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ON SOME GROUPS WHOSE SUBNORMAL SUBGROUPS ARE CONTRANORMAL-FREE

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ABSTRACT. If G is a group, a subgroup H of G is said to be contranormal in G if $H^G = G$, where H^G is the normal closure of H in G. We say that a group is contranormal-free if it does not contain proper contranormal subgroups. Obviously, a nilpotent group is contranormal-free. Conversely, if G is a finite contranormal-free group, then G is nilpotent. We study (infinite) groups whose subnormal subgroups are contranormal-free. We prove that if G is a group which contains a normal nilpotent subgroup A such that G/A is a periodic Baer group, and every subnormal subgroup of G is contranormal-free, then G is generated by subnormal nilpotent subgroups; in particular G is a Baer group. Furthermore, if G is a group which contains a normal nilpotent subgroup A such that the 0-rank of A is finite, the set $\Pi(A)$ is finite, G/A is a Baer group, and every subnormal subgroup of G is contranormal-free, then G is a Baer group.

1. **Introduction**

Let G be a group, and let H and K be subgroups of G, with $H \leq K$. We denote, as usual, by H^K the subgroup of G generated by $H^x = \{x^{-1}hx \mid h \in H\}$, $x \in K$. If $K = G$, then H^G is called the normal closure of the subgroup H in the group G, and it is the smallest normal subgroup of G which contains H. Therefore the subgroup H is normal in G if and only if $H = H^G$.

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The subgroup H of a group G is called *contranormal* in G, if $G = H^G$. The term contranormal subgroup has been introduced by J.S. Rose in his paper [16].

Obviously, a contranormal subgroup of a group G is normal (or subnormal) in G if and only if $H = G$. Thus contranormal subgroups are antagonist to normal (subnormal) subgroups.

When G is a nilpotent group, then $H^G < G$, for every proper subgroup H of G, therefore G does not contain proper contranormal subgroups. We say that a group G is *contranormal-free*, if G contains no proper contranormal subgroups. Contranormal-free groups have been recently studied in many papers, for example [1], [2], [6], [7], [10], [11], [12], [17]. Obviously, every nilpotent group is contranormal-free. The converse is not true, see, for example, [1]. However, every finite contranormal-free group is nilpotent, in fact, every maximal subgroup of such a group must be normal, and finite groups, whose maximal subgroups are normal, are nilpotent. The situation for infinite groups is different: there exist contranormal-free groups that are not even locally nilpotent. Nevertheless, contranormal-free groups are close to be locally nilpotent, for example the groups from the above cited papers are generalized nilpotent groups. If G is a nilpotent group, then every subgroup of G is nilpotent, thus every subgroup of a nilpotent group is contranormal-free. Conversely, let G be a finitely generated soluble-by-finite contranormal-free group. Then G is nilpotent (see, for example, Lemma 2.2). Hence every locally (soluble-by-finite) group, whose subgroups are contranormal-free, is locally nilpotent. This result shows that it is natural to consider groups, in which some influential family of subgroups consists of contranormal-free subgroups. In this paper we study groups whose subnormal subgroups are contranormal-free.

Our main results are given in the following theorems.

Theorem A. *Let G be a group, and let A be a normal nilpotent subgroup of G, such that G/A is a Baer group. Suppose that A has finite 0-rank and that the set $\Pi(A)$ is finite. If every subnormal subgroup of G is contranormal-free, then G is generated by subnormal nilpotent subgroups. In particular, G is a Baer group.*

Corollary $\mathbf{A_1}$. *Let G be a group, and let A be a normal nilpotent subgroup of G, such that G/A is a nilpotent group. Suppose that A has finite 0-rank and the set $\Pi(A)$ is finite. If every subnormal subgroup of G is contranormal-free, then G is generated by subnormal nilpotent subgroups. In particular, G is a Baer group.*

Recall that a group G is called *polynilpotent* if G has a finite series

$$\langle 1 \rangle = G_0 \leq G_1 \leq \cdots \leq G_{n-1} \leq G_n = G$$

of subgroups such that G_j is normal in G_{j+1} and the factors G_{j+1}/G_j are nilpotent for every j, $0 \leq j \leq n-1$.

