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AVERAGE ORDER IN REGULAR WREATH PRODUCTS

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Abstract. We obtain an exact formula for the average order of elements of regular wreath product of two finite groups. Then focussing our attention on p-groups for primes p, we give an estimate for the average order of a wreath product $A \wr B$ in terms of maximum order of elements of A and average order of B and an exact formula for the distribution of orders of elements of $A \wr B$. Finally, we show how wreath products can be used to find several rational numbers which are limits of average orders of a sequence of p-groups with cardinalities going to infinity.

1. Introduction

For a finite group G, define the average order of elements to be $a(G) = \frac{\sum_{g \in G} order(g)}{|G|}$ and denote the maximum order of an element of G by $m(G)$. It is well known that among all groups of a fixed cardinality, the cyclic group has the largest average order (see [2]). In [1] the authors study minimum value of average order among finite groups of same size. Other authors have studied the average order function with a view to deriving characterization theorems for nilpotency and solvability (see [4, 5]).

Wreath products of groups often provide examples and counter-examples for various group theoretic questions. The famous *Krasner-Kaloujnine embedding theorem* shows that for any two groups A and H, any extension of A by H is isomorphic to a subgroup of the wreath product $A \wr H$ (see [3]).

In this article, we obtain an exact formula for the average order of a wreath product of two finite groups. Following that, we focus our attention on p-groups for primes p. We give an estimate for the

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average order of a wreath product $A \wr B$ in terms of $m(A)$ and $a(B)$. In addition, we obtain an exact formula for the distribution of orders of elements of $A \wr B$, in terms of distributions of orders of elements of A and B. Finally, we show how wreath products can be used to find several rational numbers which are limits of average orders of a sequence of p-groups with cardinalities going to infinity.

2. Notations and Conventions

(1) Given two groups A and B, let $K = \prod_{b \in B} A$. The group B naturally acts on K by $x \cdot (\alpha_b)_b = (\alpha_{x^{-1}b})_b$, for $x \in B$, $(\alpha_b)_b \in K$. The semidirect product $K \rtimes B$ is called the regular wreath product of A by B and is denoted by $A \wr B$.

(2) $\mu : \mathbf{N} \to \mathbf{Z}$ denotes the Mobius function. Recall that if n is square-free, $\mu(n) = (-1)^m$, where m is the number of distinct prime divisors of n. If n is not square-free, $\mu(n) = 0$.

(3) For a natural number n, define $\tau(n) = \prod_{p|n, p\ prime}(1-p)$. It is easy to see that $\tau(n) = \sum_{d|n} d\mu(d)$.

(4) For real valued functions f and g we adopt the notation $g = O(f)$ to mean that there is a constant $M > 0$ such that $|g| \leq M|f|$ eventually.

(5) Define $\psi(A, B) = \frac{a(A \wr B)}{m(A) \cdot a(B)}$, for p-groups A and B.

(6) Let A be a p-group of cardinality p^a and $m(A) = p^d$. For $k \in \mathbf{Z}$ define $r_{A,k} = \frac{1}{p^a} \times$ Number of elements of order at most p^{d-k}.

This essentially denotes the cumulative distribution function of the order distribution. So, $r_{A,k}$ is a non-increasing function of k, and $r_{A,k} = 1$ for $k \leq 0$, $r_{A,k} = 0$ for $k > d$.

(7) Fix a prime p. Let us call a real number β an *"Average Order Limit"* if there is a sequence of p-groups G_n with $|G_n| \to \infty$ and $a(G_n) \to \beta$.

(8) For a positive integer k we denote the group $\mathbf{Z}/k\mathbf{Z}$ by C_k.

