is H-invariant.

If A is not of type (p,p), then contains a H-invariant subgroup of order p. Otherwise $P \cong M_p(2,1)$ or $M_p(1,1,1)$, in either case, $Z(P) \leqslant A$ is H-invariant of order p. So in any cases P has a H-invariant subgroup of order p. Therefore $H/\text{core}_G(H)$ is embedded to $\mathbb{Z}_{p-1}$.

In case (i), P has a subgroup L of type (p,p) that H does not acts irreducibly on L. As any Sylow subgroup of H is non-normal in G, L is maximal nilpotent subgroup of LH. By Lemma 4.3, H is cyclic, for LH is $\mathcal{NAC}$-group.

Now assume that $L \leqslant P$ is of type (p,p) then $\mathcal{C}_Q(L) = \text{core}_G(Q)$, since for any $x \in \mathcal{C}_Q(L)$, $\langle x \rangle \trianglelefteq G$. Therefore $\text{core}_G(Q)$ is Dedekindian. Similarly $\mathcal{C}_H(L) = \text{core}_G(H) \trianglelefteq G$ is cyclic.

Finally, any abelian subgroup of $\mathcal{C}_N(K)$ is cyclic by Lemma 3.3.

The converse is provided by Lemma 3.4. □

It is now natural to ask whether or not all the p-groups presented in Theorem 2.1, can be occurred in Theorem 5.2? In the following, we give some examples of groups such that their Sylow p-subgroups are present in Theorem 2.1

(i) $G = \text{SmallGroup}(648, 123) \cong \mathrm{M}_3(3,1) \rtimes \mathrm{Q}_8$.
(ii) $G = \text{SmallGroup}(648, 485) \cong (\mathrm{M}_3(1,1,1) * \mathbb{Z}_9) \rtimes \mathrm{Q}_8$.
(iii) $G = \text{SmallGroup}(648, 188) \cong \mathrm{P}_{81} \rtimes \mathrm{Q}_8$.

Since for any p-group P presented in Theorem 2.1, $\text{Aut}(P)$ is solvable, then $H/\mathcal{C}_H(P)$ is solvable. As $\mathcal{C}_H(P)$ is Dedekindian, because commute with a subgroup of type (p,p), then H is solvable and so any $\mathcal{NAC}$-group characterized in Theorem 2.1, is solvable. In the following remark we described solvability of $\text{Aut}(P)$.

Remark 5.3. *If $P \cong P_{81}$, then P is a 3-group of maximal class, so by* [1, Lemma 1.1], *Aut(P) is solvable. Now assume that P is one of the group $M_p(m,n)$ or $M_p(1,1,1) \star C_{p^n}$ and set the central subgroup $N = \langle b^p \rangle$ or $N = \langle c^p \rangle$ respectively. Then P/N is a p-group of maximal class, for $P/N \cong M_p(m,1)$ or $M_P(1,1,1)$. Hence* $\text{Aut}(P/N)$ *is solvable (*[1, Lemma 1.1]*). Now consider the natural homomorphism $\rho : \text{Aut}(P) \to \text{Aut}(P/N)$. The kernel of ρ is a subgroup of central automorphisms of*

P. By [4, Proposition 4.3], as P is purely non-abelian in any case, the central automorphisms of P is nilpotent. Consequently $\mathrm{Aut}(P)$ *is solvable.*

Now, we can assume that any odd-order Sylow subgroup of G is cyclic.

