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A CHARACTERIZATION OF A_5 BY ITS AVERAGE ORDER

MARIUS TĂRNĂUCEANU

Abstract. Let $o(G)$ be the average order of a finite group G. M. Herzog, P. Longobardi and M. Maj [M. Herzog, P. Longobardi and M. Maj, Another criterion for solvability of finite groups, *J. Algebra*, **597** (2022) 1-23.] showed that if G is non-solvable and $o(G) = o(A_5)$, then $G \cong A_5$. In this note, we prove that the equality $o(G) = o(A_5)$ does not hold for any finite solvable group G. Consequently, up to isomorphism, A_5 is determined by its average order.

1. Introduction

Given a finite group G, we denote by $\psi(G)$ the sum of element orders of G and by $o(G)$ the average order of G, that is

$$\psi(G) = \sum_{x\in G} o(x) \text{ and } o(G) = \frac{\psi(G)}{|G|}\,.$$

In the last years there has been a growing interest in studying the properties of these functions and their relations with the structure of G (see for example [1]-[8], [10],[11], [13] and [15]-[17]).

In [10], A. Jaikin-Zapirain uses the average order to determine a lower bound for the number of conjugacy classes of a finite p-group/nilpotent group. He also suggests the following question: "*Let G be a finite $(p$-$)$group and N be a normal $($abelian$)$ subgroup of G. Is it true that $o(G) \geq o(N)^{\frac{1}{2}}$?*". Recently, E.I. Khukhro, A. Moretó and M. Zarrin proved the following result (see [11, Theorem 1.2]):

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Theorem A. *Let $c > 0$ be a real number and $p \geq \frac{3}{c}$ be a prime. Then there exists a finite p-group G with a normal abelian subgroup N such that $o(G) < o(N)^c$.*

Note that Theorem A provides a negative answer to Jaikin-Zapirain's question even if we replace the exponent $\frac{1}{2}$ with any positive real number c. In the same paper [11], the authors posed the following conjecture:

Conjecture. *Let G be a finite group and suppose that $o(G) < \frac{211}{60} = o(A_5)$. Then G is solvable.*

It has been confirmed by M. Herzog, P. Longobardi and M. Maj [7].

Theorem B. *Let G be a finite group. If $o(G) < \frac{211}{60}$, then G is solvable. Moreover, if G is a non-solvable finite group, we have $o(G) = \frac{211}{60}$ if and only if $G \cong A_5$.*

This constitutes the starting point for the current work. Our main result is as follows.

Theorem 1.1. *If G is a finite solvable group, then $o(G) \neq \frac{211}{60}$.*

It is now clear that Theorems B and 1.1 lead to the following characterization of A_5.

Corollary 1.2. *Up to isomorphism, A_5 is the unique finite group G such that $o(G) = \frac{211}{60}$.*

Note that the same result was obtained by M. Herzog, P. Longobardi and M. Maj [8]. Our proof is different from that in [8] and based upon the next theorem.

Theorem C. *Let G be a finite group and $n_d(G)$ be the number of elements of order d in G, $\forall\, d \in \mathbb{N}$. Then the following statements hold:*

1) [12] *If p is a prime divisor of $|G|$ and G is not a p-group, then*
$$n_p(G) \leq \frac{p}{p+1}\,|G| - 1.$$
2) [14] *If G is non-abelian, then*
$$n_2(G) \leq \frac{3}{4}\,|G| - 1.$$

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [9].

2. Proofs of the main results

We start with the following easy but important lemma.

Lemma 2.1. *Let G be a finite group of order n and $n_d(G)$ be the number of elements of order d in G, $\forall\, d \in \mathbb{N}^*$. If $o(G) = \frac{211}{60}$, then*

$$4n_2(G) + 3n_3(G) + 2n_4(G) + n_5(G) \geq \frac{149}{60}\,n - 5. \tag{2.1}$$

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Proof. Since $n = \sum_{d \in \mathbb{N}^*} n_d(G)$, we deduce that

$$\begin{aligned} \psi(G) &= 1 + 2n_2(G) + \cdots + 6n_6(G) + \ldots \\ &\geq 1 + 2n_2(G) + \cdots + 5n_5(G) + 6(n_6(G) + n_7(G) + \ldots) \\ &= 1 + 2n_2(G) + \cdots + 5n_5(G) + 6(n - 1 - n_2(G) - \cdots - n_5(G)) \\ &= 6n - 5 - (4n_2(G) + 3n_3(G) + 2n_4(G) + n_5(G)). \end{aligned}$$

Since $o(G) = \frac{211}{60}$, it follows that $\psi(G) = \frac{211}{60}\, n$. Therefore we have

$$\frac{211}{60}\, n \geq 6n - 5 - (4n_2(G) + 3n_3(G) + 2n_4(G) + n_5(G)),$$

which is equivalent to (1), completing the proof. □

Our second lemma shows that the average order of a finite nilpotent group which is not a p-group cannot be certain rational numbers.