3. Main Results

Theorem 3.1. *Let A, B be finite groups with at least 2 elements. For $m \geq 1$, let s_m be the number of elements of A whose m-th power is 1. For $n|b$, let d_n be the number of elements of B of order n. Then*

$$a(A \wr B) = \sum_{m \Big| |A|, n \Big| |B|} \frac{m}{n} \left(\frac{s_m}{|A|}\right)^n d_{\frac{|B|}{n}} \tau\left(\frac{|A|}{m}\right).$$

Let p be a prime. From now on, assume A, B are p-groups, $|A| = p^a$, $|B| = p^b$, a, $b \geq 1$.

Theorem 3.2. *With notations as in theorem 3.1,*

$$a(A \wr B) = p^d a(B) - (p-1)a(B) \sum_{n \leq b} k_n \cdot \Big[\sum_{m \leq d-1} p^m \left(\frac{s_{p^m}}{p^a}\right)^{p^n}\Big],$$

where $k_n = \frac{p^{-n} d_{p^{b-n}}}{a(B)}$. Note that $k_n \geq 0$ $\forall n$, $\sum_{n \leq b} k_n = 1$.

Theorem 3.3. $a(B) \leq a(A \wr B) \leq p^d a(B)$.

Note that Theorem 3.3 implies that $0 \le \psi(A,B) \le 1$.

Theorem 3.4. *With notations as in theorem 3.1,*

$$a(A \wr B) = p^d a(B) - (p-1)a(B).\sum_{n\le b} k_n.[p^{d-1}(\frac{s_{p^{d-1}}}{p^a})^{p^n}] + O(p^{d-1}a(B)).$$

Here, the implicit constant in $O(p^{d-1}a(B))$ is independent of p as well as of the groups. Now, we explicitly write the order distribution of $A \wr B$ in terms of the order distributions of A and B.

Theorem 3.5. *Let $m(A)$=p^d, $m(B)$=p^e. Then we have $m(A \wr B) = p^{d+e}$, and $r_{A\wr B,k} = \sum_{i=0}^{e}(r_{B,i} - r_{B,i+1})r_{A,k-i}^{p^{p^{b-e+i}}}$.*

Corollary 3.6. $r_{A\wr C_p,k} = (1-p^{-1})r_{A,k} + \frac{r^p_{A,k-1}}{p}$.

In view of theorem 3.4, it is natural to ask whether $a(A \wr B) = p^d a(B) + O(p^{d-1}a(B))$, i.e., whether "$\psi(A,B) = 1 + O(\frac{1}{p})$". We shall show that this is not true in general. However, if we assume A is abelian, then it is true.

Corollary 3.7. *Let A be a p-group. Define p-groups A_n recursively by $A_0 = A$, $A_n = A_{n-1} \wr C_p$. Let B be a p-group. Then $\psi(A_n, B) \to 0$ as $n \to \infty$.*

Note that corollary 3.7 shows that "$\psi(A,B) = 1 + O(\frac{1}{p})$" cannot be true in general. Now we show how the situation differs if we assume A to be abelian.

Theorem 3.8. *For abelian p-groups A, define $t(A)$ to be the unique positive integer such that $A \cong C_{p^{d_1}} \oplus C_{p^{d_2}} \oplus \cdots \oplus C_{p^{d_k}} \oplus C_{p^d}^{t(A)}$, where $d_1 \le d_2 \le \cdots \le d_k < d$. Then,*

(i) *For each noncyclic group B, $1 - p^{-t(A)p} \le \psi(A,B) \le 1$.*

(ii) *For cyclic groups B, $\psi(A,B) = 1 - p^{-t(A)} + O(p^{-t(A)-1})$.*

In particular, $\psi(A,B) = 1 + O(\frac{1}{p})$ holds if A is abelian.

In [6] limiting value of average orders of a sequence of finite groups whose sizes go to infinity has been studied. Now we prove a result in this area.

