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A GENERALIZATION OF THE CHERMAK–DELGADO MEASURE ON SUBGROUPS AND ITS ASSOCIATED LATTICE

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Abstract. We generalize the Chermak–Delgado measure of a subgroup of a finite group G, $\mu(H) = |H||C_G(H)|$, and its associated lattice of subgroups with maximal measure. We consider mappings M of the lattice of all subgroups $\mathrm{Sub}(G)$ into itself and define a measure associated to M by setting $\mu(H) = |H||M(H)|$. We investigate under what conditions on M the subgroups with maximal measure form a sublattice of $\mathrm{Sub}(G)$. In particular, our focus is on the case where $M(H)$ is a centralizer-like subgroup.

1. Introduction

In 1989, Chermak and Delgado [4] introduced a family of measures for subgroups of a finite group G that acts on another finite group. In the special case where G acts on itself by conjugation, then for any positive real number r and for every $H \leq G$ we define

$$\mu_r(H) = |H|^r|C_G(H)|.$$

We then let $m_r = \max\{\mu_r(H) \mid H \leq G\}$, and consider all subgroups H for which $\mu_r(H) = m_r$. This collection forms a lattice of subgroup of G, and Chermak and Delgado prove a number of interesting results relating to this lattice. Particular attention has been paid to the lattice for $r = 1$, and we will

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refer to μ_1 and the corresponding lattice as the *Chermak–Delgado measure* and the *Chermak–Delgado lattice* of a group, respectively.

Many of the interesting properties of the Chermak–Delgado lattice are described in Isaacs' book [7], and several authors have explored the properties of the Chermak–Delgado lattice. See e.g., [1–3, 5, 10–13].

We seek to explore generalizations of the function μ_1 by considering subgroups other than $C_G(H)$. Let G be a finite group, and $\mathrm{Sub}(G)$ be the lattice of subgroups of G. Consider $M\colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$, a mapping of the lattice of subgroups of G into itself. We may then use M to define a measure $\mu\colon \mathrm{Sub}(G) \to \mathbb{Z}^+$, the positive integers, by $\mu(H) = |H|\,|M(H)|$. The question arises under what conditions on M will the subgroups with maximal measure form a lattice. Theorem 2.2 gives an answer to this question. The three examples following that theorem give cases where M leads to a lattice.

Chermak and Delgado in [4] (see also [7]) show that the least element in the Chermak–Delgado lattice is a characteristic abelian subgroup of G. As shown in Example 2.6, the least element in these more general settings may be neither characteristic nor normal.

Our main focus is the case where $M(H)$ is a centralizer-like subgroup, as given in Theorem 3.1. Not in all cases will this lead to a lattice, as Example 3.8 shows. However, in some situations we do obtain a lattice, e.g., for the word $w(x, y) = x^a y^b$, where a and b are integers (see Proposition 3.5). We explore various other words, where we obtain lattices, sometimes under additional assumptions on the structure of G.

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2. CD-Admissible Functions and CD-measures

Let G be a finite group, and denote by $\mathrm{Sub}(G)$ the lattice of subgroups of G. In this section we introduce CD-measures, i.e., mappings from $\mathrm{Sub}(G)$ to the positive integers with the property that the subgroups of maximal measure form a sublattice of $\mathrm{Sub}(G)$. We also define CD-admissible functions, which are functions $M\colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ for which $\mu(H) = |H|\,|M(H)|$ is a CD-measure. In Theorem 2.2 we establish a sufficient condition for M to be CD-admissible.

Definition 2.1. *Let G be a group, and let $\mu\colon \mathrm{Sub}(G) \to \mathbb{Z}^+$ be a function assigning a positive integer to every subgroup of G. Let $m = \max\{\mu(H) \mid H \in \mathrm{Sub}(G)\}$. We say that μ is a CD-measure on the subgroups of G if and only if the set*

$$\mathrm{CD}_\mu(G) = \{H \in \mathrm{Sub}(G) \mid \mu(H) = m\}$$

forms a sublattice of $\mathrm{Sub}(G)$. In that case, we say that $\mathrm{CD}_\mu(G)$ is the CD-lattice of G associated to μ.

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To avoid a proliferation of indices, if μ is clear from context, we will denote the CD-lattice associated to μ simply by $CD(G)$.

We are interested in CD-measures μ that are derived in a manner analogous to the Chermak–Delgado measure: by associating to each subgroup H of G some subgroup $M(H)$, and then defining $\mu(H) = |H|\,|M(H)|$. Moreover, we would like M to have properties that allow us to generalize properties of the Chermak–Delgado lattice to this more general setting. The following theorem gives sufficient conditions that will ensure a function M has these properties.

Theorem 2.2. *Let G be a finite group and let $M\colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ be a function such that for all subgroups $H, K \in \mathrm{Sub}(G)$ we have:*

(i) *If $H \leq K$, then $M(K) \leq M(H)$; that is, M reverses inclusions.*
(ii) *$M(H) \cap M(K) \leq M(\langle H, K\rangle)$.*

Then $\mu(H) = |H|\,|M(H)|$ is a CD-measure on G. Moreover, given subgroups $H, K \in \mathrm{CD}_\mu(G)$ we have:

(iii) *$\langle H, K\rangle = HK$;*
(iv) *$M(H \cap K) = \langle M(H), M(K)\rangle$;*
(v) *$\langle M(H), M(K)\rangle = M(H)M(K)$.*

Proof. First, consider arbitrary subgroups H and K of G (not necessarily of maximal measure). Note that $H \cap K \leq H, K \leq \langle H, K\rangle$ and therefore we have

$$M(\langle H, K\rangle) \leq M(H) \cap M(K) \leq \langle M(H), M(K)\rangle \leq M(H \cap K), \tag{2.1}$$

since M reverses inclusions (condition (i)). Together with condition (ii), this yields for every H and K

$$M(H) \cap M(K) = M(\langle H, K\rangle). \tag{2.2}$$

Next we will show that

$$\mu(H \cap K)\mu(\langle H, K\rangle) \geq \mu(H)\mu(K) \tag{2.3}$$

always holds for any subgroups H and K.