Lemma 2.2. *Let G be a finite nilpotent group. If G is not a p-group, then $o(G) \notin \left\{\frac{61}{15}, \frac{211}{60}\right\}$.*

Proof. We write G as a direct product of its Sylow subgroups

$$G = G_1 \times \cdots \times G_k,$$

where $k \geq 2$ and $G_i \in \mathrm{Syl}_{p_i}(G)$, for all $i = 1, \ldots, k$. Since the function o is multiplicative, we get

$$o(G) = o(G_1) \cdots o(G_k).$$

Let $|G_i| = p_i^{n_i}$, $i = 1, \ldots, k$, and assume that $p_1 < \cdots < p_k$. Then for each i we have

$$o(G_i) \geq \frac{1 + p_i(p_i^{n_i} - 1)}{p_i^{n_i}} = p_i - \frac{p_i - 1}{p_i^{n_i}} \geq p_i - \frac{p_i - 1}{p_i} > 1.$$

For $k \geq 3$, we get

$$o(G) \geq o(G_1)o(G_2)o(G_3) \geq \left(2 - \frac{1}{2}\right)\left(3 - \frac{2}{3}\right)\left(5 - \frac{4}{5}\right) = \frac{147}{10} > \frac{61}{15} > \frac{211}{60}.$$

If $k = 2$ and $(p_1, p_2) \neq (2, 3)$, then

$$o(G) = o(G_1)o(G_2) \geq \min\left\{\left(3 - \frac{2}{3}\right)\left(5 - \frac{4}{5}\right), \left(2 - \frac{1}{2}\right)\left(5 - \frac{4}{5}\right)\right\} = \frac{63}{10} > \frac{61}{15}.$$

Thus, we can suppose that $(p_1, p_2) = (2, 3)$. If $(n_1, n_2) \neq (1, 1)$, then

$$o(G) = o(G_1)o(G_2) \geq \min\left\{\left(2 - \frac{1}{4}\right)\left(3 - \frac{2}{3}\right), \left(2 - \frac{1}{2}\right)\left(3 - \frac{2}{9}\right)\right\} = \frac{49}{12} > \frac{61}{15}.$$

Finally, for $(n_1, n_2) = (1, 1)$ we have $G \cong C_6$ and so

$$o(G) = \frac{7}{2} \notin \left\{\frac{61}{15}, \frac{211}{60}\right\},$$

as desired. □

We are now able to prove our main result. Note that a search with GAP shows that there is no solvable group G of order less than 1000 such that $o(G) = \frac{211}{60}$.

Proof of Theorem 1.1. Assume that there exists a finite solvable group G of order n with $o(G) = \frac{211}{60}$. Then $60\psi(G) = 211n$, implying that $n = 2^2 3^{\alpha_2} 5^{\alpha_3} \cdots$ with $\alpha_2, \alpha_3 \geq 1$. Let H be a Sylow 2-subgroup of G. We distinguish the following two cases.

Case 1. $H \cong C_4$ Then G has a normal 2-complement K of order $m = 3^{\alpha_2} 5^{\alpha_3} \cdots$. First of all, we prove that

$$n_4(G) = 2m = \frac{1}{2}\,n\,. \tag{2.2}$$

Suppose that $n_4(G) < 2m$. Since $\frac{n_4(G)}{2}$ divides m, it follows that $\frac{n_4(G)}{2} \leq \frac{m}{3}$ and so $n_4(G) \leq \frac{2}{3}\,m = \frac{1}{6}\,n$. This implies that $n_2(G) \leq \frac{1}{12}\,n$. Also, since all elements of odd order of G are contained in K, we get $n_3(G), n_5(G) < m = \frac{1}{4}\,n$. Thus we have

$$4n_2(G) + 3n_3(G) + 2n_4(G) + n_5(G) < \left(\frac{1}{3} + \frac{3}{4} + \frac{1}{3} + \frac{1}{4}\right) n = \frac{5}{3}\,n,$$

which together with (1) lead to

$$\frac{149}{60}\,n - 5 < \frac{5}{3}\,n,$$

that is $n \leq 6$, a contradiction.

Next we prove that

$$n_2(G) = m = \frac{1}{4}\,n\,. \tag{2.3}$$

Suppose that $n_2(G) < m$. Since every subgroup of order 2 of G is contained in at least three Sylow 2-subgroups, we infer that $n_2(G) \leq \frac{m}{3} = \frac{1}{12}\,n$. Similarly, we get

$$4n_2(G) + 3n_3(G) + 2n_4(G) + n_5(G) < \left(\frac{1}{3} + \frac{3}{4} + 1 + \frac{1}{4}\right) n = \frac{7}{3}\,n$$

and so

$$\frac{149}{60}\,n - 5 < \frac{7}{3}\,n$$

by (1). Thus $n \leq 33$, a contradiction.