Theorem 3.9. *Suppose B_n is a sequence of p-groups with $a(B_n) \to \beta$. Then $a((C_p)^n \wr B_n) \to p\beta$. So, if β is an Average Order Limit, then $p\beta$ is also. Hence $p^r\beta$ is an Average Order Limit for each nonnegative integer r.*

Note that whatever the sizes of B_n be, sizes of $(C_p)^n \wr B_n$ always go to infinity. So, taking $\{B_n\}_n$ to be a constant sequence B, we see that $p^r a(B)$ is an *Average Order Limit*, for any $r \ge 2$. This guarantees the existence of many non-integral *Average Order Limits*. For example, taking $B = (C_p)^b$ and $2 \le r \le b-1$, we get $p^{r+1} - \frac{p-1}{p^{b-r}}$ as an *Average Order Limit*.

Corollary 3.10. *Given any p-group B, and integer $r \ge 2$, there are sequences G_n, H_n of p-groups with $|G_n| \to \infty$, $|H_n| \to \infty$ and $a(G_n \wr H_n) \to p^r a(B)$.*

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4. Proofs of the Theorems

We start with two preliminary lemmas.

Lemma 4.1. *For any semidirect product $H \rtimes K$, we have $a(H \rtimes K) \geq a(K)$. Also, for $(h,k) \in H \rtimes K$, we have $order(k) | order((h,k))$.*

Proof. For any $(h,k) \in H \rtimes K$, $(h,k)^m$ has k^m as 2nd coordinate. So, $(h,k)^m = 1 \implies k^m = 1$. So, order$(k)|order((h,k))$, in particular order$((h,k)) \geq order(k)$, for all $h \in H$. So, a$(H \rtimes K) \geq \frac{|H|}{|H|\cdot|K|}\sum_{k\in K} order(k) = \frac{1}{|K|}\sum_{k\in K} order(k) = a(K)$. □

Lemma 4.2. *Let p be a prime and $b \geq 1$ an integer. Then $a(C_{p^b}) = p^b + O(p^{b-1})$.*

Proof. In C_{p^b}, there are exactly $\phi(p^n) = p^n(1-p^{-1})$ elements of order p^n, for each $1 \leq n \leq b$. So,

$$a(C_{p^b}) = \frac{1 + \sum_{n=1}^{b} p^n \cdot p^n \cdot (1-p^{-1})}{p^b} = \frac{1 + p(p-1)\sum_{n=0}^{b-1} p^{2n}}{p^b}$$

$$= \frac{1 + (p^2-p)\frac{p^{2b}-1}{p^2-1}}{p^b} = \frac{1 + \frac{p}{p+1}(p^{2b}-1)}{p^b}$$

$$= p^{-b} + \frac{p}{p+1}(p^b - p^{-b}) = p^b + O(p^{b-1}).$$

□

Let us recall some standard facts before starting the proof of theorem 3.1. For groups H, K, and injective group homomorphism ϕ: $K \to Aut(H)$, we have an injective group homomorphism $\psi : H \rtimes_\phi K \to Perm(H)$ defined by $\psi(h,k)(x) = h\phi(k)(x)$. So, $H \rtimes_\phi K$ can be regarded as a subgroup of Perm(H) consisting of the transformations $T_{h,k}$ defined by $T_{h,k}(x) = h\phi(k)(x)$.

So, if A and B has at least 2 elements, $A \wr B$ is the subgroup of Perm$(\prod_{b\in B} A)$, consisting of the permutations $T_{\underline{\alpha},x}$, $\underline{\alpha} \in \prod_{b\in B} A$, $x \in B$, where $T_{\underline{\alpha},x}(\underline{a}) = (\alpha_b a_{x^{-1}b})_b$, $(\underline{a} = (a_b)_b)$.

Proof of Theorem 3.1. Note that lemma 4.1 yields order$(x) \mid$ order $(T_{\underline{\alpha},x})$ for all $(\underline{\alpha}, x) \in A \wr B$. Fix $x \in B$. Let d=order(x). For $m \geq 1$, we have $T^m_{\underline{\alpha},x^{-1}}(\underline{a}) = (\alpha_b \alpha_{xb} \cdots \alpha_{x^{m-1}b} a_{x^m b})_b$.