By (2.1), we obtain

$$\mu(H \cap K) = |H \cap K|\,|M(H \cap K)| \geq |H \cap K|\,|\langle M(H), M(K)\rangle|. \tag{2.4}$$

Since $|\langle M(H), M(K)\rangle| \geq |M(H)M(K)|$, we conclude that

$$|H \cap K|\,|\langle M(H), M(K)\rangle| \geq |H \cap K|\,|M(H)M(K)|, \tag{2.5}$$

and hence (2.4) implies

$$\mu(H \cap K) \geq |H \cap K|\,|M(H)M(K)|. \tag{2.6}$$

We have

$$|H\cap K|\,|M(H)M(K)| = \frac{|H|\,|K|}{|HK|}\cdot\frac{|M(H)|\,|M(K)|}{|M(H)\cap M(K)|}; \tag{2.7}$$

since $|M(\langle H,K\rangle)| \geq |M(H)\cap M(K)|$ by (ii), then by (2.1) and $|HK| \leq |\langle H,K\rangle|$ we obtain

$$\frac{|H|\,|K|}{|HK|}\cdot\frac{|M(H)|\,|M(K)|}{|M(H)\cap M(K)|} \geq \frac{|H|\,|K|}{|\langle H,K\rangle|}\cdot\frac{|M(H)|\,|M(K)|}{|M(\langle H,K\rangle)|}. \tag{2.8}$$

We observe that (2.6), (2.7), and (2.8) imply

$$\mu(H\cap K) \geq \frac{\mu(H)\mu(K)}{\mu(\langle H,K\rangle)}, \tag{2.9}$$

which proves (2.3).

To prove that μ is a CD-measure, let $H,K \in \mathrm{CD}_\mu(G)$, i.e., $\mu(H) = \mu(K) = m$. Then it follows from (2.3) that

$$m^2 = \mu(H)\mu(K) \leq \mu(H\cap K)\mu(\langle H,K\rangle) \leq m^2,$$

and we conclude that $\mu(H\cap K) = \mu(\langle H,K\rangle) = m$. Therefore, if $H,K\in\mathrm{CD}_\mu(G)$, then we also have $\langle H,K\rangle, H\cap K \in \mathrm{CD}_\mu(G)$, proving that $\mathrm{CD}_\mu(G)$ is a sublattice of $\mathrm{Sub}(G)$. Thus, μ is a CD-measure, as claimed.

Next we will prove that (iii) holds. If $H,K\in\mathrm{CD}_\mu(G)$, equality holds in (2.9), and therefore in (2.8). Cancellation and inversion lead to

$$|HK|\,|M(H)\cap M(K)| = |\langle H,K\rangle|\,|M(\langle H,K\rangle)|.$$

Using (2.2), we obtain $|HK| = |\langle H,K\rangle|$. Since $HK \subseteq \langle H,K\rangle$, we conclude that $HK = \langle H,K\rangle$, proving (iii).

To obtain (iv), we observe that (2.5), (2.7), (2.8), and (2.9) yield

$$|H\cap K|\,|\langle M(H),M(K)\rangle| \geq \frac{\mu(H)\mu(K)}{\mu(\langle H,K\rangle)} = m = \mu(H\cap K) = |H\cap K|\,|M(H\cap K)|.$$

We conclude that $|\langle M(H),M(K)\rangle| \geq |M(H\cap K)|$. This, together with (2.1) shows that (iv) holds.

Finally, we will show that (v) holds. By (2.7) and (2.8) it follows that

$$|H\cap K|\,|M(H)M(K)| \geq m = |H\cap K|\,|M(H\cap K)|.$$

Using (iv), this leads to

$$|H\cap K|\,|M(H)M(K)| \geq |H\cap K|\,|\langle M(H),M(K)\rangle|.$$

Since $M(H)M(K) \subseteq \langle M(H),M(K)\rangle$, we must have

$$|M(H)M(K)| \leq |\langle M(H),M(K)\rangle|.$$

Hence we conclude that $M(H)M(K) = \langle M(H),M(K)\rangle$, establishing (v). $\square$

For simplicity, let us say that a function $M\colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ that satisfies properties (i) and (ii) in Theorem 2.2 is *CD-admissible* (or just "admissible") on G. To illustrate the theorem, we give some examples of admissible functions M and their corresponding CD-measures and CD-lattices.

Example 2.3. *The choice $M(H) = C_G(H)$ satisfies conditions (i) and (ii), and yields the original Chermak–Delgado measure and Chermak–Delgado lattice of a group.*

Example 2.4. *We can pick an arbitrary subgroup A of G, and define $M(H) = A$ for all H. This satisfies conditions (i) and (ii), and therefore yields a CD-measure on G; in this case, however, the resulting CD-lattice consists only of G itself.*

Example 2.5. *Let G be any group, and let A and B be subgroups of G. Define*

$$M(H) = \begin{cases} B & \text{if } H \leq A, \\ \{e\} & \text{otherwise.} \end{cases}$$

Note that M reverses inclusion: if $H \leq K$, either $M(K) = \{e\}$, hence is contained in $M(H)$, or $M(K) = B$, which means $H \leq K \leq A$, and $M(H) = B = M(K)$. Thus, M satisfies property (i). For property (ii), if $M(H) \cap M(K)$ is trivial, then it is contained in $M(\langle H, K\rangle)$. If $M(H) \cap M(K)$ is not trivial, then we must have $M(H) = M(K) = B$, and hence $H, K \leq A$. This implies $\langle H, K\rangle \leq A$, so $M(H) \cap M(K) = B = M(\langle H, K\rangle)$. Thus, M defines a CD-measure on G, which is explicitly given by:

$$\mu(H) = \begin{cases} |H|\,|B| & \text{if } H \leq A, \\ |H| & \text{otherwise.} \end{cases}$$

From this we have that if $|A|\,|B| < |G|$, then the corresponding CD-lattice contains only the whole group G. If $|A|\,|B| > |G|$, then $\mathrm{CD}_\mu(G) = \{A\}$. And if $|A|\,|B| = |G|$, then the CD-lattice contains precisely A and G.

