http://ijgt.ui.ac.ir/

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. x No. x (202x), pp. xx-xx.

www.ui.ac.ir

GENERALIZED NILPOTENT BRACES AND NILPOTENT GROUPS

MARTYN R. DIXON*, LEONID A. KURDACHENKO AND IGOR YA. SUBBOTIN

Abstract. The authors give a brief survey of some results concerning nilpotent braces and their generalizations. Various results concerning $\star$-hypercentral and locally $\star$-nilpotent braces are given.

1. Introduction

A *left brace (or skew left brace of abelian type)* is a set A together with two binary operations, addition denoted by $+$ and multiplication denoted by $\cdot$ (which is often omitted), satisfying the following conditions:

(LB1) A is an abelian group under addition;
(LB2) A is a group under multiplication;
(LB3) $a(b+c) = ab + ac - a$ for all $a, b, c \in A$.

In this paper, all braces will be left braces in this sense. In some sense braces represent a type of fusion of two groups such that the zero element of the additive group is the identity element of the multiplicative group and we denote this common neutral element by 0 or 1, as appropriate. We shall sometimes denote the additive group of A by $(A, +)$ and the multiplicative group of A by $(A, \cdot)$.

Braces were first introduced by W. Rump in [10] as a generalization of Jacobson radical rings in order to help study involutive set-theoretic solutions of the Yang-Baxter equation. Since then braces have been studied for their algebraic properties and have also been linked with other research areas.

MSC(2010): Primary: 16N80; Secondary 16T25, 16N40, 20F19.

Keywords: Brace, nilpotent brace, $\star$-hypercentral brace, locally $\star$-nilpotent brace.

Communicated by Patrizia Longobardi.

Article Type: Ischia Group Theory 2022.

Received: 22 July 2023, Accepted: 08 November 2023.

*Corresponding author.

http://dx.doi.org/10.22108/ijgt.2023.138490.1860 .

The reader is referred to [3] for some elementary properties and the connections with Yang-Baxter equations, which we do not address here.

If A is a left brace, then a subset S of A is called a *subbrace* (more precisely a *left subbrace*) if S is closed under addition and multiplication and is a left brace by restriction of these operations to S. In other words, a subset S of A is a subbrace of A if and only if S is closed under addition and multiplication and under addition S is an additive subgroup of A and under muliplication S is a multiplicative subgroup of A.

For the left brace A and for each element $a \in A$ we define a map $\lambda_a : A \longrightarrow A$ by $\lambda_a(x) = ax - a$ for all $x \in A$. Furthermore we define $a \star b = ab - a - b$ for all $a, b \in A$. It is easy to see that $a \star b = \lambda_a(b) - b$. The operation $\star$ plays a very important role in the study of left braces.

A left brace A is called *trivial* or *abelian* if $a \star b = 0$ for all $a, b \in A$, or equivalently, $a + b = ab$. In the case of abelian braces the addition and multiplication coincide.

If A is a left brace, then a subbrace L of A is called a *left ideal* of A if $a \star b \in L$ for all elements $a \in A$ and $b \in L$. This means that if $x, y \in L$, then $x - y \in L$ and if $a \in A$ and $z \in L$, then $\lambda_a(z) \in L$, as in [3]. We recall that a subbrace L of a left brace A is an *ideal* if $a \star z, z \star a \in L$ for all $a \in A, z \in L$.

As with other algebraic structures it is possible to study the concept of nilpotency. In the theory of braces there are several different approaches to this concept and we refer the reader to the papers [3, 4, 5, 9, 12, 13]. In group theory many generalizations of nilpotency arise and so it is with braces also. Our approach to nilpotency is based on the following concept, first introduced by Bonatto and Jedlička [2] and Jespers, Van Antwerpen and Vendramin [9]

Let A be a left brace. The set

$$\begin{aligned}\zeta(\star, A) &= \{a \in A | a \star x = x \star a = 0 \text{ for all } x \in A\}\\ &= \{a \in A | ax = a + x = xa \text{ for all } x \in A\}\\ &= \{a \in A | \lambda_a(x) = x \text{ and } \lambda_x(a) = a \text{ for all } x \in A\}\end{aligned}$$

is here called the $\star$-*center* of A. We show in Proposition 2.3 that $\zeta(\star, A)$ is an ideal of A which is contained in the center, $\zeta(A)$, of $(A, \cdot)$; this known result also appears in [2] and [9] but we give a proof for the sake of completeness.

By analogy with groups we construct the *upper $\star$-central series*

$$0 = \zeta_0(\star, A) \leq \zeta_1(\star, A) \leq \cdots \zeta_\alpha(\star, A) \leq \zeta_{\alpha+1}(\star, A) \leq \cdots \zeta_\gamma(\star, A)$$

as follows. We let

$$\begin{gathered}\zeta_1(\star, A) = \zeta(\star, A) \text{ and}\\ \zeta_{\alpha+1}(\star, A)/\zeta_\alpha(\star, A) = \zeta(\star, A/\zeta_\alpha(\star, A))\end{gathered}$$

for all ordinals α; as usual for limit ordinals λ we set $\zeta_\lambda(\star, A) = \cup_{\mu<\lambda}\zeta_\mu(\star, A)$.

http://dx.doi.org/10.22108/ijgt.2023.138490.1860

The definition, together with Proposition 2.3, implies that each term of this series is an ideal of A. The last term $\zeta_\infty(\star, A) = \zeta_\gamma(\star, A)$ of this series is called the *upper $\star$-hypercenter* of A and we denote the length of the upper $\star$-central series of A by $\mathrm{zl}(A)$. Furthermore, if $A = \zeta_\infty(\star, A)$, then A is called a *$\star$-hypercentral* brace; in this case, if $\mathrm{zl}(A)$ is finite, then we say that A is *$\star$-nilpotent*. This series has also been introduced in [2, 7] and [14] but there this series is called the *upper annihilator series* and the last term is then called the *hyper-annihilator*; a $\star$-hypercentral brace is called in [14] *annihilator hypercentral* and in [2, 9] $\star$-nilpotency is called *central nilpotency (and respecitively) annihilator nilpotency*.

If A is a left brace and K, L are subbraces, then we let $K \star L$ denote the subgroup of $(A, +)$ generated by the elements $x \star y$, where $x \in K, y \in L$.

When A is a left brace and B, C are ideals of A with $B \leq C$, then the factor C/B is called *$\star$-central* if $A \star C, C \star A \leq B$. Thus for all $a \in A, c \in C$ we have $a \star c, c \star a \in B$.

Let

$$0 = C_0 \leq C_1 \leq \cdots C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma$$

be an ascending series of ideals of the the brace A. This series is called *$\star$-central* if every factor of this series is $\star$-central; in other words $A \star C_{\alpha+1}, C_{\alpha+1} \star A \leq C_\alpha$ for all $\alpha < \gamma$. It is clear that the upper $\star$-central series is $\star$-central.

We define two further canonical series of the left brace A. First, let $A^{(1)} = A$ and recursively define $A^{(\alpha+1)} = A^{(\alpha)} \star A$ for all ordinals α and as usual set $A^{(\lambda)} = \cap_{\mu<\lambda} A^{(\mu)}$ for all limit ordinals λ. As in [4] A is called *right nilpotent* if $A^{(n)} = 0$ for some natural number n.

Similarly, let $A^1 = A$, $A^{\alpha+1} = A \star A^\alpha$, for all ordinals α and $A^\lambda = \cap_{\mu<\lambda} A^\mu$ for all limit ordinals μ. As in [4] A is *left nilpotent* if $A^m = 0$ for some natural number m. We note that a left nilpotent brace need not be right nilpotent and vice-versa (see [1] and [3], for example). Indeed Rump [11] gave an example of a left brace of cardinality 6 such that $A^{(3)} = 0$, but $A^m \neq 0$ for all m.

We shall say that A is *Smoktunowicz-nilpotent* if there are natural numbers n, k such that $A^{(n)} = A^k$. Such braces were introduced by A. Smoktunowicz in the paper [12]. In [12, Theorem 1.3] a single criterion was given for these conditions and left braces satisfying this single criterion were in [13] called *strongly nilpotent*.