So,

$$T^{dm}_{\underline{\alpha},x^{-1}} = 1 \iff \alpha_b \alpha_{xb} \cdots \alpha_{x^{dm-1}b} = 1 \;\forall\; b \in B$$

$$\iff (\alpha_b \alpha_{xb} \cdots \alpha_{x^{d-1}b})^m = 1 \;\forall\; b \in B.$$

Name this condition (1). □

Multiplication by x divides B into $\frac{|B|}{d}$ orbits of size d, and (1) is equivalent to saying that product of α_b's, for b running in each orbit in cyclic order, has m'th power= 1.

Number of $\underline{\alpha} \in \prod_{b\in B} A$ satisfying (1) is exactly $(|A|^{d-1}s_m)^{\frac{|B|}{d}}$. The reason is as follows. Fix a point p in a orbit. For each $b \neq p$ in that orbit, α_b can be chosen to be anything in A. After that α_p has exactly

s_m choices. Altogether we get α_b's for b running over a fixed orbit has exactly $|A|^{d-1}s_m$ choices. There are $\frac{|B|}{d}$ orbits. So in total we have $(|A|^{d-1}s_m)^{\frac{|B|}{d}}$ choices for $\underline{\alpha}$.

From (1), it is clear that $T^{d|A|}_{\underline{\alpha},x^{-1}} = 1$. So, $\frac{order(T_{\underline{\alpha},x^{-1}})}{d} \mid |A|$

For $m \geq 1$, let g_m= number of $\underline{\alpha} \in \prod_{b\in B} A$ with order$(T_{\underline{\alpha},x^{-1}}) = dm$. We have just shown that unless $m \mid |A|$, we have $g_m = 0$. Also, for all $k \geq 1$, $\sum_{m|k} g_m$ = number of $\underline{\alpha}$ with $(T_{\underline{\alpha},x^{-1}})^{dk} = 1 = (|A|^{d-1}s_k)^{\frac{|B|}{d}}$.

By Mobius inversion, $g_m = |A|^{\frac{|B|(d-1)}{d}} \sum_{n|m} \mu(\frac{m}{n}) s_n^{\frac{|B|}{d}}$ for all $m \geq 1$. So, $\sum_{\underline{\alpha}\in\prod_{b\in B} A} order(T_{\underline{\alpha},x^{-1}})$

$= \sum_{m\big||A|} dmg_m = d|A|^{\frac{|B|(d-1)}{d}} \sum_{m\big||A|,n|m} m\mu(\frac{m}{n}) s_n^{\frac{|B|}{d}}$

$= d|A|^{\frac{|B|(d-1)}{d}} \sum_{n\big||A|,c\big|\frac{|A|}{n}} nc\mu(c) s_n^{\frac{|B|}{d}}$ $(m = nc)$

$= d|A|^{\frac{|B|(d-1)}{d}} \sum_{m\big||A|,c\big|\frac{|A|}{m}} mc\mu(c) s_m^{\frac{|B|}{d}}$ (rename n by m)

$= d|A|^{\frac{|B|(d-1)}{d}} \sum_{m\big||A|} ms_m^{\frac{|B|}{d}} [\sum_{c\big|\frac{|A|}{m}} c\mu(c)]$

$= d|A|^{\frac{|B|(d-1)}{d}} \sum_{m\big||A|} ms_m^{\frac{|B|}{d}} \tau(\frac{|A|}{m})$

$= |A|^{|B|} d|A|^{-\frac{|B|}{d}} \sum_{m\big||A|} ms_m^{\frac{|B|}{d}} \tau(\frac{|A|}{m})$.