The Chermak–Delgado lattice of G has a number of interesting properties. For instance, that the least element of the lattice is a characteristic subgroup of G that contains $Z(G)$; that this subgroup is always abelian; that if H has maximal Chermak–Delgado measure, then so does $C_G(H)$, and $H = C_G(C_G(H))$. In particular, the map $H \longmapsto C_G(H)$ induces a lattice anti-automorphism of the Chermak–Delgado lattice, proving that this lattice is self-dual. We also wish to explore which, and under what conditions, these and related properties will generalize to CD-measures.

That the least element of the Chermak–Delgado lattice is a characteristic subgroup of G is a consequence of the well-known fact that for every $\phi \in \mathrm{Aut}(G)$, $\phi(C_G(H)) = C_G(\phi(H))$, and thus $\mu(H) = \mu(\phi(H))$ for all $H \in \mathrm{Sub}(G)$. That means that ϕ induces an automorphism of the Chermak–Delgado lattice, which must therefore send the least element to itself, proving that this subgroup is characteristic.

As we will see in the following examples, for an arbitrary function M this does not need to hold in general, and so we may not be able to conclude that the least element of the CD-lattice $\mathrm{CD}_\mu(G)$ will be characteristic, or even normal. First we provide an example where the least element of the lattice is not characteristic.

Example 2.6. *Let $G = C_2 \times C_2$ be the Klein 4-group, and let M be as in Example 2.5 with $A = B = C_2 \times \{e\}$. Here we have $|A|^2 = 4 = |G|$, so $\mathrm{CD}_\mu(G) = \{A, G\}$; the least element is normal, but not characteristic.*

Next we provide an example where the least element of the lattice is not even a normal subgroup.

Example 2.7. *Take $G = A_5$ and let M be as in Example 2.5 with $A = B$ the subgroup stabilizing 5; therefore $A = B \cong A_4$. Then $|A||B| = 144 > 60 = |G|$, so this time the CD-lattice consists only of A, which is not normal in G.*

In light of these examples, we note the following:

Proposition 2.8. *Let $M\colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ be CD-admissible on G with associated CD-measure μ.*

(i) *If for every $g \in G$ and $H \in \mathrm{CD}_\mu(G)$ we have $|M(H)| = |M(gHg^{-1})|$, then the least element of $\mathrm{CD}_\mu(G)$ is normal in G.*

(ii) *If for every $\phi \in \mathrm{Aut}(G)$ and $H \in \mathrm{CD}_\mu(G)$ we have $|M(H)| = |M(\phi(H))|$, then the least element of $\mathrm{CD}_\mu(G)$ is characteristic in G.*

Proof. If $H \in \mathrm{CD}_\mu(G)$ and $\phi \in \mathrm{Aut}(G)$ is such that $|M(H)| = |M(\phi(H))|$, then

$$\mu(H) = |H|\,|M(H)| = |\phi(H)|\,|M(\phi(H))| = \mu(\phi(H)),$$

hence $\phi(H) \in \mathrm{CD}_\mu(G)$.

Since automorphisms respect inclusions, the results follow. □

One additional interesting property of the least element A of the Chermak–Delgado lattice of a group G is that in addition to being abelian and characteristic, if B is *any* abelian subgroup of G, then $[G : A] \leq [G : B]^2$. The key to this result is the fact that the centralizer of a subgroup in the Chermak–Delgado lattice again lies in the lattice; for CD-admissible functions M, the corresponding property requires that if $H \in \mathrm{CD}_\mu(G)$, then $M(H) \in \mathrm{CD}_\mu(G)$. As the final theorem of this section shows, this will yield the analogous conclusion for CD-measures.

Theorem 2.9. *Let G be a finite group, and let $M\colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ be a CD-admissible function on G. Let $\mu(H) = |H|\,|M(H)|$ be the corresponding CD-measure on G, and $\mathrm{CD}_\mu(G)$ the resulting CD-lattice. If for every $H \in \mathrm{CD}_\mu(G)$ we have $M(H) \in \mathrm{CD}_\mu(G)$, then:*

(i) *The map $H \longmapsto M(H)$ defines a lattice anti-homomorphism of $\mathrm{CD}_\mu(G)$ with itself.*

(ii) *If K is the least element of $\mathrm{CD}_\mu(G)$, then $K \leq M(K)$.*

(iii) *If L is any subgroup of G such that $L \leq M(L)$, then $[G : K] \leq [G : L]^2$.*

Proof. From Theorem 2.2(iv) and (2.2) we have that $H \longmapsto M(H)$ is a lattice anti-homomorphism, which establishes (i).

If K is the least element of $\mathrm{CD}_\mu(G)$, then $M(K) \in \mathrm{CD}_\mu(G)$, and therefore $K \leq M(K)$, giving (ii).

Finally, let L be any subgroup of G satisfying $L \leq M(L)$. Then

$$\mu(K) \geq \mu(L) = |L|\,|M(L)| \geq |L|^2,$$

and therefore

$$[G:L]^2 = \frac{|G|^2}{|L|^2} \geq \frac{|G|^2}{\mu(K)} = \frac{|G|^2}{|K|\,|M(K)|} = \frac{|G|}{|K|} \cdot \frac{|G|}{M(K)} \geq \frac{|G|}{|K|} = [G:K],$$

which yields (iii). ☐

3. A family of subgroup functions

In this section we introduce a family of functions $M\colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$, and investigate under what conditions they are CD-admissible. These functions can be defined for any 2-letter word.

