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ON NUMBERS WHICH ARE ORDERS OF NILPOTENT GROUPS WITH BOUNDED CLASS

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Abstract. Let n be a positive integer. In this short note, we characterize those numbers m for which any group of order m is an n-Engel group and those numbers m for which any group of order m has all its subgroups subnormal of defect at most n.

1. Introduction

Let $n, \alpha_1, \dots, \alpha_r$ be positive integers and let $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be an integer number where each p_i is a prime. A well-known theorem of Müller [6] states that every group of order m is nilpotent of class at most n if and only if:

(a) p_i does not divide $p_j^k - 1$, for all $i \neq j$ and for all integers k such that $1 \leq k \leq \alpha_j$;

(b) $1 \leq \alpha_i \leq n+1$, for all i.

We say that a pair (m, n) of positive integers is a *Müller pair* if the conditions (a) and (b) are satisfied. Therefore every group of order m is nilpotent of class at most n if and only if (m, n) is a Müller pair. The Müller's theorem (see also [10] for an alternative proof) solves a special case of a classical problem in finite group theory asking to characterize those numbers m for which the groups of order m (and only them) satisfy a given property $\mathfrak{X}$ (usually these numbers are called $\mathfrak{X}$-*numbers*). Cyclic numbers have been characterized in [3, 4, 5], abelian numbers in [3, 8], nilpotent numbers in [6, 7], ...; we refer the interested reader to [2], where a satisfactory exposition of the known results in this area is given.

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The aim of this paper is to give a further contribution to this topic. Recall that a group G is called *n-Engel group* if

$$[x,_n y] = [\dots[[x, y], y], \dots, y] = 1$$

for all $x, y \in G$.

Theorem. *Let $n, \alpha_1, \dots, \alpha_r$ be positive integers and let $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ be an integer number where each p_i is a prime. Then the following conditions are equivalent:*

1. *(m, n) is a Müller pair;*
2. *every group of order m is an n-Engel group;*
3. *every group of order m has all subgroups subnormal with defect at most n.*

From Theorem we deduce the following corollary.

Corollary. *Let m and n be positive integers. All groups of order m are nilpotent of class at most n if and only if all groups of order m are n-Engel.*

The layout of the paper is as follows. We first provide the necessary backgrounds on finite p-groups of maximal class and then we use them to prove our theorem.

Our notation is standard and can for instance be found in [9].

2. Proof of the Theorem

For p a prime and $n > 2$, a p-group of order p^n is called a *p-group of maximal class* if its nilpotency class is $n - 1$. In the following, we collect the necessary background on p-groups of maximal class.

Lemma 2.1. *Let G be a finite non-abelian p-group of order p^n, $n > 2$, with an abelian maximal subgroup. Then the following conditions are equivalent:*

(a) *$|Z(G)| = p$;*
(b) *$|G : G'| = p^2$;*
(c) *G is of maximal class.*

Proof. Clearly, (c) implies (a) and (b). Moreover, it follows immediately from [1, Lemma 1.1] that (a) and (b) are equivalent conditions.

Assume that (a) holds. Let c be the nilpotency class of G. Now, $\gamma_c(G) \leq Z(G)$ has order p and $G/\gamma_c(G)$ satisfies (b) (and hence (a) too). It follows by induction on the order of G that $G/\gamma_c(G)$ has nilpotency class $n - 2$ and so $c = n - 1$. □

Corollary 2.2. *A finite p-group G of maximal class is 2-generator.*

Proof. Let F be the Frattini subgroup of G. Since G is a finite nilpotent group we have that $G' \leq F$ and from that $F = G'$. Then $|G : F| = p^2$ by Lemma 2.1, and so 2 is the minimum number of generators for G. □

Theorem 2.3. *Let p be a prime number and $n > 2$. Then there is a p-group $G = \langle x \rangle \ltimes A$ of maximal class and order p^n, where $x^p = 1$ and A is abelian.*

Proof. See [1, Proposition 9.15]. □

The equivalence between the condition (1) and the condition (2) of the Theorem is proved essentially by understanding which p-groups of maximal class (with an abelian maximal subgroup) are n-Engel.