This is true for any $x \in B$ of order d. There are d_n elements of order n in B, for each $n \mid |B|$. So,

$\sum_{x\in A\wr B} order(x) = \sum_{n\big||B|} d_n |A|^{|B|} n |A|^{-\frac{|B|}{n}} [\sum_{m\big||A|} ms_m^{\frac{|B|}{n}} \tau(\frac{|A|}{m})]$

$= |B||A|^{|B|} \sum_{m\big||A|,n\big||B|} d_n \frac{n}{|B|} |A|^{-\frac{|B|}{n}} ms_m^{\frac{|B|}{n}} \tau(\frac{|A|}{m})$

$= |B||A|^{|B|} \sum_{m\big||A|,n\big||B|} d_{\frac{|B|}{n}} n^{-1} |A|^{-n} ms_m^n \tau(\frac{|A|}{m})$ (by replace n with $\frac{|B|}{n}$)

$= |B||A|^{|B|} \sum_{m\big||A|,n\big||B|} \frac{m}{n} (\frac{s_m}{|A|})^n d_{\frac{|B|}{n}} \tau(\frac{|A|}{m})$.

Dividing by $|B||A|^{|B|}$ we get the desired result.

Proof of Theorem 3.2. Theorem 3.1, together with the observation that $s_{p^m} = p^a$ for $m \geq d$ yields

$\mathrm{a}(A \wr B)$

$= \sum_{n\leq b} \frac{p^a}{p^n} (\frac{s_{p^a}}{p^a})^{p^n} d_{p^{b-n}}$

$-(p-1) \sum_{m\leq a-1,n\leq b} \frac{p^m}{p^n} (\frac{s_{p^m}}{p^a})^{p^n} d_{p^{b-n}}$

$= p^{a-b} \sum_{n\leq b} p^{b-n} d_{p^{b-n}}$

$-(p-1) \sum_{d\leq m\leq a-1,n\leq b} \frac{p^m}{p^n} d_{p^{b-n}}$

$-(p-1) \sum_{m\leq d-1,n\leq b} \frac{p^m}{p^n} (\frac{s_{p^m}}{p^a})^{p^n} d_{p^{b-n}}$

$= p^a \mathrm{a}(B) - (p-1)p^d [\sum_{m\leq a-d-1} p^m] \sum_{n\leq b} p^{-n} d_{p^{b-n}}$

$-(p-1) \sum_{m\leq d-1,n\leq b} \frac{p^m}{p^n} (\frac{s_{p^m}}{p^a})^{p^n} d_{p^{b-n}}$

$= p^a \mathrm{a}(B) - (p^{a-d} - 1)p^d . a(B)$

$-(p-1) \sum_{m\leq d-1,n\leq b} \frac{p^m}{p^n} (\frac{s_{p^m}}{p^a})^{p^n} d_{p^{b-n}}$

$= p^d \mathrm{a}(B) - (p-1) \sum_{m\leq d-1,n\leq b} p^m (\frac{s_{p^m}}{p^a})^{p^n} p^{-n} d_{p^{b-n}}$

$= p^d \mathrm{a}(B) - (p-1) \sum_{n\leq b} p^{-n} d_{p^{b-n}} [\sum_{m\leq d-1} p^m (\frac{s_{p^m}}{p^a})^{p^n}]$

$= p^d \mathrm{a}(B) - (p-1) a(B) \sum_{n\leq b} k_n . [\sum_{m\leq d-1} p^m (\frac{s_{p^m}}{p^a})^{p^n}]$. □

Proof of Theorem 3.3. Theorem 3.2 proves the second inequality, and the first inequality follows from lemma 4.1. □

Proof of Theorem 3.4. Let us look at theorem 3.2 more closely. $\frac{s_{p^m}}{p^a} < 1 \; \forall m \leq d-1$. So, $\sum_{m \leq d-2} p^m (\frac{s_{p^m}}{p^a})^{p^n} \leq \sum_{m \leq d-2} p^m = \frac{p^{d-1}-1}{p-1}$, for each $n \leq b$. We also have $k_n \geq 0 \; \forall n$, $\sum_{n \leq b} k_n = 1$. Hence,

$$(p-1)a(B). \sum_{n \leq b} k_n . [\sum_{m \leq d-2} p^m (\tfrac{s_{p^m}}{p^a})^{p^n}] = O(p^{d-1} a(B)).$$