We note that according to Proposition 2.10 a left brace is $\star$-nilpotent if and only if it is Smoktunowicz-nilpotent. Thus $\star$-hypercentral braces are the natural generalization of Smoktunowicz-nilpotent braces.

Let A be a left brace and let M be a subset of A. The *subbrace B of A generated by M* is the intersection of all subbraces C of A that contain M. If M is a finite set, then B will be called *finitely generated*. We let $\mathrm{br}(M)$ denote the subbrace generated by M. We refer the reader to [7, 9] for further details and properties.

A brace A is called *locally Smoktunowicz-nilpotent* if every finitely generated subbrace of A is Smoktunowicz-nilpotent. Proposition 2.10 shows that a brace A is locally Smoktunowicz-nilpotent if and only if it is locally $\star$-nilpotent.

The main results of this paper are as follows.

Theorem 1.1. *Let A be a left brace. Then the following results hold.*

(i) *If A is $\star$-hypercentral brace then the multiplicative group of A is hypercentral.*

(ii) *A is $\star$-hypercentral if and only if for each element $a \in A$ and every pair of countable subsets $\{x_n | n \in \mathbb{N}\}, \{y_n | n \in \mathbb{N}\}$ of elements of A, there exist natural numbers k, t such that*

$$(\cdots(a \star x_1) \star x_2) \star \cdots) \star x_j = 0 \text{ for all } j \geq k$$

$$y_j \star (\cdots \star (y_3 \star (y_2 \star (y_1 \star a) \cdots) = 0 \text{ for all } j \geq t.$$

(iii) *A is $\star$-hypercentral if and only if each of its countable subbraces is $\star$-hypercentral.*

(iv) *If A is $\star$-hypercentral, then every subbrace is ascendant. In particular, if A is Smoktunowicz-nilpotent, then every subbrace is a subideal.*

(v) *If A is $\star$-hypercentral, then A is locally Smoktunowicz-nilpotent.*

We give the definition of ascendant subbrace in Section 3.

By Theorem 1.1 the multiplicative group of a $\star$-hypercentral left brace (and, in particular, of a Smoktunowicz-nilpotent left brace) is hypercentral (respectively nilpotent). Hence the multiplicative group of A has a torsion part (the greatest periodic subgroup) S. This naturally leads to the question of the relationship of S with the torsion part T of the additive group of A. Our next result gives an answer to this question.

Theorem 1.2. *Let A be a $\star$-hypercentral left brace. Then the following results hold.*

(i) *The torsion part of the additive group of A coincides with the torsion part of the multiplicative group of A.*

(ii) *The torsion part of the additive group of A is an ideal of A.*

(iii) *If the additive group of A is torsion-free, then the additive group of the factor brace $A/\zeta_\alpha(\star, A)$ is torsion-free for each ordinal α.*

(iv) *If the additive group of A is torsion-free, then the additive group of every factor of the upper $\star$-central series is torsion-free.*

Our next result gives some properties of locally Smoktunowicz-nilpotent left braces.

Theorem 1.3. *Let A be a locally Smoktunowicz-nilpotent left brace. Then the following assertions hold.*

(i) *The multiplicative group of A is locally nilpotent and the torsion part of the additive group of A coincides with the torsion part of multiplicative group of A.*

http://dx.doi.org/10.22108/ijgt.2023.138490.1860

(ii) *The torsion part of the additive group of A is an ideal of A.*

(iii) *If the additive group of A is torsion-free, then the multiplicative group of A is torsion-free.*

(iv) *If $B \leq C$ are ideals of A such that the factor C/B is A-chief, then C/B is $\star$-central in A. In particular C/B has prime order.*

(v) *If L is a maximal subbrace of A, then L is an ideal of A.*

2. A brief survey of $\star$-nilpotent braces

We begin by first discussing $\star$-nilpotent left braces but also give some well-known properties of left braces. We note the following criteria for a subset of A to be an ideal, which is well-known (see [3, 9], for example).

Proposition 2.1. *Let A be a left brace and let L be an ideal of A. Then the following conditions hold:*

(I1) *If $x, y \in L$, then $xy^{-1} \in L$;*

(I2) *if $a \in A$ and $z \in L$, then $a^{-1}za \in L$;*

(I3) *if $a \in A$ and $z \in L$, then $\lambda_a(z) \in L$.*

Conversely, if L is a non-empty subset of A satisfying (I1)-(I3), then L is an ideal of A.

We shall also need some of the following properties of $\star$ and λ_a whose proofs can be found in [3] or [9].

Lemma 2.2. *Let A be a left brace. Then*

(i) $a \star (b + c) = a \star b + a \star c$;

(ii) $(ab) \star c = a \star (b \star c) + b \star c + a \star c$;

(iii) $(a + b) \star c = a \star (\lambda_{a^{-1}}(b) \star c) + (\lambda_{a^{-1}}(b) \star c) + a \star c$;

(iv) $\lambda_y(b \star a) = yby^{-1} \star \lambda_y(a)$;

(v) $yby^{-1} = \lambda_y(\lambda_b(y^{-1}) - y^{-1} + b) = \lambda_y(b \star y^{-1} + b)$

for all elements $a, b, c, y \in A$.

The following result appears in [2] and [9] but we give the proof.

Proposition 2.3. *Let A be a left brace. The $\star$-center of A is an ideal of A. In particular, every term of the upper $\star$-central series is an ideal of A.*

Proof. We use Proposition 2.1. Let $a \in \zeta(\star, A)$ and let $x \in A$. Then $a \in \zeta(A)$ so $a^{-1} \in \zeta(A)$ and $a^{-1}x = xa^{-1}$. Furthermore, we have

$$x = a^{-1}(ax) = a^{-1}(a + x) = a^{-1}a + a^{-1}x - a^{-1} = a^{-1}x - a^{-1}.$$

It follows that $xa^{-1} = a^{-1}x = a^{-1} + x$ and hence $a^{-1} \in \zeta(\star, A)$.

Let also $b \in \zeta(\star, A)$. Then $ab \in \zeta(A)$ so

$$(ab)x = x(ab) = x(a+b) = xa + xb - x = x + a + b = x + ab$$

so that $ab \in \zeta(\star, A)$. Thus $\zeta(\star, A)$ satisfies condition (I1) of Proposition 2.1.

For each $a \in \zeta(\star, A)$ and $x \in A$ we have $\lambda_x(a) = a$ so that $\zeta(\star, A)$ satisfies condition (I3) of Proposition 2.1.

Finally $a \in \zeta(A)$ so $x^{-1}ax = a \in \zeta(\star, A)$. □

Next we give an important property of the upper $\star$-hypercenter, which is analogous to the well-known result in group theory. This result appear as [2, Proposition 2.10]. There is a similar result in [5, Proposition 2.26]

Proposition 2.4. *Let A be a left brace and suppose that K is a non-trivial ideal of A such that $K \leq \zeta_\infty(\star, A)$. Then $K \cap \zeta(\star, A)$ is non-trivial.*

Proof. Let

$$0 = \zeta_0(\star, A) \leq \zeta_1(\star, A) \leq \cdots \zeta_\alpha(\star, A) \leq \cdots \zeta_\gamma(\star, A) = \zeta_\infty(\star, A)$$

be the upper $\star$-central series of A. Since $K \leq \zeta_\infty(\star, A)$, there is an ordinal α such that $K \cap \zeta_\alpha(\star, A) \neq 0$ and we let β be the least such ordinal with this property. Clearly β is not a limit ordinal and the definition of β gives $K \cap \zeta_{\beta-1}(\star, A) = 0$. Since $S = K \cap \zeta_\beta(\star, A) \leq \zeta_\beta(\star, A)$ it follows that $S \star A, A \star S \leq \zeta_{\beta-1}(\star, A)$. On the other hand, K is an ideal of A so $S \star A, A \star S \leq K$. It follows that $S \star A, A \star S \leq \zeta_{\beta-1}(\star, A) \cap K = 0$. Thus $S \leq \zeta(\star, A)$ giving the required result. □

Corollary 2.5. *Let A be a left brace. If A is $\star$-hypercentral and K is a nontrivial ideal of A, then $K \cap \zeta(\star, A)$ is nontrivial.*

Let A be a left brace. The set

$$\begin{aligned} \mathrm{Soc}(A) &= \{a \in A | a \star x = 0 \text{ for all } x \in A\} \\ &= \{a \in A | ax = a + x \text{ for all } x \in A\} \end{aligned}$$

is called the *socle* of A. In [3] it is shown that $\mathrm{Soc}(A)$ is an ideal of A, a result attributable to Rump. Clearly $\mathrm{Soc}(A)$ contains $\zeta(\star, A)$.