Corollary $\mathbf{A_2}$. *Let G be a polynilpotent group, and suppose that G has finite 0-rank and the set $\Pi(G)$ is finite. If every subnormal subgroup of G is contranormal-free, then G is a Baer group.*

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Corollary A_3. *Let G be a soluble group, and suppose that G has finite 0-rank and the set $\Pi(G)$ is finite. If every subnormal subgroup of G is contranormal-free, then G is a Baer group.*

The following result is dual in some sense to Theorem A.

Theorem B. *Let G be a group, and A be a normal nilpotent subgroup of G, such that G/A is a periodic Baer group. If every subnormal subgroup of G is contranormal-free, then G is a Baer group.*

Corollary B_1. *Let G be a periodic group, and let A be a normal nilpotent subgroup of G such that G/A is a Baer group. If every subnormal subgroup of G is contranormal-free, then G is a Baer group.*

Corollary B_2. *Let G be a periodic nilpotent-by-nilpotent group. If every subnormal subgroup of G is contranormal-free, then G is a Baer group.*

Corollary B_3. *Let G be a periodic polynilpotent group. If every subnormal subgroup of G is contranormal-free, then G is a Baer group.*

Corollary B_4. *Let G be a periodic soluble group. If every subnormal subgroup of G is contranormal-free, then G is a Baer group.*

2. **Some preliminary results**

We begin with the following result, which is often useful.

Lemma 2.1. *Let G be a group. Then:*

(i) If C is a contranormal subgroup of G, and K is a subgroup containing C, then K is a contranormal subgroup of G.

(ii) If C is a contranormal subgroup of G, and H is a normal subgroup of G, then CH/H is a contranormal subgroup of G/H.

(iii) If H is a normal subgroup of G, and C is a subgroup of G, such that $H \leq C$ and C/H is a contranormal subgroup of G/H, then C is a contranormal subgroup of G.

(iv) If C is a contranormal subgroup of G, and D is a contranormal subgroup of C, then D is a contranormal subgroup of G.

(v) Let C be a subgroup of G, and let $\mathcal{S}$ be a family of subgroups of G, such that C is contranormal in every subgroup $H \in \mathcal{S}$. Then C is contranormal in the subgroup generated by all subgroups in $\mathcal{S}$.

Proof. The assertions (i), (ii) and (iii) are obvious.

To prove (iv), let C be a contranormal subgroup of G, and let D be a contranormal subgroup of C. If g is an element of G, then $g = (x_1^{-1}c_1x_1)\cdots(x_t^{-1}c_tx_t)$ for some elements $x_1, \dots, x_t \in G$, $c_1, \dots, c_t \in C$. On the other hand, the equality $C = D^C$ implies that $c_j = (y_{j1}^{-1}d_1y_{j1})\cdots(y_{js(j)}^{-1}d_{s(j)}y_{js(j)})$, where $y_{j1}, \dots, y_{js(j)} \in C$, $d_1, \dots, d_{s(j)} \in D$, $1 \leq j \leq t$. Then

$$x_j^{-1}c_jx_j = x_j^{-1}((y_{j1}^{-1}d_1y_{j1})\cdots(y_{js(j)}^{-1}d_{s(j)}y_{js(j)}))x_j =$$

$$(x_j^{-1}y_{j1}^{-1}d_1y_{j1}x_j)\cdots(x_j^{-1}y_{js(j)}^{-1}d_{s(j)}y_{js(j)}x_j).$$

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It follows that $x_j^{-1}c_jx_j \in D^G$. Since this is true for every j, $1 \leq j \leq t$, we obtain that $g \in D^G$. Thus the subgroup D is contranormal in G.