□

Proof of Theorem 3.5. Let $x \in B$, order $(x)=p^{e_1}$, $\underline{\alpha} \in \prod_{b \in B} A$. Multiplication by x divides B into p^{b-e_1} orbits of size p^{e_1}; let $b_1, b_2, \ldots, b_{p^{b-e_1}}$ be representatives of distinct orbits. By (1) of theorem 3.1, the order of $T_{\underline{\alpha}, x^{-1}}$ equals $p^{e_1} . \max_{1 \leq i \leq p^{b-e_1}} order(\alpha_{b_i} \alpha_{x b_i} \cdots \alpha_{x^{p^{e_1}-1} b_i}) \leq p^e . p^d = p^{d+e}$.

If $x \in B$ is of order p^e, and $y \in A$ is of order p^d, then the order of $T_{\underline{\alpha}, x^{-1}}$ equals p^{d+e}, where $\underline{\alpha}$ is defined by $\alpha_1 = y$, $\alpha_b = 1 \; \forall b \neq 1$. So, $\mathrm{m}(A \wr B) = p^{d+e}$. Now we prove the second statement.

Again fix $x \in B$, order $(x)=p^{e_1}$. Let $k \leq d+e-e_1$. Note that

$$T_{\underline{\alpha}, x^{-1}}^{p^{d+e-k}} = 1 \iff (\alpha_{b_i} \alpha_{x b_i} \cdots \alpha_{x^{p^{e_1}-1} b_i})^{p^{d+e-e_1-k}} = 1 \; \forall \; i.$$

Name this condition (2).

As we showed in the proof of theorem 3.1, the number of $\underline{\alpha} \in \prod_{b \in B} A$ satisfying (2) is

$$((p^a)^{p^{e_1}-1} s_{p^{d+e-e_1-k}})^{p^{d-e_1}} = (p^{ap^{e_1}} r_{A,k-(e-e_1)})^{p^{b-e_1}} = p^{ap^b} r_{A,k-(e-e_1)}^{p^{b-e_1}}.$$

(Note that $s_{p^{d+e-e_1-k}} = p^a r_{A,k-(e-e_1)}$.)

So, for $k \leq d+e-e_1$, $|\{\underline{\alpha} \in \prod_{b \in B} A : T_{\underline{\alpha}, x^{-1}}^{p^{d+e-k}} = 1\}| = p^{ap^b} r_{A,k-(e-e_1)}^{p^{b-e_1}}$. This is true for $k > d+e-e_1$ also, as then both sides are 0.

So, $r_{A \wr B, k}$
$= p^{-ap^b} p^{-b} \sum_{e_1=0}^{e} (\text{no. of elements of } B \text{ of order } p^{e_1}) p^{ap^b} r_{A,k-(e-e_1)}^{p^{b-e_1}}$
$= \sum_{i=0}^{e} \frac{\text{no. of elements of } B \text{ of order } p^{e-i}}{p^b} \cdot r_{A,k-i}^{p^{b-e+i}}$ (here $i = e - e_1$)
$= \sum_{i=0}^{e} (r_{B,i} - r_{B,i+1}) r_{A,k-i}^{p^{b-e+i}}$.
Corollary 3.6 is a direct consequence of theorem 3.5. To prove Corollary 3.7, we need a lemma. Write $r_{n,k} = r_{A_n,k}$. □

Lemma 4.3. *For all k, $\lim_{n \to \infty} r_{n,k} = 1$.*

Proof. We proceed by induction on k. For $k \leq 0$, we have $r_{n,k} = 1 \; \forall n$, and nothing to show. For the induction step, let $k \geq 1$ and assume $\lim_{n \to \infty} r_{n,k-1} = 1$. By corollary 3.6., $r_{n+1,k} = (1-p^{-1}) r_{n,k} + \frac{r_{n,k-1}^p}{p}$.