We note also the following well-known results.

Proposition 2.6. *Let A be a left brace and let L be a left ideal of A. Then*

(i) *$A \star L$ is a left ideal of A. Moreover, if L is an ideal of A, then $L \star A$ is an ideal of A.*

(ii) *If $Soc(A)$ contains L, then $A \star L$ is an ideal of A.*

It is shown in [3] and indeed this follows from Proposition 2.6 that A^n is always a left ideal of A and $A^{(n)}$ is an ideal of A. Using Proposition 2.6 we may generalize this to:

Proposition 2.7. *Let A be a left brace. Then A^{α} is a left ideal for each ordinal α and $A^{(\alpha)}$ is an ideal for each ordinal α.*

Next we give the relationship between $A^{(j)}, A^{j}$ and $\star$-central series, analogous to the corresponding result for groups.

Lemma 2.8. *Let A be a left brace and let*

$$0 = C_0 \leq C_1 \leq \cdots \leq C_n = A$$

be a finite $\star$-central series of A. Then

(i) $A^{(j)}, A^{j} \leq C_{n-j+1}$ *and hence* $A^{(n+1)} = 0 = A^{n+1}$*;*

(ii) $C_j \leq \zeta_j(\star, A)$ *and hence* $\zeta_n(\star, A) = A$.

In particular, $zl(A) \leq n$.

Proof. (i) We use induction on j. For $j = 2$ we have $A^2 = A^{(2)} = A \star A = C_n \star A \leq C_{n-1}$ so the result holds for $j = 2$.

Suppose now that $j > 2$ and that we have already proved $A^{(m)}, A^{m} \leq C_{n-m+1}$ for all $m < j$. Then $A^{(j)} = A^{(j-1)} \star A \leq C_{n-j+1+1} \star A \leq C_{n-j+1}$ and similarly $A^{j} = A \star A^{j-1} \leq A \star C_{n-j+1+1} \leq C_{n-j+1}$.

(ii) We again use induction on j. For $j = 1$ we have $C_1 \star A = 0 = A \star C_1$ and hence $C_1 \leq \zeta_1(\star, A)$. Suppose that $j > 2$ and we have already proved that $C_m \leq \zeta_m(\star, A)$ for all $m < j$. Since C_j/C_{j-1} is $\star$-central we have $C_j \star A, A \star C_j \leq C_{j-1} \leq \zeta_{j-1}(\star, A)$, by the induction hypothesis. Let $a \in A, c \in C_j$. Then $ca - c - a = c \star a \in \zeta_{j-1}(\star, A)$ so that

$$\begin{aligned}\zeta_{j-1}(\star, A) &= (ca - c - a + \zeta_{j-1}(\star, A)) \\ &= (c + \zeta_{j-1}(\star, A)) \star (a + \zeta_{j-1}(\star, A))\end{aligned}$$

In a similar way we also obtain $(a + \zeta_{j-1}(\star, A)) \star (c + \zeta_{j-1}(\star, A)) = \zeta_{j-1}(\star, A)$. This means that

$$(C_j + \zeta_{j-1}(\star, A))/\zeta_{j-1}(\star, A) \leq \zeta(\star, A/\zeta_{j-1}(A)) \leq \zeta_j(\star, A)/\zeta_{j-1}(\star, A)$$

so that $C_j \leq \zeta_j(\star, A)$ as required. □

The next result follows from [9, Corollary 2.15].

Lemma 2.9. *Let A be a Smoktunowicz-nilpotent left brace. Then A has a finite $\star$-central series.*

Using Lemmas 2.8 and 2.9 we have:

Proposition 2.10. *Let A be a left brace. Then A has a finite $\star$-central series if and only if A is Smoktunowicz-nilpotent.*

We may easily establish the following natural corollary and give its very easy proof.

Corollary 2.11. *Let A be a $\star$-hypercentral left brace.*

http://dx.doi.org/10.22108/ijgt.2023.138490.1860

(i) *If S is a subbrace of A, then S is $\star$-hypercentral and $zl(S) \leq zl(A)$;*

(ii) *if L is an ideal of A, then A/L is $\star$-hypercentral and $zl(A/L) \leq zl(A)$.*

Proof. (i) Let

$$0 = Z_0 \leq Z_1 \leq Z_2 \leq \cdots Z_\alpha \leq Z_{\alpha+1} \leq \cdots Z_\gamma = A$$

be the upper $\star$-central series of A and set $C_\alpha = Z_\alpha \cap S$ for $0 \leq \alpha < \gamma$. Then, for each α, C_α is an ideal of S. We have $S \star C_{\alpha+1} \leq S \star Z_{\alpha+1} \leq Z_\alpha$ and $C_{\alpha+1} \star S \leq Z_{\alpha+1} \star S \leq Z_\alpha$ for $0 \leq \alpha < \gamma$. On the other hand, since S is a subbrace we have $S \star C_{\alpha+1} \leq S$ and $C_{\alpha+1} \star S \leq S$ so that

$$S \star C_{\alpha+1} \leq S \cap Z_\alpha = C_\alpha, C_{\alpha+1} \star S \leq S \cap Z_\alpha = C_\alpha$$

for $0 \leq \alpha < \gamma$. These inclusions show that the series

$$0 = C_0 \leq C_1 \leq C_2 \leq \cdots C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma = S$$

is a $\star$-central series of ideals of S. Hence S is $\star$-hypercentral and a result analogous to Lemma 2.8 shows that $\mathrm{zl}(S) \leq \mathrm{zl}(A)$.

(ii) We note that $Z_\alpha + L$ is an ideal of A and $(Z_\alpha + L)/L$ is an ideal of A/L for $0 \leq \alpha < \gamma$. Since $(a + L) \star (b + L) = (a \star b) + L$ we have

$$(A/L) \star (Z_{\alpha+1} + L)/L = (A \star Z_{\alpha+1} + L)/L \leq (Z_\alpha + L)/L,$$
$$(Z_{\alpha+1} + L)/L \star (A/L) = (Z_{\alpha+1} \star A + L)/L \leq (Z_\alpha + L)/L$$

for $0 \leq \alpha < \gamma$. In turn, these inclusions show that the series

$$0 = (Z_0 + L)/L \leq (Z_1 + L)/L \leq \cdots (Z_\alpha + L)/L \leq (Z_{\alpha+1} + L)/L$$
$$\leq \cdots (Z_{\gamma+1} + L)/L = A/L$$

is a $\star$-central series of ideals for A/L. Thus A/L is $\star$-hypercentral and a result analogous to Lemma 2.8 shows that $\mathrm{zl}(A/L) \leq \mathrm{zl}(A)$ as required. □

We shall use the phrase "class of left braces" to be synonymous with the notion of "class of groups" from group theory. Let $\mathfrak{N}_S$ denote the class of left braces which are Smoktunowicz-nilpotent. More precisely, we let $\mathfrak{N}_S(n, k)$ denote the class of left braces A such that $A^{(n)} = 0 = A_k$ and n, k are least with these properties.

Let A be a left brace and let $\mathfrak{X}$ be a class of left braces. Let

$$A^{\mathfrak{X}} = \cap\{H | H \text{ is an ideal of } A \text{ and } A/H \in \mathfrak{X}\},$$

an ideal which is called, as usual, the $\mathfrak{X}$-*residual* of A.

There are two cases to consider, when $A/A^{\mathfrak{X}} \in \mathfrak{X}$ and $A/A^{\mathfrak{X}} \notin \mathfrak{X}$.

In the case when $A/A^{\mathfrak{X}} \in \mathfrak{X}$, it follows that $A^{\mathfrak{X}}$ is the smallest ideal of A such that $A/A^{\mathfrak{X}}$ is an $\mathfrak{X}$-brace.

Remak's theorem gives us the following very useful result.