In order to prove (v), write K the subgroup generated by all subgroups belonging to the family $\mathcal{S}$. If x is an element of K, then $x = h_1h_2 \cdots h_n$, where $h_j \in H_j \in \mathcal{S}$, $1 \leq j \leq n$. Since C is contranormal in H_j, we have $H_j = C^{H_j}$. Therefore $x \in \langle C^{H_1}, \dots, C^{H_n} \rangle \leq C^K$. Hence $K \leq C^K$.

On the other hand, the inclusion $C \leq K$ implies that $C^K \leq K$. Then $K = C^K$, as required. □

The following result follows easily.

Lemma 2.2. *Let G be a finitely generated soluble-by-finite group. If G does not contain proper contranormal subgroups, then G is nilpotent.*

Proof. Let S be a normal subgroup of G of finite index. If H/S is a proper contranormal subgroup of G/S, then, by Lemma 2.1, H is a proper contranormal subgroup of G. Therefore the group G/S does not contain proper contranormal subgroups. Then G/S is nilpotent, since it is finite. Therefore, every finite quotient of G is nilpotent, and G is nilpotent (see [14]). □

We often use the following result.

Lemma 2.3. *Let G be a group, and let A, C be subgroups of G, such that A is normal in G and $C = SA$, for some subgroup S of G. Then $S^C = S[S, A]$.*

Proof. Let x be an element of C and v be an element of S. Then $x^{-1}vx = v[v, x]$, thus $S^C \leq S[S, C]$. The equality $C = SA$ implies that $x = au$ for some $a \in A$, $u \in S$. Then we have $[v, x] = [v, au] = [v, u]u^{-1}[v, a]u = [v, u][u^{-1}vu, u^{-1}au] \in S[S, A]$. Hence $[S, C] \leq S[S, A]$ and then $S^C = S[S, A]$. □

Corollary 2.4. *Let G be a group, and let A, C subgroups of G, such that A is normal in G and $C = SA$, for some subgroup S of G. If $A = [S, A]$, then S is contranormal in C.*

Now we can prove Theorem B.

Proof. (of Theorem B). Let S be a finitely generated subgroup of G. Since G/A is a Baer group, then $SA/A = C/A$ is subnormal in G/A. Thus C is subnormal in G. C is an extension of a nilpotent normal subgroup by a finite nilpotent group, and it is contranormal-free, then C is nilpotent (see, for example [17]). Hence S is subnormal in C, and so S is subnormal in G. Therefore G is a Baer group. □

3. Nonperiodic groups whose subnormal subgroups are contranormal-free

In this section we investigate not necessarily periodic groups, whose subnormal subgroups are contranormal-free.

We start with some module-theoretical notions and some module-theoretical results.

Let R be an integral domain, and let A be a module over R. If S is a subset of R, then we define the *annihilator of* S *in* A by the rule

$$Ann_A(S) = \{a \in A \mid ax = 0, \forall x \in S\}.$$

It is not difficult to prove that if the ring R is commutative, then the annihilator of every subset S of R is an R-submodule of A.

Dually, if B is a subset of A, then we define the *annihilator of* B *in* R by the rule

$$Ann_R(B) = \{x \in R \mid bx = 0, \forall b \in B\}.$$

It is not hard to see that if the ring R is commutative, then the annihilator of every subset B of A is an ideal of R.

We put

$$Tor_R(A) = \{a \in A \mid Ann_R(a) \neq \langle 0 \rangle\}.$$

It is possible to prove that $Tor_R(A)$ is an R-submodule of A, called the *R-periodic part of A*. The R-module A is said to be *R-periodic* if $A = Tor_R(A)$, and R-torsion-free if $Tor_R(A) = \langle 0 \rangle$.