Example 2.4. *The dihedral group $D_{2^{n+2}}$ of order 2^{n+2} is nilpotent of class $n+1$ and it is not an n-Engel group.*

Proof. Put $t = 2^{n+1}$, then

$$D_{2^{n+2}} = \langle x, y \ : \ x^t = 1, \, y^2 = 1, \, (xy)^2 = 1 \rangle.$$

By induction we see that $[x,_m y] = x^{(-1)^m 2^m}$ for every m. On the other hand, $[x,_m y] = x^{(-1)^m 2^m} = 1$ if and only if $m \geq n+1$ and we are done. □

Lemma 2.5. *Let G be a p-group of maximal class of order p^{n+2} having an abelian maximal subgroup A. Then G it is not an n-Engel group.*

Proof. By Corollary 2.2, $G = \langle a, g \rangle$, $a \in A$ and $g \notin A$. We proceed by induction on the order of the group.

If $n = 1$ then $[a, g] \neq 1$ since G is not abelian. It follows that G is not a 1-Engel group. Put $n > 1$ and suppose that the statement is true for $n-1$. We can observe that $[a,_n g] = [a,_{n-1} g]^{-1}[a,_{n-1} g]^g = 1$ if and only if $[a,_{n-1} g] \in Z(G)$ and this is true if and only if $[aZ(G),_{n-1} gZ(G)] = 1$ in $G/Z(G)$. On the other hand $|G/Z(G)| = p^{n+1}$ by Lemma 2.1 and $p^{n+1} = p^{(n-1)+2}$. Thus, the group of maximal class $G/Z(G)$ satisfies the inductive hypothesis and so

$$[aZ(G),_{n-1} gZ(G)] \neq 1$$

By induction, the statement is true for all $n \in \mathbb{N}$. □

Lemma 2.6. *Let $G = \langle x \rangle \ltimes A$ be a p-group of maximal class whose order is p^{n+1}, where $x^p = 1$ and A is abelian. Then $\langle x \rangle$ is subnormal of defect at most n.*

Proof. Let

$$\langle x \rangle = H_0 \lhd H_1 \lhd \cdots \lhd H_d = G$$

be the normal closure series of $\langle x \rangle$ in G. We start proving that

$$H_1 = \langle x \rangle Z(G) = \langle x \rangle \gamma_n(G).$$

The subgroups $\langle x \rangle$ and $H_1 \cap A$ are both normal in H_1 and we have also that

$$\langle x \rangle \cap (H_1 \cap A) = 1,$$

so $H_1 = \langle x \rangle \times (H_1 \cap A)$. By Lemma 2.1, $|Z(G)| = p$ and also $H_1 \cap A \neq 1$. It follows that $H_1 \cap A = Z(G)$ and so $H_1 = \langle x \rangle Z(G) = \langle x \rangle \gamma_n(G)$. Note that the group

$$G/\gamma_n(G) = \langle x\gamma_n(G) \rangle \ltimes A/\gamma_n(G)$$

satisfies the hypothesis and the statement follows by induction. □

Proof of the Theorem. (2) ⇒ (1) Suppose first that all groups of order m are n-Engel groups. A well-known theorem of Zorn states that all finite Engel groups are nilpotent (see [9, 12.3.4]), so Müller's theorem shows that m satisfies the condition (a). Suppose now that there is a prime number p such that p^{n+2} divides m. It follows that we can find a group of order m admitting as direct factor $D_{2^{n+2}}$ or, by Theorem 2.3, there is a direct factor K of G of type $K = \langle x \rangle \ltimes A$ of order p^{n+2}, having maximal class $n+1$, where A is an abelian group and x is such that $x^p = 1$. We have thus reached a contradiction by Lemma 2.5. This contradiction shows that m must satisfy condition (b) too.
(1) ⇒ (2) Conversely, suppose that (m, n) is a Müller pair and let G a group of order m. Then G is a nilpotent group of class at most n by Müller's theorem and so it is also a n-Engel group.
(3) ⇒ (1) Suppose that each group of order m has all subgroups subnormal of defect at most n. Let G be a group of order m nilpotent of class $c > n$. By Müller's theorem, p_i does not divide $p_j^k - 1$ for each $i \neq j$ and for each $1 \leq k \leq \alpha_j$, and $1 \leq \alpha_i \leq c+1$. It follows that there exists i with $\alpha_i \geq n+2$. It is therefore possible to assume that G is a p-group of order p^{n+2}. On the other hand, by Theorem 2.3 there exists a group K of order p^{n+2} of maximal class $K = \langle x \rangle \ltimes A$ with A abelian and such that $x^p = 1$. This is a contradiction by Lemma 2.6. □

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