Recall that a *2-letter word* (in the language of group theory) $w(x_1, x_2)$ is an element of the free group F_2 of rank 2 in the generators x_1 and x_2. A word $w(x_1, x_2)$ defines for each G a function $w\colon G\times G \to G$ that sends the pair (g_1, g_2) to the "value" $w(g_1, g_2)$, that is, to the image of $w(x_1, x_2)$ under the morphism $F_2 \to G$ induced by mapping the free generators x_1 and x_2 to g_1 and g_2, respectively.

Theorem 3.1. *Let $w(x_1, x_2)$ be a 2-letter word. Given a group G and a subgroup H, we define*

$$^*w_1(H) = \{x \in G \mid w(xg, h) = w(g, h) \text{ for all } g \in G, h \in H\},$$

$$w_1^*(H) = \{x \in G \mid w(gx, h) = w(g, h) \text{ for all } g \in G, h \in H\},$$

$$^*w_2(H) = \{x \in G \mid w(h, xg) = w(h, g) \text{ for all } g \in G, h \in H\},$$

$$w_2^*(H) = \{x \in G \mid w(h, gx) = w(h, g) \text{ for all } g \in G, h \in H\}.$$

Then $^*w_1(H)$, $w_1^*(H)$, $^*w_2(H)$, and $w_2^*(H)$ are subgroups of G.

Proof. Consider $^*w_1(H)$ for a subgroup H of G, and let e be the identity of G. Then $w(eg, h) = w(g, h)$ for all $g \in G$ and $h \in H$. Thus, $e \in {}^*w_1(H)$. Let $x, x' \in {}^*w_1(H)$. Then $w(xx'g, h) = w(x'g, h) = w(g, h)$ for all $g \in G$ and $h \in H$. Thus, $xx' \in {}^*w_1(H)$. For $x \in {}^*w_1(H)$ and arbitrary $g \in G$ and $h \in H$, we have $w(g, h) = w(xx^{-1}g, h) = w(x^{-1}g, h)$. Hence $x^{-1} \in {}^*w_1(H)$ and $^*w_1(H)$ is a subgrup of G, as claimed. The proofs for $w_1^*(H)$, $^*w_2(H)$, and $w_2^*(H)$ are similar. ☐

These subgroups were introduced in [9] as "centralizer-like subgroups" in context with the 2-Engel word $w(x_1, x_2) = [x_1, x_2, x_2]$. We call $^*w_i(H)$ a *left relative ith margin of H in G*, and $w_i^*(H)$ a *right relative ith margin of H in G.* We will refer to them collectively as *relative margins.*

Although we have restricted the definition above to 2-letter words, it is clear that they can be defined for n-letter words. The definitions are related to the concept of left ith marginal subgroup of a group G with respect to an n-letter word w, which is defined to be the subgroup of all $x \in G$ for which

$$w(g_1, \ldots, g_{i-1}, xg_i, g_{i+1}, \ldots, g_n) = w(g_1, \ldots, g_i, \ldots, g_n)$$

for all $g_j \in G$, and similarly for right ith marginal subgroup of G; these groups were introduced by Philip Hall [6].

For an n-variable word w, the *marginal subgroup of G corresponding to w* is the intersection of the left and right ith marginal subgroups, as i ranges from 1 through n. This is a characteristic subgroup of G. For example, the marginal subgroup associated to the word $w(x_1, x_2) = [x_1, x_2]$ is the center of G.

Note that for a word w, the maps $H \mapsto {}^*w_i(H)$ and $H \mapsto w_i^*(H)$ define inclusion-reversing functions $\mathrm{Sub}(G) \to \mathrm{Sub}(G)$, so they are potential candidates for CD-admissible functions. Our investigation was in fact inspired by the first example below, which yields the original Chermak–Delgado measure and also a sublattice.

Example 3.2. *Let $w(x_1, x_2) = [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$. Then*

$$\begin{aligned} {}^*w_1(H) = {}^*w_2(H) &= C_G(H), \\ w_1^*(H) = w_2^*(H) &= C_G(H^G), \end{aligned}$$

where H^G is the normal closure of H in G. In particular, we have that

$$\begin{aligned} {}^*w_i(H) \cap {}^*w_i(K) &= {}^*w_i(\langle H, K\rangle), \\ w_i^*(H) \cap w_i^*(K) &= w_i(\langle H, K\rangle), \end{aligned}$$

so that these functions are CD-admissible. The left relative margins define the Chermak–Delgado measure, and so yields the original Chermak–Delgado lattice of G. The right relative margins yield the sublattice of the Chermak–Delgado lattice consisting of those subgroups that are normal in G, and in particular has the same least element (since the least element of the Chermak–Delgado lattice is characteristic in G, hence normal).

Before proceeding, we note that if any of these relative margins yield a CD-admissible function, then they will satisfy the condition in Proposition 2.8(ii). We restrict out attention to 2-letter words and finite groups, but the results will hold for n-letter words and arbitrary groups.

Proposition 3.3. *Let $w(x_1, x_2)$ be a word, and let G be a finite group. For every $\phi \in \mathrm{Aut}(G)$, we have*

$$\phi({}^*w_i(H)) = {}^*w_i(\phi(H)) \quad \text{and} \quad \phi(w_i^*(H)) = w_i^*(\phi(H)), \qquad i = 1, 2.$$

In particular, if $M(H) = {}^*w_i(H)$ or $M(H) = w_i^*(H)$ are CD-admissible, then the least element of the corresponding CD-lattice is characteristic in G.