We define the *R-assassinator of A* by putting

$$Ass_R(A) = \{P \mid P \textit{ is a prime ideal of } R \textit{ such that } Ann_A(P) \neq \langle 0 \rangle\}.$$

If U is an ideal of R, we will write

$$A_U = \{a \in A \mid aU^n = \langle 0 \rangle, \textit{for some positive integer } n\}.$$

It is not difficult to prove that A_U is an R-submodule of A, called *the U-component of A*. If $A = A_U$, then A is called an *U-module*. Furthermore, let

$$\Omega_{U,k}(A) = \{a \in A \mid aU^k = \langle 0 \rangle\}.$$

It is easy to see that $\Omega_{U,k}(A)$ is an R-submodule and that

$$\Omega_{U,1}(A) \leq \Omega_{U,2}(A) \leq \cdots \leq \Omega_{U,k}(A) \leq \cdots,$$

$$A_U = \bigcup_{k \in \mathbb{N}} \Omega_{U,k}(A).$$

In particular, if A is an (additive) abelian group and p is a prime, then $\Omega_{p\mathbb{Z},k}(A)$ is exactly the subgroup $\{a \in A \mid |a| \leq p^k\}$ (where $|a|$ denotes the order of a in $A(+)$).

If D is a Dedekind domain and A is a D-periodic module, then $A = \bigoplus_{P \in \pi} A_P$, where $\pi = Ass_D(A)$. Moreover, if B is a D-submodule of A, then $(A/B)_P = (A_P + B)/B$, for every $P \in Ass_D(A)$, and $A/B = \bigoplus_{P \in \pi}((A_P + B))/B)$ (see, for example [13, Corollary 3.8]).

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Lemma 3.1. *Let $\langle g\rangle$ be an infinite cyclic group, and let A be a $\mathbb{Z}\langle g\rangle$-module, whose additive group is an abelian p-group, where p is a prime. If the lower layer $L = \Omega_{p\mathbb{Z},1}(A)$ of A is $\mathbb{Z}\langle g\rangle$-periodic, then A is $\mathbb{Z}\langle g\rangle$-periodic. Moreover, if $J = F_p\langle g\rangle$ is the group ring of the infinite cyclic group $\langle g\rangle$ over the prime field F_p, then $Ass_J(\Omega_{p\mathbb{Z},k+1}(A)/\Omega_{p\mathbb{Z},k}(A)) = Ass_J(L)$, for every positive integer k.*

Proof. Let $J = F_p\langle g\rangle$ be the group ring of the infinite cyclic group $\langle g\rangle$ over the prime field F_p. Put $L_k = \Omega_{p\mathbb{Z},k}(A)$, where $k \in \mathbb{N}$. The map $\varphi : L_2 \to L_1 = L$, defined by the rule $\varphi(a) = pa$, $a \in L_2$, is a $\mathbb{Z}\langle g\rangle$-endomorphism. Let $a \in L_2$. Since L_1 is J-periodic, then there exists an element $y \in J$ such that $0 = (pa)y = p(ay)$. Moreover, $Ann_J(a) = Jy = P_1^{t_1} \cdots P_n^{t_n}$, for some ideals $P_1, \dots, P_n \in Ass_J(L)$, and some positive integers $t_1, \dots, t_n$. It follows that $ay \in L_1$. Therefore $Ann_J(a + L_1) = Ann_J(a)$. Hence the J-module L_2/L_1 is J-periodic and $Ass_J(L_2/L_1) = Ass_J(L_1)$. Furthermore, $Ann_{\mathbb{Z}\langle g\rangle}(a) \neq \langle 0\rangle$, hence L_2 is $\mathbb{Z}\langle g\rangle$-periodic. The result follows by induction. □

Corollary 3.2. *Let $\langle g\rangle$ be an infinite cyclic group, and let A be a $\mathbb{Z}\langle g\rangle$-module whose additive group is an abelian p-group, where p is a prime. Put $A_k = \Omega_{p\mathbb{Z},k}(A)$, for every $k \in \mathbb{N}$. If the lower layer A_1 of A is a P-module for some prime ideal P of the ring $F_p\langle g\rangle$, then every factor A_{k+1}/A_k is a P-module.*

Let A be an R-module. The intersection $Mon_R(A)$ of all non-zero R-submodules of A is called the **R-monolith** of A. An R-module A is said to be **R-monolithic** if its R-monolith is non-trivial. In this case the R-monolith of A is the unique simple R-submodule of A.