Proposition 2.12. *Let A be a left brace and let $\mathfrak{X}$ be a class of left braces closed with respect to taking subbraces and Cartesian products. Then $A/A^{\mathfrak{X}} \in \mathfrak{X}$.*

In particular, if $\mathfrak{X} = \mathfrak{A}$ the class of all abelian braces, then the $\mathfrak{A}$-residual $A^{\mathfrak{A}}$ is exactly the derived subbrace $A^{(2)}$. In this case we have $A/A^{\mathfrak{A}} \in \mathfrak{A}$.

Proposition 2.13. *Let n, k be fixed natural numbers and let $\{A_\lambda | \lambda \in \Lambda\}$ be a family of left braces such that $A_\lambda \in \mathfrak{N}_S(n,k)$ for each $\lambda \in \Lambda$. Then $C = \underset{\lambda\in\Lambda}{Cr}\, A_\lambda \in \mathfrak{N}_S(n,k)$.*

Proof. Clearly $C^{(j)} = \underset{\lambda\in\Lambda}{\mathrm{Cr}}\, A_\lambda^{(j)}$ and $C^j = \underset{\lambda\in\Lambda}{\mathrm{Cr}}\, A_\lambda^j$ for all natural numbers j. It follows that $C^{(n)} = 0 = C^k$, so that $C \in \mathfrak{N}_{n,k}$. □

Corollary 2.14. *The class $\mathfrak{N}_S(n,k)$ is a variety of braces for all natural numbers n, k.*

3. Some properties of $\star$-hypercentral braces

In this section we discuss sums and products of ideals in left braces. These results will be well-known but we give the proofs. We begin with the following result.

Lemma 3.1. *Let A be a left brace, let L be an ideal of A and let S be a subbrace of A. Then*

$$S + L = \{s + u | s \in S, u \in L\} = \{su | s \in S, u \in L\} = SL = LS.$$

Proof. We know from [3] that if $a \in A$, then $a + L = aL = La$. Thus if $x \in S + L$, then $x = s + u \in s + L = sL$ so $x \in SL$. Conversely, if $x \in SL$, then $x = su \in sL = s + L$ so $x = s + v \in S + L$. □

Lemma 3.2. *Let A be a left brace and let L be an ideal of A.*

(i) *If S is a subbrace of A, then $S + L$ is a subbrace of A;*
(ii) *if S is a left ideal of A, then $S + L$ is a left ideal of A;*
(iii) *if S is an ideal of A, then $S + L$ is an ideal of A.*

Proof. (i) We have to show that $(S + L, +)$ and $(S + L, \cdot)$ are subgroups. Let $s + u, t + v \in S + L$ with $s, t \in S, u, v \in L$. Then

$$(s + u) - (t + v) = (s - t) + (u - v) \in S + L.$$

Also since L is an ideal we have $s + u \in s + L = sL$ so $s + u = sw$ and likewise $t + v = tx$, for some $w, x \in L$. Then

$$(s + w)(t + v)^{-1} = sw(tx)^{-1} = swx^{-1}t^{-1} = st^{-1}(twx^{-1}t^{-1}).$$

Since L is an ideal, $(L, \cdot)$ is a normal subgroup of $(A, \cdot)$ so we have $twx^{-1}t^{-1} \in L$ and since S is a subbrace we have $st^{-1} \in S$. Thus $(s + w)(t + v)^{-1} \in SL = S + L$. It follows that $S + L$ is a subbrace.

(ii) Suppose S is a left ideal. We have to show further that $\lambda_a(x) \in S + L$ whenever $a \in A$ and $x \in S + L$. Let $x = s + u$ for $s \in S, u \in L$. We have

$$\lambda_a(x) = ax - a = a(s + u) - a = as + au - a - a = \lambda_a(s) + \lambda_a(u)$$

and since S, L are both left ideals we have $\lambda_a(s) + \lambda_a(u) \in S + L$. Thus $S + L$ is a left ideal.

(iii) Suppose now that S is an ideal also. It remains to show that $(S + L, \cdot)$ is a normal subgroup of $(A, \cdot)$. But $S + L = SL$ and both $(S, \cdot), (L, \cdot)$ are normal subgroups of $(A, \cdot)$, so $(SL, \cdot)$ is a normal subgroup of $(A, \cdot)$ and this completes the proof. □

Lemma 3.3. *Let A be a left brace and let S be a subbrace of A. For each ordinal α, $S + \zeta_\alpha(\star, A)$ is an ideal of $S + \zeta_{\alpha+1}(\star, A)$.*

Proof. Without loss of generality we may assume that $\alpha = 1$. We already know from Lemma 3.2 that $S + \zeta_1(\star, A)$ is a subbrace of A. Furthermore, $S + \zeta_1(\star, A) = S\zeta_1(\star, A)$ is a normal subgroup of $S + \zeta_2(\star, A) = S\zeta_2(\star, A)$ so Proposition 2.1 shows that we only need to prove that $\lambda_a(z) \in S + \zeta_1(\star, A)$ for all $a \in S + \zeta_2(\star, A), z \in S + \zeta_2(\star, A)$. However $z = s + x$ for some $s \in S, x \in \zeta_1(\star, A)$ and $\lambda_a(s + x) = \lambda_a(s) + \lambda_a(x)$. Since $\zeta_1(\star, A)$ is an ideal of A we have $\lambda_a(x) \in \zeta_1(\star, A)$. Also, we may write $a = tw$ for some $t \in S, w \in \zeta_2(\star, A)$ so $\lambda_a(s) = tws - tw = t(ws - w) - t$. But $ws = w + s$ (mod $\zeta_1(\star, A)$) so $\lambda_a(s) \in S + \zeta_1(\star, A)$ as required. This concludes the proof. □

A subbrace S of a brace A will be called *ascendant* in A if there is a series of subbraces

$$S = A_0 \leq A_1 \leq A_2 \leq \cdots A_\alpha \leq A_{\alpha+1} \leq \cdots A_\gamma = A$$

in which A_α is an ideal of $A_{\alpha+1}$ for all $\alpha < \gamma$. If this series is finite, then we shall say that S is a *subideal* of A. Part (ii) of the next result is Proposition 5.6 of [9].

Lemma 3.4. *Let A be a left brace.*

(i) *If A is $\star$-hypercentral, then every subbrace is ascendant.*
(ii) *If A is Smoktunowicz-nilpotent, then every subbrace is a subideal.*
(iii) *If A is $\star$-hypercentral, then every maximal subbrace of A is an ideal.*

Proof. Let S be a subbrace of A and let

$$0 = A_0 \leq A_1 \leq A_2 \leq \cdots A_\alpha \leq A_{\alpha+1} \leq \cdots A_\gamma = A$$

be the upper $\star$-central series of A. Then

$$S = S + A_0 \leq S + A_1 \leq S + A_2 \leq \cdots S + A_\alpha \leq S + A_{\alpha+1} \leq \cdots A_\gamma = A.$$

By Lemma 3.3 each term of this series is an ideal of the next term. This implies both (i) and (ii). Assertion (iii) is immediate from (i). □

In preparation for the next result we prove a preliminary lemma.

Lemma 3.5. *Let A be a left brace and let $a, y \in A$. If $a \star y + y = 0$, then $y = 0$.*

Proof. We have $0 = a \star y + y = \lambda_a(y)$. Thus $ay = a$ and hence $y = 1 = 0$. □

We leave as an open question whether a sum of two $\star$-nilpotent ideals of a left brace is itself $\star$-nilpotent and whether the corresponding analogue of a result of P. Hall (see [8, Lemma 1]) holds. It also seems to be unclear whether there is an analogue of the Hirsch-Plotkin radical. We have been able, however, to prove the following result in pursuit of such goals.