So, $\liminf_{n \to \infty} r_{n+1,k} = (1-p^{-1}) \liminf_{n \to \infty} r_{n,k} + p^{-1}$, as induction hypothesis implies $\lim_{n \to \infty} r_{n,k-1} = 1$. But $\liminf_{n \to \infty} r_{n+1,k} = \liminf_{n \to \infty} r_{n,k}$. Hence, $\liminf_{n \to \infty} r_{n,k} = (1-p^{-1}) \liminf_{n \to \infty} r_{n,k} + p^{-1}$; that is, $\liminf_{n \to \infty} r_{n,k} = 1$. Since we always have $0 \leq r_{n,k} \leq 1$, we get $\lim_{n \to \infty} r_{n,k} = 1$. Induction completes the proof. □

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Proof of Corollary 3.7. Let $m(A) = p^d$, $|B| = p^b$. By theorem 3.5, $m(A_n) = p^{d+n}$ for all n. By theorem 3.2,

$\psi(A_n, B) = 1 - \frac{p-1}{p^{d+n}} \sum_{r \le b} k_r (\sum_{m=0}^{d+n-1} p^m r_{n,d+n-m}^{p^r})$
$= 1 - (1-p^{-1}) \sum_{r \le b} k_r (\sum_{m=0}^{d+n-1} p^{-(d+n-1-m)} r_{n,(d+n-1-m)+1}^{p^r})$
$= 1 - (1-p^{-1}) \sum_{r \le b} k_r (\sum_{m=0}^{d+n-1} p^{-m} r_{n,m+1}^{p^r})$ (replace m by $d+n-1-m$).

Fix $M \in \mathbf{N}$. For all sufficiently large n, we have $d+n-1 > M$, hence $\psi(A_n, B) \le 1 - (1 - p^{-1}) \sum_{r \le b} k_r (\sum_{m=0}^{M} p^{-m} r_{n,m+1}^{p^r})$. Since $\lim_{n\to\infty} r_{n,m+1} = 1$ by lemma 3.3, we get
$\limsup_{n\to\infty} \psi(A_n, B) \le 1 - (1-p^{-1}) \sum_{r \le b} k_r \sum_{m=0}^{M} p^{-m}$
$= 1 - (1-p^{-1}) \sum_{m=0}^{M} p^{-m}$ (as $\sum_{r \le b} k_r = 1$) for all $M \in \mathbf{N}$.
Taking $M \to \infty$, we get $\limsup_{n\to\infty} \psi(A_n, B) \le 0$, that is, $\lim_{n\to\infty} \psi(A_n, B) = 0$. □

Proof of Theorem 3.8. We use the notation of theorem 3.2, and write t instead of $t(A)$. Projection gives a surjective group homomorphism $\phi : A \to (C_{p^d})$. Note that the subgroup of p^{d-1}-torsion elements of A is $\phi^{-1}((p\mathbf{Z}/p^d\mathbf{Z})^t)$, and for $0 \le m \le d-2$, the subgroup of p^{d-m}-torsion elements of A is contained in $\phi^{-1}((p^m\mathbf{Z}/p^d\mathbf{Z})^t)$. So, $\frac{s_{p^{d-1}}}{p^a} = p^{-t}$, and for each $0 \le m \le d-2$, we have $\frac{s_{p^m}}{p^a} \le p^{-mt} \le p^{-2t}$. By theorem 3.2,