Let G be a group, and let g be an element of $\zeta(G)$. Let A be a $\mathbb{Z}G$-module, whose additive group is a p-group for some prime p. Let $\langle x_g\rangle$ be an infinite cyclic group. Then we can consider A as $\mathbb{Z}\langle x_g\rangle$-module by putting $ax_g = ag$, for each element $a \in A$. Now let $J_g = F_p\langle x_g\rangle$ be the group ring of the group $\langle x_g\rangle$ over the prime field F_p. Note that J_g is a principal ideal domain. In a natural way we can consider every factor $\Omega_{p\mathbb{Z},k+1}(A)/\Omega p\mathbb{Z}, k(A)$ as a J_g-module.

Lemma 3.3. *Let G be a finitely generated nilpotent group, and let A be a $\mathbb{Z}G$-module. Suppose that B, C are $\mathbb{Z}G$-submodules of A, such that $B \leq C$, the additive group of C/B is an elementary abelian p-group for some prime p and C/B is a P-module for some prime ideal P of the group ring J_g, where $g \in \zeta(G)$. Then A contains a $\mathbb{Z}G$-submodule D, such that the additive group of A/D is a p-group, A/D is a $\mathbb{Z}\langle g\rangle$-periodic, and every factor $\Omega_{p\mathbb{Z},k+1}(A/D)/\Omega_{p\mathbb{Z},k}(A/D)$ is a P-module, for all positive integers k.*

Proof. Without loss of generality, we may assume that $B = \langle 0\rangle$. Then $C \leq \Omega_{p\mathbb{Z},1}(A) = L$. We have $L = \bigoplus_{Q\in\pi} L_Q$, where $\pi = Ass_J(L)$ and $C \leq L_P$. Set $E = \bigoplus_{Q\in\pi\setminus\{P\}} L_Q$, then $L = L_P \oplus E$. Clearly E is a $\mathbb{Z}G$-submodule of A. Consider the factor module A/E. Then, again without loss of generality, we may assume that $E = \langle 0\rangle$. Hence L is a P-module. Put $U = \Omega_{P,1}(L)$. Then $U = \bigoplus_{\lambda\in\Lambda} U_\lambda$, where $U_\lambda \simeq_{J_g} J_g/P$, in particular, U_λ is a finite simple J_g-module, $\lambda \in \Lambda$. Fix some λ, and choose an element $0 \neq a \in U_\lambda$. Let

$$\mathcal{M} = \{X \mid X \text{ is a } \mathbb{Z}G - \text{submodule of } A \text{ such that } a \notin X\}.$$

Let D be a maximal element of the family $\mathcal{M}$, ordered by inclusion. Then A/D is a monolithic $\mathbb{Z}G$-module. Denote by M/D the $\mathbb{Z}G$-monolith of A/D. Being a simple $\mathbb{Z}G$-module, M/D is finite (see [4, Theorem 5.1]). Clearly M/D is an elementary abelian p-subgroup. Suppose that A/D has an element $b + D$ such that the $\mathbb{Z}G$-submodule B/D, generated by $b + D$, is infinite. Since A/D is a monolithic $\mathbb{Z}$-module, we obtain $M/D \leq B/D$. On the other hand, the natural semidirect product $(B/D) \rtimes G$ is a finitely generated abelian-by-nilpotent group, then it is residually finite (see [4, Theorem 1]). Therefore, since M/D is finite, B/D contains a $\mathbb{Z}G$-submodule Y/D such that $(M/D) \cap (Y/D) = \langle 0 \rangle$ and the index $|B : Y|$ is finite. In particular, Y/D is non-zero, and we obtain a contradiction. This contradiction shows that every element of A/D generates a finite $\mathbb{Z}G$-submodule. In particular, the additive group of A/D is periodic. Moreover, the fact that a is a p-element implies that A/D is a p-group. Also we obtain that $\Omega_{p\mathbb{Z},1}(A/D)$ is J_g-periodic. Since every Q-component of $\Omega_{p\mathbb{Z},1}(A/D)$ is a $\mathbb{Z}G$-submodule for each prime ideal Q of J_g, we obtain that in this case $\Omega_{pZ,1}(A/D)$ is a P-module. The result follows from Corollary 3.2. □