Theorem 3.6. *Let A be a left brace and let B be an ideal of A. Then the $\star$-center of B is a normal subgroup of A under multiplication.*

Proof. Let a be an arbitrary element of A, let b be an arbitrary element of B and let z be an arbitrary element of $\zeta(\star, B)$. Then $b \star z = z \star b = 0$. Since B is an ideal of A we have $a^{-1}ba \in B$ and it follows that $(a^{-1}ba) \star z = 0$. On the other hand,

$$\begin{aligned}(a^{-1}ba) \star z &= a^{-1} \star (ba \star z) + ba \star z + a^{-1} \star z \\ &= a^{-1} \star (b \star (a \star z) + a \star z + b \star z) + b \star (a \star z) \\ &+ a \star z + b \star z + a^{-1} \star z \\ &= a^{-1} \star (b \star (a \star z) + a \star z) + b \star (a \star z) + a \star z \\ &+ a^{-1} \star z \\ &= a^{-1} \star (b \star (a \star z)) + a^{-1} \star (a \star z) + b \star (a \star z) \\ &+ a \star z + a^{-1} \star z.\end{aligned} \tag{3.1}$$

We also have

$$0 = 0 \star z = 1 \star z = a^{-1}a \star z = a^{-1} \star (a \star z) + a \star z + a^{-1} \star z. \tag{3.2}$$

It follows from equations (3.1) and (3.2) that

$$0 = (a^{-1}ba) \star z = a^{-1} \star (b \star (a \star z)) + b \star (a \star z)$$

and it follows from Lemma 3.5 that $b \star (a \star z) = 0$. We note that the center of the multiplicative group of B contains $\zeta(\star, B)$ and we now obtain

$$\begin{aligned}a^{-1}za \star b &= a^{-1} \star (za \star b) + za \star b + a^{-1} \star b \\ &= a^{-1} \star (z \star (a \star b) + a \star b + z \star b) + z \star (a \star b) + a \star b \\ &+ z \star b + a^{-1} \star b.\end{aligned} \tag{3.3}$$

Since B is an ideal of A we have $a \star b \in B$, so $z \star (a \star b) = 0$ and also $z \star b = 0$. Hence equation (3.3) becomes

$$a^{-1}za \star b = a^{-1} \star (a \star b) + a \star b + a^{-1} \star b = (a^{-1}a) \star b = 0 \star b = 0.$$

It follows that $(a^{-1}za)b = a^{-1}za + b = b + a^{-1}za$. Furthermore, since $aba^{-1} \in B$ we have

$$b(a^{-1}za) = a^{-1}(aba^{-1})za = a^{-1}z(aba^{-1})a = (a^{-1}za)b.$$

Consequently, $a^{-1}za \in \zeta(\star, B)$. This proves the result. □

Proposition 3.7. *Let A be a left brace. If A is $\star$-hypercentral, then A is locally Smoktunowicz-nilpotent.*

Proof. This follows using 2.10 and [14]. □

Proposition 3.8. *Let A be a left brace. Then A is $\star$-hypercentral if and only if for each element $a \in A$ and every pair of countable subsets $\{x_n | n \in \mathbb{N}\}, \{y_n | n \in \mathbb{N}\}$ of elements of A, there exist natural numbers k, t such that*

$$(\cdots(a \star x_1) \star x_2) \star \cdots) \star x_j = 0 \text{ for all } j \geq k$$

$$y_j \star (\cdots \star (y_3 \star (y_2 \star (y_1 \star a) \cdots) = 0 \text{ for all } j \geq t.$$

Proof. Let A be a $\star$-hypercentral brace and let

$$0 = C_0 \leq C_1 \leq \cdots C_\alpha \leq C_{\alpha+1} \leq \cdots C_\gamma = A$$

be the upper $\star$-central series of A. Let $a_j = (\cdots(a \star x_1) \star x_2) \star \cdots) \star x_j$ for all natural numbers j and suppose, for a contradiction, that $a_j \neq 0$ for all such j. It follows that $a_j \notin C_n$ for all $j, n \in \mathbb{N}$. Hence $a_j \notin C_\omega$ for all j. Let μ be the largest ordinal such that $a_j \notin C_\mu$ for all $j \in \mathbb{N}$. The choice of μ implies there is a natural number m such that $a_m \in C_{\mu+1}$. Then by definition of the upper $\star$-central series we have $a_{m+1} = a_m \star x_{m+1} \in C_\mu$ and we obtain a contradiction. Hence there is a natural number k such that

$$(\cdots(a \star x_1) \star x_2) \star \cdots) \star x_j = 0 \text{ for all } j \geq k.$$

Using a similar argument we also deduce the existence of a natural number t such that

$$y_j \star (\cdots \star (y_3 \star (y_2 \star (y_1 \star a) \cdots) = 0 \text{ for all } j \geq t.$$

To prove sufficiency of the condition we suppose, for a contradiction, that the $\star$-center of A is trivial. Clearly this will be sufficient to then prove Proposition 3.8. Let $a = a_0$ be an arbitrary non-zero element of A and suppose that $a \notin \mathrm{Soc}(A)$. Then there is an element $x_1 \in A$ such that $a_1 = a \star x_1 \neq 0$. If $a_1 \notin \mathrm{Soc}(A)$, then there is an element $x_2 \in A$ such that $a_2 = a_1 \star x_2 = (a \star x_1) \star x_2 \neq 0$. If we have constructed $x_1, x_2, \dots, x_n$ and $a_1, \dots, a_n$ such that $a_{i+1} = a_i \star x_{i+1} \neq 0$ for $i \leq n-1$ and if $a_n \notin \mathrm{Soc}(A)$, then there exists $x_{n+1} \in A$ such that $a_{n+1} = a_n \star x_{n+1} \neq 0$ and the construction proceeds. However, our condition implies that there exists n and elements $x_1, \dots, x_n$ such that

$$(\cdots(a \star x_1) \star x_2) \star \cdots) \star x_n \neq 0 \text{ and}$$

$$(\cdots(a \star x_1) \star x_2) \star \cdots) \star x_n) \star x = 0$$

for all elements $x \in A$. Thus

$$a_n = (\cdots(a \star x_1) \star x_2) \star \cdots) \star x_n \in \mathrm{Soc}(A).$$

Since $a_n \notin \zeta(\star, A)$, there is an element y_1 such that $b_1 = y_1 \star a_n \neq 0$. Since $\mathrm{Soc}(A)$ is an ideal of A this means $b_1 \in \mathrm{Soc}(A)$ and hence $b_1 \star x = 0$ for each element $x \in A$. Since $b_1 \notin \zeta(\star, A)$ there exists $y_2 \in A$ such that $b_2 = y_2 \star b_1 \neq 0$. In this way we construct an infinite family $\{y_n | n \in \mathbb{N}\}$ of elements such that $y_n \star (\cdots \star (y_3 \star (y_2 \star (y_1 \star a_n) \cdots) \neq 0$ for all natural numbers n. This contradicts our conditions. This contradiction shows that A has nontrivial $\star$-center. By transfinite induction this shows that A is a $\star$-hypercentral brace. □

Corollary 3.9. *Let A be a left brace. Then A is $\star$-hypercentral if and only if each of its countable subbraces is $\star$-hypercentral.*

Proof. If A is a $\star$-hypercentral brace, then every subbrace is $\star$-hypercentral. Hence every countable subbrace of A is $\star$-hypercentral.

Conversely, suppose that every countable subbrace of A is $\star$-hypercentral. Let a, x_n, y_n be arbitrary elements of A for $n \in \mathbb{N}$. Then the subbrace of A generated by the subset $\{a, x_n, y_n | n \in \mathbb{N}\}$ is countable and hence $\star$-hypercentral. By Proposition 3.8 there are natural numbers k, t such that

$$(\cdots(a \star x_1) \star x_2) \star \cdots) \star x_j = 0 \text{ for all } j \geq k$$

$$y_j \star (\cdots \star (y_3 \star (y_2 \star (y_1 \star a) \cdots) = 0 \text{ for all } j \geq t.$$

Then Proposition 3.8 implies that A is a $\star$-hypercentral brace. □

This result is a natural analogue of the corresponding result in group theory obtained by S. N. Chernikov in [6].

The proof of Theorem 1.1 is now easy to accomplish.

Proof of Theorem 1.1. We note that the center of the multiplicative group of a left brace A contains the $\star$-center of A. It follows that the upper $\star$-central series of A is an ascending central series of the multiplicative group of A. It follows that the multiplicative group of a $\star$-hypercentral left brace is hypercentral. Assertion (ii) follows from Proposition 3.8 while (iii) is immediate from Corollary 3.9. Assertion (iv) follows from Lemma 3.4. Assertion (v) follows from Proposition 3.7. □

4. The torsion subgroups of the additive and multiplicative groups of left braces

Suppose that A is a $\star$-hypercentral left brace. The upper $\star$-central series is a central series of the multiplicative group of A and hence the multiplicative group of A is a hypercentral group. Thus the set of elements of finite order in the multiplicative group of A is a characteristic subgroup of $(A, \cdot)$. The set of elements of finite order in the additive group of A is also a characteristic subgroup of $(A, +)$ and there is a natural question concerning the relationship between these two subgroups.