Theorem 5.4. *Let G be a non-nilpotent group such that whose odd-order Sylow subgroups are cyclic. Assume that Q is a non-abelian non-normal Sylow 2-subgroup of G. Then G is $\mathcal{NAC}$-group if and only if G is isomorphic to one of the following groups.*

- (i) *$G \cong N \rtimes Q$, where N is cyclic of odd order and Q is one of the following group. Furthermore $|Q/\mathcal{C}_Q(N)| = 2$.*
  - (i-1) $\langle a, b, c \,|\, a^8, b^2a^4, c^2, [a, b]c, [c, a]a^4, [c, b]\rangle$
  - (i-2) *$M_{2^{\ell+2}}$ the modular 2-group of order $2^{\ell+2}$, where $\ell \geq 2$.*
  - (i-3) *$\langle a, c \,|\, a^{2^\ell}, a^{2^{\ell-1}}c^4, [a, c]a^2\rangle$, where $\ell \geq 2$.*
- (ii) *$G \cong N \rtimes Q_{2^n}$, where N is $\mathcal{NAC}$-group with cyclic Sylow subgroups of odd order.*
- (iii) *$G \cong N \rtimes (QR)$, where N is $\{2, 3\}'$-Hall $\mathcal{NAC}$-group with cyclic Sylow subgroups, $Q \cong Q_8$ or Q_{16} and $R \cong \mathbb{Z}_{3^n}$ for some n. If $Q \cong Q_8$ then $QR \cong (Q_8 \rtimes R)$ otherwise QR contains a subgroup K of index 2, such that $K \cong Q_8 \rtimes R$.*
- (iv) *$G \cong N \rtimes H$, where N is odd order cyclic subgroup and H contains a subgroup H_1 such that $|H : H_1| \leq 2$ and $H_1 \cong Z \times \mathrm{SL}(2, q)$ for some prime number q and all Sylow subgroups of Z are cyclic.*

Proof. We consider two general cases in which Q contains a subgroup of type $(2, 2)$ or not.

Case 1: Assume that Q contains a subgroup $L = \langle x, y\rangle$ of type $(2, 2)$.
By Proposition 3.7 (iv), G is 2-nilpotent with cyclic 2-complement. So $G = NQ$ where N is cyclic normal Hall subgroup of G. By Proposition 3.7 (iv) and (vi), $|Q| \geq 16$ and $\mathcal{C}_Q(L)$ is maximal in Q. If $\mathcal{C}_Q(L)$ is non-abelian then $\mathcal{C}_Q(L) \cong Q_8 \times \mathbb{Z}_2$, and $|Q| = 32$. The 2-groups with $\mathcal{NAC}$ property are listed in [9, Theorem 3.5]. And just the group (9) of list is 2-group with above property, therefore Q is group (i-1).

Let $\mathcal{C}_Q(L) = \langle a, y\rangle$ be an abelian group of type $(2^\ell, 2)$. Then $(\mathcal{C}_Q(L))^p$ is cyclic normal subgroup of Q, therefore $a^{2^{\ell-1}} \in Z(Q)$. Now we cam assume that $x = a^{2^{\ell-1}}$ and there exist an element $c \notin \mathcal{C}_Q(L)$ such that $1 \neq c^2 \in \mathcal{C}_Q(L)$, since $L = \Omega_1(Q)$. If $|c| = 2^{\ell+1}$, then $c^y = ca^{2^{\ell-1}} = c^{1+2^\ell}$, for $[c, y] = x$. Therefore $Q = \langle c, y\rangle \cong M_{2^{\ell+2}}$ and Q is group (i-2).

Assume that $|c| \leq 2^\ell$. If $y \notin \Phi(Q)$, then $\langle y\rangle$ has an complement H in Q. Since $|\Omega_1(H)| = 2$, so $H \cong Q_{2^{\ell+1}}$. As $[c^2, a] = 1$, then $c^2 = x$ and so $\langle c, y\rangle \cong D_8$. Now we reach to contradiction $D_8 = \Omega_1(D_8) \leqslant \Omega_1(Q) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence $y \in \Phi(Q)$ and $Q = \langle c, a\rangle$. So Q is a group of two generators, by checking the list of $\mathcal{NAC}$-groups given in the [9, Theorem 3.5] with the 2-groups of order at most 2^6 and [9, Theorem 3.6] with 2-groups of order greater than 2^6, immediately we see that Q has the presentation (i-3) of Theorem.