Lemma 3.4. *Let $\langle g \rangle$ be an infinite cyclic group, and let A be an $F_p\langle g \rangle$-module. Suppose that A is a P-module for some prime ideal P of the ring $F_p\langle g \rangle$. If $P \neq (g–1)F_p\langle g \rangle$, then $A = A(g - 1)$.*

Proof. Let $J = F_p\langle g \rangle$. If a is an element of A, then $Ann_J(a) = P^k$, for some positive integer k. Since $(g - 1)J$ is a maximal ideal of J, then $P^k + (g - 1)J = J$. It follows that $1 = (g - 1)x + y$, for some elements $x \in J$, $y \in P^k$. Then $a = a \cdot 1 = a((g - 1)x + y) = ax(g - 1)$. Thus $a \in A(g - 1)$, which proves the equality $A = A(g - 1)$. □

Corollary 3.5. *Let $\langle g \rangle$ be an infinite cyclic group, and let A be a $\mathbb{Z}\langle g \rangle$-module whose additive group is an abelian p-group, where p is a prime. Suppose that $\Omega_{p\mathbb{Z},1}(A)$ is a P-module for some prime ideal P of the ring $F_p\langle g \rangle$. If $P \neq (g - 1)F_p\langle g \rangle$, then $A = A(g - 1)$.*

Proof. Put $A_k = \Omega_{pZ,k}(A)$, $k \in \mathbb{N}$. By Corollary 3.2 every factor A_{k+1}/A_k is a P-module. Lemma 3.4 implies that $A_{k+1}/A_k = (A_{k+1}/A_k)(g - 1)$. Furthermore, $A_2/A_1 = (A_2/A_1)(g - 1) = (A_2(g - 1) + A_1)/A_1$.

The last equality, together with $A_1(g - 1) = A_1$, implies that $A_2(g - 1) = A_2$. By induction, we obtain that $A_k = A_k(g-1)$, for every positive integer k. Therefore $A = \bigcup_{k\in\mathbb{N}} A_k = \bigcup_{k\in\mathbb{N}} A_k(g-1) = (\bigcup_{k\in\mathbb{N}} A_k)(g - 1) = A(g - 1)$. □

Corollary 3.6. *Let G be a finitely generated nilpotent group, and let A be a $\mathbb{Z}G$-module. Suppose that B, C are $\mathbb{Z}G$-submodules of A, such that $B \leq C$, the additive group of C/B is an elementary abelian p-group for some prime p, and C/B is a P-module for some prime ideal P of the group ring J_g, where $g \in \zeta(G)$. If $P \neq (g - 1)F_p\langle g \rangle$, then A contains a $\mathbb{Z}G$-submodule D such that $A/D = (A/D)(g - 1)$.*

Corollary 3.7. *Let G be a finitely generated nilpotent group, and let A be a $\mathbb{Z}G$-module. Suppose that B, C are $\mathbb{Z}G$-submodules of A, such that $B \le C$ and the additive group of C/B is an abelian p-group for some prime p. Suppose that the lower layer $\Omega_{p\mathbb{Z},1}(C/B)$ is not $F_p\langle g\rangle$-periodic for some element $g \in \zeta(G)$. Then A contains a $\mathbb{Z}G$-submodule U such that $A/U = (A/U)(g-1)$.*