Proposition 4.1. *Let A be a left brace. Then the torsion part of tthe additive group of A is a left ideal of A.*

Proof. Let T be the torsion part of the additive group of A and let x be an arbitrary element of T. There is a natural number n such that $nx = 0$. Since $a(b+c) = ab + ac - a$ and $a^{-1}a = 0$ for all $a, b, c \in A$ we have

$$0 = x^{-1}x = x^{-1}((n+1)x) = -nx^{-1}$$

and it follows that $x^{-1} \in T$.

Suppose in addition that $y \in T$ and $ky = 0$ for some $k \in \mathbb{N}$. Then $x \star nky = x \star 0 = 0$. On the other hand, using Lemma 2.2

$$x \star nky = nk(x \star y) = nk(xy - x - y) = nk(xy).$$

Thus $nk(xy) = 0$, so $xy \in T$ and it follows that T is a group under multiplication. Thus T is a subbrace of A. Moreover, by Lemma 2.2, $n(a \star x) = a \star nx = a \star 0 = 0$ so $a \star x \in T$ for all $a \in A$. Thus T is a left ideal of A, as required. □

Lemma 4.2. *Let A be a left brace and let x be an element of A such that $x \in \zeta_k(\star, A)$, for some natural number k. If x has finite order n in the additive group of A, then x has finite order dividing n^k in the multiplicative group of A.*

Proof. We use induction on k. If $k = 1$, then $\zeta_1(\star, A)$ is a trivial brace so the order of the an element x in the additive group of A coincides with the order of x in the multiplicative group of A.

Suppose now that $k > 1$ and let T denote the torsion part of the additive group of A. It follows from Proposition 4.1 that $x^r \in T$ for all integers r. Using Lemma 2.2 we obtain

$$0 = x^r \star nx = n(x^r \star x) = n(x^{r+1} - x^r - x)$$

for all $r \in \mathbb{N} \cup \{0\}$ so that for all such r we have $nx^r = nx^{r+1}$. Hence $0 = nx = nx^2 = \cdots = nx^r$. Thus x^r has additive order dividing n. Since $x \in \zeta_k(\star, A)$ we have $x^n \equiv nx \pmod{\zeta_{k-1}(\star, A)}$ and since $nx = 0$ we have $x^n \in \zeta_{k-1}(\star, A)$. As we noted above $nx^n = 0$ so we may apply the induction hypothesis which now gives the result. □

We here record the following extension of this result to $\star$-hypercentral braces as follows.

Corollary 4.3. *Let A be a $\star$-hypercentral left brace and let T, S be the respective torsion parts of the additive and multiplicative groups of A. Then T is a subset of S.*

Proof. We use transfinite induction on the length of the upper $\star$-central series for A. We let $\zeta_\beta(\star, A)$ denote the terms of the upper $\star$-central series. Let x be an element of T of additive order $n \in \mathbb{N}$. It is clear that if $x \in \zeta_1(\star, A)$, then $x^n = 1$, so $x \in S$ in this case.

Suppose next that α is an ordinal and that $x \in \zeta_\alpha(\star, A)$. Suppose further that if $\beta < \alpha$, then $T \cap \zeta_\beta(\star, A) \subseteq S \cap \zeta_\beta(\star, A)$.

If α is a limit ordinal, then $x \in \zeta_\beta(\star, A)$ for some $\beta < \alpha$ so inductively it follows that $x \in S$. Thus we may assume that $\alpha - 1$ exists. Then, by Proposition 4.1, $x^n \in T$. But also $x^n \equiv nx \pmod{\zeta_{\alpha-1}(\star, A)}$ so $x^n \in \zeta_{\alpha-1}(\star, A)$. Thus, by the induction hypothesis, $x^n \in S$ and it follows that there exists $k \in \mathbb{N}$ such that $x^{nk} = 1$. Hence $x \in S$, which proves the result. □

It is clear that if A is a trivial left brace, then the torsion part of the additive group of A coincides with the torsion part of the multiplicative group of A. We next see that this also holds for Smoktunowicz-nilpotent braces, a result due to Jespers, van Antwerpen and Vendramin [9], who prove it in the more general case of skew braces. Our proof is a little different. We note that in general if n, k are natural numbers and if $a, b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_k$ are elements of a left brace A, then

$$a(b_1 + \cdots + b_n - c_1 - \cdots - c_k) = ab_1 + ab_2 + \cdots + ab_n - ac_1 - \cdots - ac_k + (k - n + 1)a.$$

We use this fact in the proof of the next result. We also note that in the binomial expansion of $(1+(-1))^k$, if we write $l_i = \binom{k}{i}$ for $i = 0, 1, \dots, k$, then for odd values of k we have $-l_1 + l_2 - \cdots + l_{k-1} = -l_0 + l_k = 0$.

Theorem 4.4. *Let A be Smoktunowicz-nilpotent left brace. Then the torsion part of the additive group of A coincides with the torsion part of the multiplicative group of A.*

Proof. Let T denote the torsion part of the additive group of A and let S denote the torsion part of the multiplicative group of A. We have already seen in Corollary 4.3 that T is a subset of S.

Let $x \in S$ and suppose that $x \in \zeta_k(\star, A)$. Choose $n \geq k$ such that $x^n = 1$ (this avoids special cases later). Then it follows that $nx \in \zeta_{k-1}(\star, A)$. Thus, $x \star nx = n(x \star x) = n(x^2 - 2x)$. Let $x_1 = x \star x$. Then $nx_1 \in \zeta_{k-2}(\star, A)$. In general, for each r with $1 \leq r \leq k$, let $x_r = x \star x_{r-1}$ and note that $nx_r = x \star nx_{r-1} \in \zeta_{k-r-1}(\star, A)$. Using the definition of $\star$ an easy induction argument shows that

$$x_{r-1} = x^r - rx^{r-1} + \binom{r}{2} x^{r-2} - \binom{r}{3} x^{r-3} + \cdots + (-1)^{r-1} rx,$$

where $\binom{r}{s}$ is a binomial coefficient. However $nx_{k-1} = 0$ which implies that

$$nx^k = nkx^{k-1} - n\binom{k}{2} x^{k-2} + n\binom{k}{3} x^{k-3} + \cdots + (-1)^k nkx.$$

Let $l_i = \binom{k}{i}$ for $1 \leq i \leq k$ so that we have

$$nx^k = nl_1 x^{k-1} - nl_2 x^{k-2} + nl_3 x^{k-3} + \cdots + (-1)^k nl_{k-1} x. \tag{4.1}$$

Note that $x(nx^k) = nx^{k+1} - (n-1)x$ and

$$x(nl_1x^{k-1} - nl_2x^{k-2} + nl_3x^{k-3} + \cdots + (-1)^k nl_{k-1}x) =$$
$$nl_1x^k - nl_2x^{k-1} + nl_3x^{k-2} + \cdots + (-1)^k nl_{k-1}x^2 + tx$$

where $t = 1$ if k is odd or $t = 1 - 2n$ if k is even, using the binomial expansion of $(1 + (-1))^k$. In turn this implies that

$$nx^{k+1} = nl_1x^k - nl_2x^{k-1} + \cdots + (-1)^k nl_{k-1}x^2 + (n-1+t)x.$$

Using Equation (4.1) we may replace nx^k in this expression. Thus we may write $nx^{k+1} = np_k(x)$ where $p_k(x)$ is an integer polynomial in x of degree at most $k-1$. Continuing in this manner we eventually obtain $nx^n = np_{n-1}(x)$ where again, $p_{n-1}(x)$ is a polynomial of degree at most $k-1$. However $x^n = 1$ so $np_{n-1}(x) = 0$. Using this equation we may therefore write some positive multiple of nx^{k-1} in the form $nmx^{k-1} = nq(x)$ where $q(x)$ is now a polynomial in x of degree at most $k-2$. We repeat the process above, reducing the degree of the polynomial at each stage. Eventually this process terminates with an equation of the form $Lx^2 = Mx$, where $L, M \in \mathbb{Z}$. We repeat the process of multiplying both sides by x and replacing another $n-2$ times if necessary before ending with an equation of the form $Nx = 0$ for some $N \in \mathbb{N}$ so that x has finite order as an element of T. This shows that every element of S is an element of T and this, together with Corollary 4.3, proves that $S = T$. ☐