Proof. The key observation is that for $x, y \in G$ we have $\phi(w(x,y)) = w(\phi(x), \phi(y))$. For arbitrary elements $g \in G$ and $h \in H$, we have $x \in {}^*w_1(H)$ if and only if $w(xg, h) = w(g, h)$. This holds exactly when $\phi(w(xg,h)) = \phi(w(g,h))$, which is equivalent to $w(\phi(xg), \phi(h)) = w(\phi(g), \phi(h))$. Replacing $\phi(xg)$ with $\phi(x)\phi(g)$, as g ranges over all $g \in G$ and h over all $h \in H$, we see that this equality holds if and only if $w(\phi(x)g', h') = w(g', h')$, for arbitrary $g' \in G$ and $h' \in \phi(H)$. Hence we conclude that $x \in {}^*w_1(H)$ if and only if $\phi(x) \in {}^*w_1(\phi(H))$, as claimed. Similar arguments establish the result for $w_1^*(H)$, ${}^*w_2(H)$, and $w_2^*(H)$. □

As an easy corollary to Proposition 3.3, we obtain the following Theorem:

Theorem 3.4. *Let $w(x_1, x_2)$ be a two letter word. If ${}^*w_i(H)$ (resp. $w_i^*(H)$) is CD-admissible on G, then the least element of the corresponding CD-lattice of G is a characteristic subgroup that contains the left ith margin (resp. right ith margin) of G, and in particular contains the marginal subgroup of G corresponding to w.*

We now give other examples where the relative margins yield CD-admissible functions. As we see in the next proposition, for some words we have that all four of ${}^*w_i$ and w_i^* are admissible and define CD-measures, but yield only a trivial lattice.

Proposition 3.5. *Let a and b be integers, and let $w(x_1, x_2) = x_1^a x_2^b$. Then the left and right ith relative margins ${}^*w_i$ and w_i^* for $i = 1, 2$ define CD-admissible functions in any group G, whose associated CD-lattices contain only G itself.*

Proof. Note that $(xg)^a h^b = g^a h^b$ if and only if $(xg)^a = g^a$ for all $g \in G$. This condition is independent of H, and thus ${}^*w_1(H) = {}^*w_1(K)$ for all subgroups H and K of G. Similarly, this holds for w_1^*, ${}^*w_2$ and w_2^*.

Since constant functions satisfy conditions (i) and (ii) from Theorem 2.2, it follows that the relative margins are all CD-admissible. Since $\mu(H)$ is just a constant multiple of $|H|$, the same nonzero constant for every subgroup, it follows that the corresponding lattice is $\mathrm{CD}_\mu(G) = \{G\}$. □

Our two examples so far are words for which all four relative margins yield CD-admissible functions. The next proposition and Example 3.8 below show that this is not the case in general, so it is possible for some relative margins of a word to yield CD-admissible functions, while others do not.

Proposition 3.6. *Let $n \geq 1$, and let $w(x_1, x_2) = [x_1^n, x_2]$. Then $M(H) = {}^*w_1(H)$ and $M(H) = w_1^*(H)$ are both CD-admissible.*

Proof. It is enough to show that we have $M(H) \cap M(K) \subseteq M(\langle H, K \rangle)$ in each of the two cases. Recall that for all elements a, b, c we have $[a, bc] = [a, c][a, b]^c$. If a, b are elements of G and $x \in G$ satisfies

$w(xg, a) = w(g, a)$ and $w(xg, b) = w(g, b)$ for all $g \in G$, then

$$\begin{aligned} w(xg, ab) &= [(xg)^n, ab] = [(xg)^n, b][(xg)^n, a]^b \\ &= w(xg, b)w(xg, a)^b \\ &= w(g, b)w(g, a)^b \\ &= [g^n, b][g^n, a]^b \\ &= [g^n, ab] = w(g, ab). \end{aligned}$$

In particular, if $x \in {}^*w_1(H) \cap {}^*w_1(K)$, then $x \in {}^*w_1(\langle H, K\rangle)$, which is what we wanted to prove. The same argument shows the result for w_1^*. □

However, the left and right relative second margins, ${}^*w_2$ and w_2^*, do not define CD-admissible functions in general. We have the following description of the left and right relative second margins.

Proposition 3.7. *Let $n \geq 1$, and let $w(x_1, x_2) = [x_1^n, x_2]$. If H is a subgroup of G, then:*

(i) ${}^*w_2(H) = C_G(\langle h^n \mid h \in H\rangle)$;
(ii) $w_2^*(H) = C_G(\langle h^n \mid h \in H\rangle^G)$.

Proof. If $x \in {}^*w_2(H)$, then for every $h \in H$ we have

$$[h^n, x] = w(h, xe) = w(h, e) = [h^n, e] = e,$$

so x centralizes h^n for every $h \in H$. Conversely, if $x \in C_G(\langle h^n \mid h \in H\rangle)$, then for every $h \in H$ and $g \in G$ we have

$$w(h, xg) = [h^n, xg] = [h^n, g][h^n, x]^g = [h^n, g] = w(h, g),$$

so we conclude that $x \in {}^*w_1(H)$. This proves (i).