$$\begin{aligned}
&p^d a(B) - a(A \wr B) - (p-1)a(B)k_0 \cdot [\sum_{m \le d-1} p^m \frac{s_{p^m}}{p^a}] \\
&= (p-1)a(B) \sum_{1 \le n \le b} k_n \cdot [\sum_{m \le d-1} p^m (\frac{s_{p^m}}{p^a})^{p^n}] \\
&\le (p-1)a(B) \sum_{1 \le n \le b} k_n \sum_{m \le d-1} \frac{p^m}{p^{tp^n}} \\
&= (p-1)a(B)(\sum_{1 \le n \le b} k_n p^{-tp^n}) \cdot \frac{p^d - 1}{p-1} \\
&\le a(B) \sum_{1 \le n \le b} p^{d-tp^n} k_n \\
&\le a(B)p^{d-tp} \sum_{1 \le n \le b} k_n \le a(B)p^{d-tp}.
\end{aligned}$$

If B is noncyclic, $k_0 = 0$; so $p^d a(B) - a(A \wr B) \le a(B)p^{d-tp}$. Dividing by $p^d a(B)$, we get $1 - p^{-tp} \le \psi(A, B)$. If B is the cyclic group C_{p^b}, we have $k_0 = \frac{\phi(p^b)}{a(B)} = \frac{p-1}{p} \cdot \frac{p^b}{a(B)} = (1 + O(\frac{1}{p})) \cdot (1 + O(\frac{1}{p})) = 1 + O(\frac{1}{p})$. Here, we used lemma 4.2, and the observation that $(1 + O(p^{-1}))^{-1} = 1 + O(p^{-1})$. So,

$$\sum_{m \le d-1} p^m \frac{s_{p^m}}{p^a} = p^{d-1-t} + O(\sum_{m \le d-2} p^{m-2t}) = p^{d-1-t} + O(\frac{p^{d-1}-1}{p-1} \cdot p^{-2t}) = p^{d-1-t} + O(p^{d-2-2t}).$$

Hence,

$$\frac{(p-1)a(B)k_0 \cdot [\sum_{m \le d-1} p^m \frac{s_{p^m}}{p^a}]}{p^d a(B)} = \frac{(p-1)(1+O(p^{-1}))(p^{d-1-t}+O(p^{d-2-2t}))}{p^d} = \frac{p^{d-t}+O(p^{d-1-t})}{p^d} = p^{-t} + O(p^{-t-1}).$$

We have shown $0 \le p^d a(B) - a(A \wr B) - (p-1)a(B)k_0 \cdot [\sum_{m \le d-1} p^m \frac{s_{p^m}}{p^a}] \le a(B)p^{d-tp}$. So, dividing by $p^d a(B)$ we get

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$$\psi(A,B) = 1 - \frac{(p-1)a(B)k_0 \cdot [\sum_{m \le d-1} p^m \frac{s_{p^m}}{p^a}]}{p^d a(B)} + O(p^{-tp}) = 1 + p^{-t} + O(p^{-t-1}) + O(p^{-tp}) = 1 + p^{-t} + O(p^{-t-1}).$$

□

Proof of Theorem 3.9. Let $|B_n| = p^{b_n}$. By theorem 3.2,

$$a((C_p)^n \wr B_n) = p \cdot a(B_n) - (p-1)a(B_n) \sum_{m \le b_n} k_{m,n} (\frac{1}{p^n})^{p^m},$$

for some $0 \le k_{m,n} \le 1$, with $\sum_{m \le b_n} k_{m,n} = 1$ for each n. So, for all sufficiently large n, so that $a(B_n) \le \beta + 1$; we have

$$|a((C_p)^n \wr B_n) - p\beta| \le p|a(B_n) - \beta| + (p-1)(\beta+1)p^{-n} \to 0.$$

□

Proof of Corollary 3.10. Let $G_n = (C_p)^n$, $H_n = (C_p)^n \wr ((C_p)^n \wr (\cdots ((C_p)^n \wr B)))$, there are $r-1$ $(C_p)^n$'s here. Now corollary 3.10 follows by repeated application of theorem 3.9. □

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