Corollary 4.5. *Let A be a $\star$-hypercentral left brace. Then the torsion part of the additive group of A coincides with the torsion part of the multiplicative group of A.*

Proof. Let T, S be the respective torsion parts of the additive and multiplicative groups of A. We saw in Corollary 4.3 that $T \subseteq S$. On the other hand let $x \in S$ and let C be the subbrace of A generated by x. Then Theorem 1.1(v) shows that C is $\star$-nilpotent. But it is clear that x lies in the torsion part of the multiplicative group of C and using Theorem 4.4 we deduce that x is an element of the torsion part of the additive group of C. Hence $x \in T$ and the result follows. ☐

Corollary 4.6. *Let A be a $\star$-hypercentral left brace. Then the torsion part of the additive group of A is an ideal of A.*

Proof. Let T be the torsion part of the additive group of A. By Proposition 4.1 T is a left ideal of A. It follows that T is a subgroup of the multiplicative group of A. Theorem 1.1 shows that the multiplicative group of A is hypercentral. By Corollary 4.5 T is the torsion part of the multiplicative group of A. It follows that T is a characteristic subgroup of the multiplicative group of A and in particular T is a normal subgroup of A. This proves that T is an ideal of A. ☐

We deduce the following result which occurs in [9].

http://dx.doi.org/10.22108/ijgt.2023.138490.1860

Corollary 4.7. *Let A be a Smoktunowicz-nilpotent left brace. Then the torsion part of the additive group of A is an ideal of A.*

Theorem 4.8. *Let A be a $\star$-hypercentral left brace. If the additive group of A is torsion-free, then the additive group of the factor brace $A/\zeta_\alpha(\star, A)$ is torsion-free for each ordinal α.*

Proof. Let

$$0 = Z_0 \leq Z_1 \leq Z_2 \leq \cdots Z_\alpha \leq Z_{\alpha+1} \leq \cdots Z_\gamma = A$$

be the upper $\star$-central series of A.

Let $\alpha \geq 1$ and suppose that we have already proved that the additive group of A/Z_β is torsion-free for every ordinal $\beta < \alpha$.

Suppose, on the contrary, that the additive group of the factor brace A/Z_α is not torsion-free. Let T/Z_α denote the torsion part of the additive group of A/Z_α. Since A/Z_α is $\star$-hypercentral Corollary 4.6 implies that T/Z_α is an ideal of A/Z_α. Using Corollary 2.5 we deduce that the intersection $T_1/Z_\alpha = T/Z_\alpha \cap Z_{\alpha+1}/Z_\alpha$ is nonzero. It follows that there exists $x \in Z_{\alpha+1} \setminus Z_\alpha$ such that $x + Z_\alpha \in T/Z_\alpha \cap Z_{\alpha+1}/Z_\alpha$. Then $nx \in Z_\alpha$ for some natural number n.

Suppose first that α is a limit ordinal. Then $nx \in Z_\beta$ for some ordinal $\beta < \alpha$. Then $x \notin Z_\beta$ but $nx \in Z_\beta$ so the additive group of A/Z_β has an element of finite order, contrary to our assumption. Hence α is not a limit ordinal. Then $\alpha - 1$ exists and the induction hypothesis gives that the additive group of $A/Z_{\alpha-1}$ is torsion-free.

If $a \in A$, then $a \star x \in Z_\alpha$ and Lemma 2.2 implies that $n(a \star x) = a \star nx \in Z_{\alpha-1}$. Since $A/Z_{\alpha-1}$ is torsion-free it follows that $a \star x \in Z_{\alpha-1}$.

Now consider $x \star a$. Since $x \star x \in Z_{\alpha-1}$, by the above, it follows that $x^2 \equiv 2x \pmod{Z_{\alpha-1}}$ and hence that $x^n \equiv nx \pmod{Z_{\alpha-1}}$. Thus $nx = x^n u$ for some element $u \in Z_{\alpha-1}$. Using Lemma 2.2 we obtain

$$nx \star a = x^n u \star a = x^n \star (u \star a) + (x^n \star a) + (u \star a) \equiv x^n \star a \pmod{Z_{\alpha-1}}$$

On the other hand, again using Lemma 2.2,

$$x^2 \star a = x \star (x \star a) + 2(x \star a) \equiv 2(x \star a) \pmod{Z_{\alpha-1}}$$

and a straightforward induction shows that for each natural number j we have

$$x^j \star a \equiv j(x \star a) \pmod{Z_{\alpha-1}}.$$

In particular, we obtain

$$nx \star a \equiv x^n \star a \equiv n(x \star a) \pmod{Z_{\alpha-1}}.$$

Thus if $x \star a \notin Z_{\alpha-1}$, then$x \star a + Z_{\alpha-1}$ is an element of finite order in $A/Z_{\alpha-1}$ contrary to our hypothesis. Thus $x \star a \in Z_{\alpha-1}$ also and we see that actually $x \in Z_\alpha$, contrary to our choice of x. The result now follows. □

Corollary 4.9. *Let A be a $\star$-hypercentral left brace. If the additive group of A is torsion-free, then the additive group of every factor of the upper $\star$-central series is torsion-free.*

***Proof of Theorem* 1.2.** Assertion (i) follows from Corollary 4.5. Assertion (ii) follows from Corollary 4.6.and (iii) follows from Theorem 4.8. Finally assertion (iv) follows from Corollary 4.9. □

5. Locally Smoktunowicz-nilpotent braces

By Proposition 2.10 every $\star$-nilpotent left brace is Smoktunowicz-nilpotent and conversely every Smoktunowicz-nilpotent left brace is $\star$-nilpotent. Here we use both terms and likewise we shall use the terms locally $\star$-nilpotent and locally Smoktunowicz-nilpotent interchangeably.

Proposition 5.1. *Let A be a locally $\star$-nilpotent left brace. Then the multiplicative group of A is locally nilpotent and the torsion part of the additive group of A coincides with the torsion part of the multiplicative group of A.*

Proof. Let T, S be the respective torsion parts of the additive and multiplicative groups of A.

Let M be an arbitrary finite subset of A and let B denote the subgroup of the multiplicative group of A, generated by M. Let C denote the subbrace of A generated by M. Then C is $\star$-nilpotent and hence the multiplicative group of C is nilpotent. But B is a subgroup of the multiplicative group of C so is nilpotent. Hence the multiplicative group of A is locally nilpotent.

Suppose that $x \in T$ and let D denote the subbrace generated by x. Then D is $\star$-nilpotent. Let T_D and S_D denote the respective additive and multiplicative torsion parts of the brace B. Then $x \in T \cap D$. But clearly $T \cap D \subseteq T_D = S_D$, using Theorem 4.4. Hence x has finite multiplicative order so $x \in S$ and it follows that $T \subseteq S$. On the other hand, if $x \in S$ again consider the subbrace D generated by x. We have $x \in S \cap D \subseteq S_D \leq T_D$ and hence $x \in T$. The result follows. □

This has the following consequence which is proved for Smoctunowicz-nilpotent braces in [9].

Theorem 5.2. *Let A be locally $\star$-nilpotent left brace. The torsion part of the additive group of A is an ideal of A.*

Proof. Let T denote the torsion part of the additive group of A. By Proposition 4.1 T is a left ideal of A. It follows that T is a subgroup of the multipicative group of A. Proposition 5.1 shows that the multiplicative group of A is locally nilpotent. It follows that T is a characteristic subgroup of the multiplicative group of A and in particular T is a normal subgroup. The result now follows. □

The next result, generalizing a result in [9], follows immediately from Proposition 5.1.