The equalities (2) and (3) show that any two Sylow 2-subgroups of G intersect trivially and any element of G of even order is a 2-element. We deduce that G is a Frobenius group and consequently its kernel K is nilpotent. On the other hand, we have

$$\frac{211}{60}\,n = \psi(G) = \psi(K) + 2n_2(G) + 4n_4(G) = \psi(K) + \frac{5}{2}\,n,$$

implying that

$$\psi(K) = \frac{61}{60}\,n = \frac{61}{15}\,m,$$

or equivalently

$$o(K) = \frac{61}{15}\,.$$

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This contradicts Lemma 2.2.

Case 2. $H \cong C_2^2$ Since G is solvable, it has a Hall 2-subgroup K of order $m = 3^{\alpha_2}5^{\alpha_3}\cdots$. We have the next two subcases.

Subcase 2.1. K is normal in G

Then K contains all elements of odd order of G. By using Theorem C, 1), for $p = 3$, it follows that $n_3(G) \leq \frac{3}{4}m = \frac{3}{16}n$. Also, we have $n_3(G) + n_5(G) \leq m - 1 = \frac{n}{4} - 1$. Then

$$3n_3(G) + n_5(G) = 2n_3(G) + (n_3(G) + n_5(G)) \leq \frac{3}{8}n + \frac{1}{4}n - 1 = \frac{5}{8}n - 1,$$

implying that

$$4n_2(G) + 3n_3(G) + 2n_4(G) + n_5(G) \leq 4n_2(G) + \frac{5}{8}n - 1.$$

From (1) we get

$$n_2(G) \geq \frac{223}{480}n - 1. \tag{2.4}$$

Let H_i, $i = 1, 2, 3$, be the subgroups of order 2 of H. Then $G = KH_1 \cup KH_2 \cup KH_3$ and so $n_2(G) = 3n_2(KH_i)$, for any $i = 1, 2, 3$. On the other hand, we have $n_2(KH_i) \mid m$ by Sylow's theorems. If $n_2(KH_i) < m$, then $n_2(KH_i) \leq \frac{m}{3}$, which lead to

$$n_2(G) \leq m = \frac{1}{4}n. \tag{2.5}$$

Now, from (4) and (5), we obtain

$$\frac{223}{480}n - 1 \leq \frac{1}{4}n,$$

that is $n \leq 4$, a contradiction. Thus $n_2(KH_i) = m$, for all $i = 1, 2, 3$, and $n_2(G) = 3m = \frac{3}{4}n$. Then, by Theorem C, 2), it follows that G is abelian. Since $o(G) = \frac{211}{60}$, this contradicts Lemma 2.2.

Subcase 2.2. K is not normal in G

Then K has two or four conjugates in G. Let $C = \mathrm{Core}_G(K)$. Since $[G : C]$ divides $4! = 24$, we get $[G : C] = 12$ and so $[K : C] = 3$. We observe that A_4 is the unique group of order 12 containing more than one subgroup of order 3. Thus $G/C \cong A_4$ and K has four conjugates in G, say K_i, $i = 1, .., 4$. Then G possesses

$$\left|\bigcup_{i=1}^{4} K_i\right| = m + 3\left(m - \frac{m}{3}\right) = 3m$$

elements of odd order. Also, we have

$$\begin{aligned} n_3(G) = n_3\left(\bigcup_{i=1}^{4} K_i\right) &\leq n_3(K_1) + 3\left|\bigcup_{i=2}^{4}(K_i \setminus K_1)\right| \\ &\leq \frac{3m}{4} + 3\left(m - \frac{m}{3}\right) = \frac{11}{4}m = \frac{11}{16}n \end{aligned} \tag{2.6}$$

by applying Theorem C, 1), for the group K_1 and $p = 3$.

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Assume that $n_2(G) = m$. Then the number s_2 of Sylow 2-subgroups of G is at least $\frac{m}{3}$. Since all Sylow 2-subgroups of G are contained in CH and $|CH| = 4 \cdot \frac{m}{3}$, we get $s_2 \mid \frac{m}{3}$ and therefore $s_2 = \frac{m}{3}$. This implies again that G is a Frobenius group, a contradiction. Thus

$$n_2(G) \leq \frac{m}{3} = \frac{n}{12}\,.$$

Since C includes all Sylow 5-subgroups of G, it results that

$$n_5(G) \leq |C| - 1 = \frac{m}{3} - 1 = \frac{n}{12} - 1\,.$$

Then

$$4n_2(G) + 3n_3(G) + 2n_4(G) + n_5(G) \leq \frac{n}{3} + 3n_3(G) + \frac{n}{12} - 1 = 3n_3(G) + \frac{5}{12}\,n - 1$$

which together with (1) lead to

$$3n_3(G) \geq \left(\frac{149}{60} - \frac{5}{12}\right) n - 4,$$

that is

$$n_3(G) \geq \frac{31}{45}\,n - \frac{4}{3}\,. \tag{2.7}$$

Now, from (6) and (7), we obtain

$$\frac{31}{45}\,n - \frac{4}{3} \leq \frac{11}{16}\,n,$$

that is $n \leq 960$, a contradiction. □

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Marius Tărnăuceanu

Faculty of Mathematics, Al.I. Cuza University, Iaşi, Romania

Email: tarnauc@uaic.ro

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