To prove (ii), we observe that if $x \in w_2^*(H)$, then for all $h \in H$ and $g \in G$ we have

$$h^{-n}(gx)^{-1}h^n(gx) = [h^n, gx] = w(h, gx) = w(h, g) = [h^n, g] = h^{-n}g^{-1}h^n g.$$

Cancelling h^{-n}, we conclude that $x^{-1}(g^{-1}h^n g)x = g^{-1}h^n g$, so x commutes with $(h^n)^g$. Thus, $x \in C_G(\langle h^n \mid h \in H\rangle^G)$. Conversely, if x commutes with all conjugates of nth powers of elements of h, then for all $h \in H$ and $g \in G$ we have:

$$w(h, gx) = h^{-n}(h^n)^{gx} = h^{-n}\big((h^n)^g\big)^x = h^{-n}(h^n)^g = [h^n, g] = w(h, g).$$

Thus, $x \in w_2^*(H)$, as claimed. □

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Example 3.8. *Consider $w(x_1, x_2) = [x_1^6, x_2]$, and let $G = A_5$. Let $\mu(H) = |H|\,|{}^*w_2(H)|$. Using Proposition 3.7, we obtain the following values on the isomorphism types of subgroups of G:*

H	${}^*w_2(H)$	$\mu(H)$
$\{e\}$	G	60
C_2	G	120
C_3	G	180
C_5	C_5	25
K_4	G	240
S_3	G	360
D_{10}	C_5	50
A_4	G	720
G	$\{e\}$	60

Thus, the subgroups of maximal μ-measure are precisely the copies of A_4 in G (the point stabilizers). However, these subgroups do not form a lattice, so μ is not a CD-measure. Therefore, ${}^*w_2$ does not define a CD-admissible function. We can also see from the table that this function does not satisfy condition (ii) of Theorem 2.2: if we let H and K be distinct copies of A_4, then the relative marginals of H and K are both equal to G, but

$$G = {}^*w_2(H) \cap {}^*w_2(K) \not\leq {}^*w_2(\langle H, K\rangle) = {}^*w_2(G) = \{e\}.$$

For the right relative second margin, note that because of the simplicity of A_5 we have that if $H \leq A_5$, then $C_G(\langle h^6 \mid h \in H\rangle^G)$ is either G (if H has exponent 6), or $\{e\}$ (if H does not have exponent 6). Defining $\mu(H) = |H|\,|w_2^*(H)|$, we obtain the following values:

H	$w_2^*(H)$	$\mu(H)$
$\{e\}$	G	60
C_2	G	120
C_3	G	180
C_5	$\{e\}$	5
K_4	G	240
S_3	G	360
D_{10}	$\{e\}$	10
A_4	G	720
G	$\{e\}$	60

So once again, the subgroups of maximal measure are the copies of A_4, which do not form a lattice. Thus, μ is not a CD-measure and we conclude that w_2^* also does not define a CD-admissible function for A_5.

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As we will show in Proposition 4.6, ${}^*w_2$ and w_2^* define CD-admissible functions under additional conditions on the group G.

It may be the case that none of the relative margins define CD-admissible functions in general, but it is possible that if we put restrictions on the group G, we may obtain a natural class of groups on which some of them become CD-admissible. For instance, we have the following:

Proposition 3.9. *Let $w(x_1, x_2) = [x_1, x_2, x_2]$ be the 2-Engel word. If G is nilpotent of class at most 3, then ${}^*w_2$ and w_2^* both define CD-admissible functions on G.*

Proof. If G is a nilpotent group of class at most 3, the commutator bracket $[a, b, c]$ is multilinear in G; that is, we have that for all $x, y, z, w \in G$:

$$[xw, y, z,] = [x, y, z][w, y, z],$$

$$[x, yw, z] = [x, y, z][x, w, z],$$

$$[x, y, zw] = [x, y, z][x, y, w].$$

Assume that $x, a, b \in G$ are such that $w(a, xg) = w(a, g)$ and $w(b, xg) = w(b, g)$ for all $g \in G$. Then

$$\begin{aligned} w(ab, xg) &= [ab, xg, xg] \\ &= [a, xg, xg][b, xg, xg] \\ &= w(a, xg)w(b, xg) \\ &= w(a, g)w(b, g) \\ &= [a, g, g][b, g, g] \\ &= [ab, g, g] = w(ab, g). \end{aligned}$$

Therefore, ${}^*w_2(H) \cap {}^*w_2(K) \subseteq {}^*w_2(\langle H, K\rangle)$, so ${}^*w_2$ defines a CD-measure on G. Similar calculations hold for w_2^*. □

4. Generalizing properties of the Chermak–Delgado lattice

As noted above, the Chermak–Delgado lattice of a group and its minimal element have interesting properties. One such property is that if H lies in the Chermak–Delgado lattice of G, then so does $C_G(H)$. This means that taking centralizers induces a lattice anti-automorphism of the Chermak–Delgado lattice, which must therefore be self-dual.

Another interesting property is that if we view the Chermak–Delgado measure as being determined by the left relative margins of the word $w(x_1, x_2) = [x_1, x_2]$, as noted in Example 3.2, then because the centralizer of the least element A of the Chermak–Delgado lattice also lies in the lattice, it follows that $A \leq C_G(A)$, so that A is abelian; that is, A satisfies the word w that defines the lattice (for all $a, a' \in A$, we have $w(a, a') = e$). Another important property of A was noted above in the paragraph before Theorem 2.9.

We do not have parallel results for arbitrary CD-measures that are derived from words, but in some instances we may obtain similar conclusions. In this section, we explore some of these questions, with particular attention to the measures determined by the the relative margins ${}^*w_1$ and w_1^* associated to $w(x_1, x_2) = [x_1^n, x_2]$, as in Proposition 3.6. We also note examples where we know the corresponding properties do not hold for arbitrary CD-measures, even those that are derived by the relative margins of group words.

An easy observation is that the minimal element of a CD-lattice derived from a word need not satisfy the word:

Example 4.1. *Let a and b be positive integers, and let $w(x_1, x_2) = x_1^a x_2^b$. As we noted in Proposition 3.5, the relative margins of this word define a CD-measure whose associated lattice contains only the whole group G. This (unique) element of the CD-lattice need not satisfy the word $w(x_1, x_2)$.*

It is also not necessarily the case that if a relative margin defines a CD-measure, and H is the least element of the corresponding CD-lattice, then the relative margin of H will also lie in the lattice. This can be seen in the following example.