Proof. Without loss of generality, we may assume $B = \langle 0\rangle$. Let $J = F_p\langle g\rangle$. Since $C_1 = \Omega_{p\mathbb{Z},1}(C)$ is not J-periodic, then the element g has infinite order. Hence the group ring J is a principal ideal domain. We will consider C_1 as a JG-module. Since C_1 is not J-periodic, it has an element d such that $Ann_J(d) = \langle 0\rangle$. Then the J-submodule dJ generated by the element d is isomorphic with J. Denote by D the JG-submodule of C generated by the element d. Then D is infinite. D contains a free J- submodule E such that D/E is J-periodic, moreover $Ass_J(D/E)$ is finite (see, for example [8, Theorem 1.7]). Clearly the set of prime ideals of J is infinite, thus there exists a prime ideal P of J such that $P \notin Ass_J(D/E)$ and $P \neq (g-1)J$. Write $S = EP$. Since E is a free J-submodule, then $S \neq E$. The choice of P shows that E/S is the P-component of D/S. Then $D/S = E/S \oplus W/S$, where W/S is the P'-component of D/S. Hence $(D/S)P \le W/S$, in particular, $D/S \neq (D/S)P$. We have $(D/S)P = (DP + S)/S$, and we obtain that $D \neq DP$. Since DP is a $\mathbb{Z}G$-submodule of D, D/DP is a non-zero $\mathbb{Z}G$-factor, whose additive group is an elementary abelian p-group and $Ann_J(D/DP) = P$. The result follows from Corollary 3.6. □

Let G be a group, let R be a ring, and A an RG-module. We define the *G-center* $\zeta_G(A)$ of the module A by the following rule:

$$\zeta_G(A) = \{a \mid a \in A,\ ag = a, \forall g \in G\}.$$

It is easy to prove that the G-center of A is an RG-submodule.

Then we can construct the *upper G-central series*:

$$\langle 0\rangle = \zeta_{G,0}(A) \le \zeta_{G,1}(A) \le \cdots \le \zeta_{G,\alpha}(A) \le \zeta_{G,\alpha+1}(A) \le \cdots \le \zeta_{G,\gamma}(A)$$

of the module A by the following rule:

$$\zeta_{G,1}(A) = \zeta_G(A),\ \zeta_{G,\alpha+1}(A)/\zeta_{G,\alpha}(A) = \zeta_G(A/\zeta_{G,\alpha}(A)), \textit{for all ordinals } \alpha,$$

$$\zeta_{G,\lambda}(A) = \bigcup_{\beta<\lambda} \zeta_{G,\beta}(A), \textit{for a limit ordinal } \lambda,\ \textit{and the } G-\textit{center of } A/\zeta_{G,\gamma}(A) \textit{ is trivial}.$$

Notice that every term of this series is an RG-submodule.

A module A is said to be *G-hypercentral* if the last term $\zeta_{G,\gamma}(A)$ of this series coincides with A.

A module A is said to be *G-nilpotent* if $A = \zeta_{G,n}(A)$, for some positive integer n.

Lemma 3.8. *Let G be an abelian finitely generated group, and let A be a $\mathbb{Z}G$-module. If A is $\langle g\rangle$-hypercentral for every element $g \in G$, then A is G-hypercentral.*

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Proof. Since G is a finitely generated abelian group, $G = \langle g_1 \rangle \times \cdots \times \langle g_k \rangle$, where $\langle g_j \rangle$ is a non-trivial cyclic p-group for some prime p or $\langle g_j \rangle$ is an infinite cyclic subgroup, $1 \leq j \leq k$, and the number k is an invariant for the group G. We will use induction on k. If $k = 1$, there is nothing to prove.