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In the either case, since $|Z(\mathcal{C}_Q(L))| > 2$, by Lemma 3.1, $\mathcal{C}_Q(L) \trianglelefteq G$. Therefore $[N, \mathcal{C}_Q(L)] = 1$ and the action of Q on N is of order 2.

Now in the following cases, we consider Q is of quaternion type.

Case 2: G is 2-nilpotent group with quaternion type Sylow 2-subgroup.

In this case, $G \cong N \rtimes Q$, where N is $\mathcal{NAC}$-group of odd order.

Case 3: G is solvable group with quaternion type Sylow 2-subgroup but is not 2-nilpotent.

By Proposition 3.7 (iv) and Theorem 2.3, $3 \mid |G|$ and $Q \cong Q_8$ or Q_{16}.

By the solvability, G has a Sylow system $\{P_1, \ldots, P_\ell, Q\}$ for some $\ell \geq 1$. Let Q_1 is maximal cyclic subgroup of Q of order 4, then $M = P_1 \cdots P_\ell Q_1$ is a maximal subgroup of index 4. Therefore $G/C \cong L/C \rtimes M/C$, where $C = \mathrm{core}_G(M)$. As L/C is of type $(2,2)$, $2 \mid |C|$. Since $C \cap Q \trianglelefteq Q$ and $C \cap Q \leqslant Q_1$, we get $|C \cap Q| = 2$. Assume that N is $\{2,3\}'$-Hall subgroup of G and $R \in \mathcal{S}y\ell_3(G)$. Then $C = C_1 \rtimes (C \cap Q)$. As 3 is the smallest factors of $|C_1|$ and $R \cap C$ is cyclic, so $C_1 \cong H \rtimes (R \cap Q)$. Thus $H \trianglelefteq G$, for $C_1 \trianglelefteq G$. Therefore $G \cong H \rtimes (RQ)$. Since G is $\mathcal{NAC}$-group, for any subgroup K of QR, if $K \ntrianglelefteq G$, then $\mathcal{C}_H(K)$ is cyclic.

If $Q \cong Q_8$, then $L = CQ \trianglelefteq G$ and $M/C \cong \mathbb{Z}_3$. then $G \cong H \rtimes (Q \rtimes R)$, for $R \leqslant \mathcal{N}_G(Q)$.

Now Assume that $Q \cong Q_{16}$. Then $M/C \cong S_3$ and $|G : LR| = 2$. Hence $LR \cap QR = (L \cap Q)R \cong Q_8 \rtimes R$. Therefore $K = (L \cap Q)R$ is a subgroup of index 2 of QR.

Case 4: G is non-solvable group with quaternion type Sylow 2-subgroup.

By [2, Theorem 1], G is not simple. Assume that N is products of all normal Sylow subgroups of G with complement H. By assumption $|N|$ is odd and any odd order Sylow subgroups of H is cyclic. By [7, Theorem C], H contains a normal subgroup H_1 such that $|H : H_1| \leq 2$ and $H_1 \cong Z \times \mathrm{SL}(2,q)$ for some prime q.

Converse: Assume that S is an abelian non-normal subgroup of G. So S is the direct product of its Sylow subgroups. As odd-order Sylow subgroups of G are cyclic, and any abelian subgroup of the quaternion group is cyclic too, thus S is cyclic in the cases (ii)-(iv).

Let G be group (i) and $S = LK \trianglelefteq G$ be abelian. Since $L \trianglelefteq N$ is cyclic, we assume that $K \leqslant Q$ is non-cyclic. Since Q is $\mathcal{NAC}$-group, $K \trianglelefteq Q$, thus there exists a maximal abelian subgroup A of Q such that $K \leqslant A$. Therefore $K \trianglelefteq G$, and so $S \trianglelefteq G$ is a contradiction. □

As examples of groups in the above theorem, we can consider the following groups of the GAP library.