Corollary 5.3. *Let A be a locally $\star$-nilpotent left brace. Then the additive group of A is torsion-free if and only if the multiplicative group of A is torsion-free.*

Lemma 5.4. *Let A be a locally $\star$-nilpotent left brace. If M is a minimal ideal of A, then the $\star$-center of A contains M. In particular, M has prime order.*

Proof. Suppose that the $\star$-center of A does not contain M. Then there exist elements $a \in M$ and $x \in A$ such that either $b = a \star x \neq 0$ or $c = x \star a \neq 0$. Suppose first that $b \neq 0$. Since M is an ideal $b \in M$ and the fact that M is a minimal ideal implies that the ideal W, generated by b, coincides with M. In particular the ideal generated by b contains a. It follows that there are elements $y_1, y_2, \dots, y_n$ such that a belongs to the ideal generated by b in the subbrace S generated by $b, x, y_1, \dots, y_n$. Let D be the ideal generated by a in S. Clearly $b = a \star x \in D \star S$. Proposition 2.6 implies that $D \star S$ contains the ideal generated by b in S. Hence $a \in D \star S$, so $D = D \star S$. On the other hand D is an ideal of the $\star$-nilpotent brace S so D has a finite series of ideals of S whose factors are $\star$-central in S. In particular it follows that $D \neq D \star S$ and we obtain a contradiction.

Suppose now that $a \star x = 0$ for all elements $a \in M$ and $x \in A$. In particular, the socle of A contains M. In this case $c = x \star a \neq 0$. Since M is an ideal we have $c \in M$. Since M is a minimal ideal the ideal generated by c in A coincides with M. In particular the ideal generated by c contains a. It follows that there are elements $y_1, \dots, y_n$ such that a belongs to the ideal generated by c in the subbrace S generated by $c, x, y_1, \dots, y_n$. Let L be the ideal in S generated by a. Then $\mathrm{Soc}(A)$ contains L.

However, $c = x \star a \in S \star L$. Proposition 2.6 implies that $S \star L$ is an ideal of S. Then $S \star L$ contains the ideal generated by c in the subbrace S. It follows that $a \in S \star L$. Then $L = S \star L$. On the other hand, L is an ideal of the $\star$-nilpotent subbrace S. Hence L has a finite series of ideals whose factors are all $\star$-central in A. Thus $L \neq S \star L$ and we obtain a contradiction. The results follow. □

A factor C/B of a left brace A is *A-chief* if it contains no nontrivial proper ideals of A/B. We note the following analogue of a well-known result from group theory.

Theorem 5.5. *Let A be a locally $\star$-nilpotent left brace.*

(i) *If $B \leq C$ are ideals of A such that the factor C/B is A-chief, then C/B is $\star$-central in A. In particular C/B has prime order.*

(ii) *If L is a maximal subbrace of A, then L is an ideal of A.*

Proof. (i) Clearly C/B is a minimal ideal of the factor brace A/B. Then Lemma 5.4 implies that C/B is $\star$-central and has prime order.

(ii) Clearly we may suppose that L is not an ideal of A. Since L is a maximal subbrace of A it follows that A is generated by L and every element $x \notin L$.

Assume now that L is not an ideal of A. Then clearly L does not contain $A \star A$ and hence there exists an element $x \in A \star A \setminus L$. The fact that L is a maximal subbrace of A implies that A is generated by L and x. Since $x \in A \star A$, there is a finite subset of M such that $x \in \mathrm{br}(M) \star \mathrm{br}(M)$. If $y \in M$, then as $A = \mathrm{br}(L \cup \{x\})$, there is a finitely generated subbrace H_y of L such that $y \in \mathrm{br}(H_y \cup \{x\})$. Let $H = \mathrm{br}(H_y | y \in M)$. Then H is finitely generated and $M \subseteq B = \mathrm{br}(H \cup \{x\})$. Since B is finitely generated it is $\star$-nilpotent. Since $H \leq L$ we have $x \notin H$. Among all the subbraces C of B containing H and not x choose a maximal subbrace D. Because $B = \mathrm{br}(H \cup \{x\})$ we see that D is a maximal subbrace of B. Since B is $\star$-nilpotent Lemma 3.4 implies that D is an ideal of B. Clearly, we then have $B \star B \leq D$. On the other hand, the inclusion $M \subseteq B$ implies that $\mathrm{br}(M) \star \mathrm{br}(M) \leq B \star B$. Since $x \in \mathrm{br}(M) \star \mathrm{br}(M)$ we have $x \in B \star B \leq D$, which contradicts the choice of D. This contradiction proves the result. □

The proof of Theorem 1.3 is now easy to accomplish.

Proof of Theorem 1.3. Assertion (i) follows from Proposition 5.1 and Assertion (ii) follows from Theorem 5.2. Assertion (iii) follows from Corollary 5.3 and (iv) follows from Theorem 5.5. □

Acknowledgments

The second author would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the Solidarity Supplementary Grant Program where work on this paper was undertaken. This work was supported by EPSRC grant no EP/R014604/1.& #34. The second author is grateful to the University of Edinburgh. The authors are sincerely grateful to Professor A. Smoktunowicz for useful consultations. The authors are also indebted to Marco Trombetti for his preprint [14] and to the referee for their careful reading of this paper and helpful suggestions.

References

[1] D. Bachiller, Classification of braces of order p^3, *J. Pure Appl. Algebra*, **219** (2015) no. 8 3568–3603.

[2] M. Bonatto and P. Jedlička, Central nilpotency of skew braces, *J. Algebra Appl.*, **22** (2023) no. 12 Paper No. 2350255.

[3] F. Cedó, Left braces: solutions of the Yang-Baxter equation, *Adv. Group Theory Appl.*, **5** (2018) 33–90.

[4] F. Cedó, T. Gateva-Ivanova and A. Smoktunowicz, On the Yang-Baxter equation and left nilpotent left braces, *J. Pure Appl. Algebra*, **221** (2017) no. 4 751–756.

[5] F. Cedó, A. Smoktunowicz and L. Vendramin, Skew left braces of nilpotent type, *Proc. Lond. Math. Soc. (3)*, **118** (2019) no. 6 1367–1392.

[6] S. N. Chernikov, On special p-groups, *Mat. Sbornik N. S.*, **27** no. 69 (1950) 185–200.

[7] I. Colazzo, M. Ferrara and M. Trombetti, On derived-indecomposable solutions of the Yang-Baxter equation, *ArXiv Math: 2210.08598*.

[8] P. Hall, The Frattini subgroups of finitely generated groups, *Proc. London Math. Soc. (3)*, **11** (1961) 327–352.

[9] E. Jespers, A. van Antwerpen and L. Vendramin, Nilpotency of skew braces and multipermutation solutions of the Yang-Baxter equation, *Commun. Contemp. Math.*, **25** 20 pp.

[10] W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation, *Adv. Math.*, **193** (2005) no. 1 40–55.

[11] ______, Braces, radical rings, and the quantum Yang-Baxter equation, *J. Algebra*, **307** (2007) no. 1 153–170.

[12] A. Smoktunowicz, On Engel groups, nilpotent groups, rings, braces and the Yang-Baxter equation, *Trans. Amer. Math. Soc.*, **370** (2018) no. 9 6535–6564.

[13] ______, Algebraic approach to Rump's results on relations between braces and pre-Lie algebras, *J. Algebra Appl.*, *21* (2022) no. 3 13 pp.

[14] M. Trombetti, The structure skew brace associated with a finite non-degenerate solution of the yang-baxter equation is finitely presented, to appear *Proc. American Math. Soc.*, (2023) 11 pp.

Martyn R. Dixon

Department of Mathematics, University of Alabama, Tuscaloosa, AL 35487-0350, U. S. A.

Email: mdixon@ua.edu

Leonid A. Kurdachenko

Department of Algebra, Facultet of mathematic and mechanik, National University of Dnepropetrovsk, Gagarin prospect 72, Dnipro 10 49010, Ukraine

Email: Lkurdachenko@i.ua, lkurdachenkogmail.com

Igor Ya. Subbotin

Department of Mathematics and Natural Sciences, National University,5245 Pacific Concourse Drive, Los Angeles, CA 90045-6904, U. S. A.

Email: isubboti@nu.edu

http://dx.doi.org/10.22108/ijgt.2023.138490.1860