Example 4.2. *Let $w(x_1, x_2) = [x_1^2, x_2]$; we know from Proposition 3.6 that the left first relative margin $M(H) = {}^*w_1(H)$ is CD-admissible for any group. Take $G = S_3$. We claim that if μ is the associated CD-measure, then $\mathrm{CD}_\mu(G) = \{H\}$, where $H = \langle(1, 2, 3)\rangle$.*

Indeed, first we note that ${}^*w_1(H) = G$. Because H has index 2 and is abelian, elements of H commute with every square. Therefore, $[(xg)^2, h] = [g^2, h] = e$ always holds. Thus, $\mu(H) = |H|\,|{}^*w_1(H)| = 3|G| = 18$.

The only subgroup that could have equal or larger μ-measure is G itself, since we always have $\mu(K) \leq |K|\,|G|$. But ${}^*w_1(G) = \{e\}$. Indeed, by taking $g = e$ we have that $[x^2, h] = [e, h] = e$ for all $h \in G$ requires every $x \in {}^*w_1(G)$ to have exponent 2. If x has order 2, then letting y be any element of order 2 distinct from x we have that xy has order 3, so $[(xy)^2, x] \neq e$ while $[x^2, x] = e$. This shows that ${}^*w_1(G) = \{e\}$. Thus, $\mu(G) = |G|\,|{}^*w_1(G)| = |G|\,|\{e\}| = 6$, so $\mu(G) < \mu(\langle(1, 2, 3)\rangle$.

Therefore, we have $\mathrm{CD}_\mu(G) = \{H\}$, with $H = \langle(1, 2, 3)\rangle$. But ${}^*w_1(H) = G$ does not lie in $\mathrm{CD}_\mu(G)$.

For the words $w(x_1, x_2) = [x_1^n, x_2]$, we do have a weaker condition, which is somewhat reminiscent of the "double centralizer" property, which tells us that $H \leq C_G(C_G(H))$ for all subgroups H of a group G:

Proposition 4.3. *Let G be a group, $n > 1$, and let $w(x_1, x_2) = [x_1^n, x_2]$. If H is a subgroup of G, then $H \leq {}^*w_2({}^*w_1(H))$.*

Proof. Let $h \in H$, and $x \in {}^*w_1(H)$. Then we have $[x^n, h] = [(xe)^n, h] = [e^n, h] = e$. Thus, for all $g \in G$ we have

$$w(x, hg) = [x^n, hg] = [x^n, g][x^n, h]^g = [x^n, g] = w(x, g).$$

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This shows that $h \in {}^*w_2({}^*w_1(H))$, as desired. ☐

To extend this further, let us restrict our attention to n-Bell groups. Recall that a group G is said to be an n-Bell group if and only if for all $g, h \in G$ we have $[g^n, h] = [g, h^n]$. See e.g. [8].

Lemma 4.4. *Fix $n > 1$, and let $w(x_1, x_2) = [x_1^n, x_2]$. If G is an n-Bell group, then for all $g_1, g_2 \in G$ we have*

$$w(g_1, g_2) = w(g_2, g_1)^{-1}.$$

Proof. We have $w(g_1, g_2) = [g_1^n, g_2] = [g_1, g_2^n] = [g_2^n, g_1]^{-1} = w(g_2, g_1)^{-1}$, as claimed. ☐

Proposition 4.5. *Fix $n > 1$, and let $w(x_1, x_2) = [x_1^n, x_2]$. If G is an n-Bell group, then for all subgroups H of G, we have ${}^*w_1(H) = {}^*w_2(H)$ and $w_1^*(H) = w_2^*(H)$.*

Proof. Let $g \in G$ and $h \in H$ be arbitrary. If $x \in {}^*w_1(H)$, then

$$w(h, xg) = (w(xg, h))^{-1} = w(g, h)^{-1} = w(h, g).$$

Therefore, $x \in {}^*w_2(H)$. Symmetrically, if $x \in {}^*w_2(H)$, then

$$w(xg, h) = (w(h, xg))^{-1} = w(h, g)^{-1} = w(g, h),$$

so $x \in {}^*w_1(H)$, as desired. A similar computation holds for $w_1^*(H)$ and $w_2^*(H)$. ☐

In Proposition 3.6 we showed that for $w(x_1, x_2) = [x_1^n, x_2]$, both $M(H) = {}^*w_1(H)$ and $M(H) = w_1^*(H)$ are CD-admissible functions. If we make the additional assumptions that G is n-Bell, then we can also obtain that $M(H) = {}^*w_2(H)$ and $M(H) = w_2^*(H)$ are likewise CD-admissible; indeed, applying Proposition 4.5 we obtain the following result.

Proposition 4.6. *Fix $n > 1$, and let $w(x_1, x_2) = [x_1^n, x_2]$. If G is an n-Bell group, then $M(H) = {}^*w_2(H)$ and $M(H) = w_2^*(H)$ are both CD-admissible functions.*

We also obtain the following result, reminiscent of the well-known "triple centralizer" equality $C_G(H) = C_G(C_G(C_G(H)))$:

Corollary 4.7. *Fix $n > 1$, and let $w(x_1, x_2) = [x_1^n, x_2]$. If G is an n-Bell group, then for all subgroups H of G we have*

$${}^*w_1(H) = {}^*w_1\Big({}^*w_1\big({}^*w_1(H)\big)\Big).$$

Proof. Let H be a subgroup of G. By Propositions 4.3 and 4.5 we have that

$$H \leq {}^*w_2\big({}^*w_1(H)\big) = {}^*w_1\big({}^*w_1(H)\big). \tag{4.1}$$

Since relative margins reverse inclusions, we therefore obtain that

$${}^*w_1(H) \geq {}^*w_1\Big({}^*w_1\big({}^*w_1(H)\big)\Big). \tag{4.2}$$

Replacing H with ${}^*w_1(H)$ in (4.1) we also obtain

$$ {}^*w_1(H) \leq {}^*w_1\Big({}^*w_1\big({}^*w_1(H)\big)\Big), \tag{4.3}$$

and combining (4.2) and (4.3) yields the desired equality. □

This allows us to prove that for $w(x_1, x_2) = [x_1^n, x_2]$ and n-Bell groups we have a result analogous to the fact that taking centralizers induces a lattice anti-isomorphism of the Chermak–Delgado lattice:

Corollary 4.8. *Fix $n > 1$, and let $w(x_1, x_2) = [x_1^n, x_2]$; let G be an n-Bell group, $M(H) = {}^*w_1(H)$, and let μ be the corresponding CD-measure. If $H \in \mathrm{CD}_\mu(G)$, then ${}^*w_1(H) \in \mathrm{CD}_\mu(G)$.*

Proof. Let $H \in \mathrm{CD}_\mu(G)$, and let $L = {}^*w_1(H)$. From (4.1) we get $H \leq {}^*w_1(L)$. Therefore,

$$\mu(L) = |L|\,|{}^*w_1(L)| \geq |L|\,|H| = |{}^*w_1(H)|\,|H| = \mu(H).$$

Since $\mu(H)$ is as large as possible, we conclude that $L \in \mathrm{CD}_\mu(G)$, as claimed. □

We can now use Theorem 2.9 to obtain the following theorem:

Theorem 4.9. *Fix $n > 1$, and let $w(x_1, x_2) = [x_1^n, x_2]$. If G is an n-Bell group, then the left relative margin ${}^*w_1(H)$ defines a CD-admissible function on G. If μ is the corresponding CD-measure and K is the least element of $\mathrm{CD}_\mu(G)$, then:*

(i) *The map $H \longmapsto {}^*w_1(H)$ defines a lattice anti-homomorphism from $\mathrm{CD}_\mu(G)$ to itself.*
(ii) *$K \leq {}^*w_1(K)$.*
(iii) *For every $k_1, k_2 \in K$, we have $w(k_1, k_2) = e$; that is, the least element of $\mathrm{CD}_\mu(G)$ satisfies the word w.*
(iv) *If H is any subgroup of G such that $H \leq {}^*w_1(H)$, then $[G : K] \leq [G : H]^2$.*

Proof. By Corollary 4.8, we have that if $H \in \mathrm{CD}_\mu(G)$, then ${}^*w_1(H) \in \mathrm{CD}_\mu(G)$. Theorem 2.9 now yields (i), (ii), and (iv).

To establish (iii), let $k_1, k_2 \in K$. Since $K \leq {}^*w_1(K)$, we have

$$w(k_1, k_2) = w(k_1 e, k_2) = w(e, k_2) = [e^n, k_2] = e,$$

which proves (iii). □

5. Future directions

The results above suggest some interesting directions for future investigations. We discuss some of them in this section, indicating how they relate to results about the Chermak–Delgado measure and lattice, and to the results we have obtained above for the more general CD-measures.

As we have seen above, it is the case that some words $w(x_1, x_2)$ will yield admissible functions for all groups with the four relative marginals they define. However, some words, such as $w(x_1, x_2) = [x_1^n, x_2]$, will yield functions that are admissible for all groups only for some of the relative marginals and not all of them. It would be useful to have some conditions on the words that ensure that all, or some, of the relative marginals will always be CD-admissible.

By the same token, there are some standard words, such as the n-Engel words $w(x_1, x_2) = [x_{1,n}\, x_2]$ and $w(x_1, x_2) = [x_1, x_{2,n-1}\, x_1]$, which may not always yield CD-admissible relative marginals but do so for some natural classes of groups (cf. Prop. 3.9). Determining the exact contours of the class of groups for which a given word defines admissible relative margins, especially when they may determine nontrivial CD-lattices, seems worth exploring.

We also note that Theorem 2.2 shows that many interesting properties of the Chermak–Delgado lattice can be recovered from properties (i) and (ii) listed there. Among them, that for any two subgroups H and K in the corresponding $\mathrm{CD}_\mu(G)$ lattice we have $\langle H, K\rangle = HK$, that $M(H \cap K) = \langle M(H), M(K)\rangle$, and that $\langle M(H), M(K)\rangle = M(H)M(K)$.

On the other hand, the Chermak–Delgado measure has additional interesting properties that do not appear to follow for general CD-measures. In particular, while the Chermak–Delgado measure has the property that if H has maximal measure then so does $C_G(H)$, if μ is a CD-measure derived from a CD-admissible function M we need not have that $H \in \mathrm{CD}_\mu(G)$ implies $M(H) \in \mathrm{CD}_\mu(G)$, as we saw in Example 4.2.

In light of Theorem 2.9, it would be interesting to determine what additional properties a CD-admissible function M should satisfy to ensure that $M(H)$ has maximal μ-measure whenever H does. Sometimes the resulting inequality is not particularly interesting: for example, consider an admissible function M as in Example 2.5, with $B = A$ and $|A|^2 > |G|$. Then $\mathrm{CD}_\mu(G) = \{A\}$, and $M(A) = A$. Since $L \leq M(L)$ means $L \leq A$, we are only saying that if $L \leq A$, then $[G : A] \leq [G : L]^2$. Although true, this is uninteresting.

Another important property of the Chermak–Delgado measure is that the subgroups of maximal Chermak–Delgado measure satisfy $H = C_G(C_G(H))$. This ensures that the anti-homomorphism of the Chermak–Delgado lattice given by $H \longmapsto C_G(H)$ is in fact an anti-isomorphism. We would like to determine when Theorem 2.9(i) can be strengthened to yield an anti-isomorphism.

Finally, the least element of the Chermak–Delgado lattice is always abelian; so if we view the lattice as being determined by the word $w(x_1, x_2) = [x_1, x_2]$ we have that the subgroup satisfies the word. While this is not the case in general, as we already noted, it would be interesting to know in which situations we can ensure that this is the case, provided that the relative marginal is CD-admissible.

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