- (i-1) $G = \mathrm{SmallGroup}(800, 20) \cong \mathbb{Z}_{25} \rtimes \mathrm{Q}$, where Q is 2-group of type (i-1).
- (i-2) $G = \mathrm{SmallGroup}(800, 19) \cong \mathbb{Z}_{25} \rtimes \mathrm{M}_3(4,1)$.
- (i-3) $G = \mathrm{SmallGroup}(800, 42) \cong \mathbb{Z}_{25} \rtimes \mathrm{Q}$, where Q is 2-group of type (i-3) with $\ell = 3$.
- (ii) $G = \mathrm{SmallGroup}(1176, 7) \cong (\mathbb{Z}_{49} \rtimes \mathbb{Z}_3) \rtimes \mathrm{Q}_8$.
- (iii) $G = \mathrm{SmallGroup}(1296, 57) \cong \mathbb{Z}_{81} \cdot \mathrm{Q}_{16}$, where $K = G' \cong Q_8 \rtimes \mathbb{Z}_{81}$ and $N = 1$.
- (iv) $G = \mathrm{SmallGroup}(672, 1045)$, that $G' \cong \mathrm{SL}(2,7)$, $|G : G'| = 2$ and $N = Z = 1$.

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We can easily compute the presentation (i-3) of the above theorem. In order not to complicate the proof, we give the tedious calculations in the following remark.

Remark 5.5. We note $y \in \Phi(Q)$ and $Q = \langle a, c\rangle$, thus $c^2 \notin \langle a\rangle$. We can assume that $c^2 = a^{2^i}y$, where $i \neq 0$. Since $c^2 \in Z(Q)$ and $y \notin Z(Q)$, so $a^{2^i} \notin Z(Q)$ thus $i < \ell - 1$, also $[a, c]^{2^i} = [c, y] = x$, for $[a^{2^i}y, c] = 1$. Hence we can write $[a, c] = a^{r2^{\ell-i-1}}y^k$, where $2 \nmid r$ and $k \in \{0, 1\}$.

First we note that if $[a, c] = a^{2r}xy^k$, for some r and k. Then by replacing a with $a_1 = ay$, $c^2 = a_1^2 y$ and $[a_1, c] = [a, c]x = a^{2r}y^k = a_1^{2r}y^k$, so we can remove x.

If $|a| = 8$ then $i = 1$ and so $c^2 = a^2y$, $[a, c]^2 = x$, thus $[a, c]$ is of order 4. Therefore $[a, c] = a^{2r}y^k$, where $k \in \{0, 1\}$ and $r = 1$ or 3. If $r = 1$ then $[a, c] = a^2y^k = a^2xy^k = a^6y^k$. Hence we can assume that $[a, c] = a^{-2}y^k$. Now suppose that $t = aca^2$, then $t^2 = yy^k$. If $k = 1$, then $t^2 = 1$ and $t \in \langle a, c\rangle$ so we get contradiction $c \in \langle a, c\rangle$. Therefore $k = 0$ and $[a, c] = a^{-2}$.

Now assume that $\ell \geq 4$. If $i = 1$, then $[a, c] = a^{r2^{\ell-2}}y^k$, where $2 \nmid r$ and $k \in \{0, 1\}$. We set $t = c^{s2^{\ell-3}}ac^{-1}y$, where $s = 2 - r$, then $t^2 = a^{s2^{\ell-2}}(ac^{-1})^2x$, (for $c^4 = a^4$), also

$$(ac^{-1})^2 = [a, c]y = a^{r2^{\ell-2}}y^{k+1},$$

hence

$$t^2 = a^{s2^{\ell-2}}a^{r2^{\ell-2}}y^{k+1}x = a^{2^{\ell-1}}y^{k+1}x = y^{k+1}.$$

By Proposition 3.7 (v), y is maximal cyclic subgroup, so $k = 1$ and $t^2 = 1$, then $t \in \langle a, y\rangle$. So we get contradiction $c \in \langle a, y\rangle$. Therefore $i \geq 2$.

Again we note that $[a, c] = a^jy^k$, where $j = r2^{\ell-i-1}$ and $k \in \{0, 1\}$, then $a^c = a^{j+1}y^k$, so

$$a = a^{c^2} = a^{(j+1)^2}y^ky^kx^k = a^{(j+1)^2}x^k.$$

Therefore $a^{j(j+2)} = x^k$ and so $2^{\ell-1+k} \mid j(j + 2)$ thus $2^{i+1-k} \mid j + 2$. If $\ell - i - 1 > 1$, then $4 \mid j$ hence $2^{i+1-k} = 2$, for $(j, j + 2) = 2$, thus $i = k = 1$, a contradiction. So $i = \ell - 2$, hence $c^2 = a^{2^{\ell-2}}y$ and $[a, c] = a^{2r}y^k$. Also $2^{\ell-1-k} \mid j + 2 = 2(r + 1)$, so $2r = u2^{\ell-1-k} - 2$, where $u = 1$ or 3, for $|a^{2r}| \leq 4$. If $u = 3$, then $[a, c] = a^{-2}a^{2^{\ell-1-k}}xy^k$. Again we can assume that $u = 1$ and $[a, c] = a^{-2}a^{2^{\ell-1-k}}y^k$.

Suppose that $t = aca^2$, then $t^2 = yy^ka^{2^{\ell-2}}a^{2^{\ell-1-k}}$. If $k = 1$, then $t^2 = 1$, hence $t \in \langle a, y\rangle$, so we get contradiction $c \in \langle a, y\rangle$. Therefore $k = 0$ and $[a, c] = a^{-2}x$. Again we can remove x, so $[a, c] = a^{-2}$.

Hence in the any case $[a, c] = a^{-2}$ and $c^4 = a^{2^{\ell-1}}$, so Q has the presentation (i-3).

Finally, we assume that G does not contain any non-cyclic abelian Sylow subgroup and all non-abelian Sylow subgroups of G are normal.

Theorem 5.6. *Let G be a non-nilpotent group such that its odd-order Sylow subgroups are cyclic and its Sylow 2-subgroup is non-abelian and normal. Then G is $\mathcal{NAC}$-group if and only if G is one of the following groups.*

(i) $Q_{2^n} \times H$ *if $n \geq 4$, where H is non-abelian with cyclic Sylow subgroups.*

(ii) $(Q_8 \times C) \rtimes H$, *where C is odd order cyclic Hall subgroup and Sylow subgroups of H are cyclic.*

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Proof. If G contains a subgroup of type $(2,2)$, then by Proposition 3.7 (iv), G is 2-nilpotent with cyclic 2-compliment, so G is nilpotent, a contradiction. Thus Q is of quaternion type. If $|Q| \geq 16$, as $\mathrm{Aut}(Q)$ is 2-group, again G is 2-nilpotent with non-abelian complement, so G is group (i).

Now assume that $Q \cong Q_8$ and N is the product of normal Sylow subgroup with complement H in G. Then $G = NH$ and any Sylow subgroup of H is cyclic, where $N \cong Q_8 \times C$.

Converse: Since any abelian subgroups of H and its normal complements are cyclic. So any abelian subgroups (in particular non-normal abelian subgroups) of G is cyclic. □

Finally, the SmallGroup$(1176, 23) \cong \mathrm{Q}_8 \rtimes (\mathbb{Z}_{49} \rtimes \mathbb{Z}_3)$ is an example of groups satisfy in the item (ii) of the above theorem.

If Sylow 2-subgroup of a group G is cyclic, then G is 2-nilpotent. According to Proposition 3.7 (iv), if an $\mathcal{NAC}$-group contains a subgroup of type $(2,2)$, again is 2-nilpotent. Consequently a non-solvable $\mathcal{NAC}$-group it can only have one of the structures presented in Theorem 4.2 (i-1) and Theorem 5.4 (iv).

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