Suppose now that $k > 1$. Let

$$\langle 1 \rangle = A_0 \leq A_1 \leq \cdots \leq A_\alpha \leq A_{\alpha+1} \leq \cdots \leq A_\gamma = A$$

be the upper $\langle g_1 \rangle$-central series of A. Consider an arbitrary factor $A_{\alpha+1}/A_\alpha$ of this series. Since A is $\langle g_j \rangle$-hypercentral for all j, $1 \leq j \leq k$, then $A_{\alpha+1}/A_\alpha$ is $\langle g_j \rangle$-hypercentral for all $j > 1$. Furthermore, $g_1 \in C_G(A_{\alpha+1}/A_\alpha)$, thus we can consider the factor $A_{\alpha+1}/A_\alpha$ as a $\mathbb{Z}(G/\langle g_1 \rangle)$-module. By induction, the factor $A_{\alpha+1}/A_\alpha$ is $\mathbb{Z}(G/\langle g_1 \rangle)$-hypercentral. The choice of the element g_1 ensures that every $\mathbb{Z}(G/\langle g_1 \rangle)$-central factor of $A_{\alpha+1}/A_\alpha$ is also $\mathbb{Z}G$-central. Since this is true for every ordinal $\alpha < \gamma$, we can construct a refinement of the series $\{A_\alpha \mid \alpha < \gamma\}$ whose factors are G-central. Therefore A is G-hypercentral. □

Lemma 3.9. *Let G be a group, and let A be a normal abelian subgroup of G, such that G/A is a finitely generated soluble-by-finite group. Suppose that B, C are G-invariant subgroups of A, such that $B \leq C$ and the additive group of C/B is an abelian p-group, for some prime p. If G is contranormal-free, then the factor C/B is G-hypercentral.*

Proof. Let S be a finitely generated subgroup of G such that $G = SA$. If $S = G$, then, by Lemma 2.2, the group G is nilpotent, and all is proved. Therefore we can assume that S is a proper subgroup of G. By Lemmas 2.1 and 2.2, the factor group G/A is nilpotent. If G/A is finite, then the result follows from [17]. Now assume that G/A is infinite. Then its center contains elements of infinite order (see, for example [4, Lemma 7]). Let $K = C/A$ and consider A as a $\mathbb{Z}K$-module. Let $g \in \zeta(K)$ and write, as before, $J_g = F_p\langle x_g \rangle$. Without loss of generality we may assume that $B = \langle 0 \rangle$. Suppose first that $C_1 = \Omega_{p\mathbb{Z},1}(C)$ is a J_g-periodic module. In particular, this is true if the element g has finite order. Then $C_1 = \bigoplus_{P \in \pi} L_P$ where $\pi = Ass_J(C_1)$. Suppose now that $Ass_J(C_1) \neq \{(x_g - 1)J_g\}$, and let P be a a prime ideal such that $P \neq (x_g - 1)J_g$. Then the P-component L_P is different from zero. Write $B = \bigoplus_{Q \in \pi,\ Q \neq P} L_Q$, then $C_1 = L_P \oplus B$ and B is a $\mathbb{Z}G$-submodule of A. We have $C_1/B \simeq_{J_g} L_P$, hence C_1/B is a P-module. Then, by Corollary 3.6, we get than A contains a $\mathbb{Z}G$-submodule D such that $A/D = (A/D)(g-1)$, and then, in multiplicative notation, $[zD, A/D] = A/D$ where $zA = g$. Therefore $A/D = [SD/D, A/D] = [S, A]D/D$, and $[S, A] = A$. Then the subgroup S is contranormal in $G = SA$, by Corollary 2.4. This contradiction shows that C_1 is a $(x_g - 1)J_g$-module. Write $R = (x_g - 1)J_g$. For every element $a \in \Omega_{R,1}(C_1)$ we have that $a(z-1) = 0$. Therefore $\Omega_{R,1}(C_1)$ is contained in the $\langle z \rangle$-center of L. Arguing similarly, we show that all the factors $\Omega_{R,n+1}(C_1)/\Omega_{R,n}(C_1)$ are $\langle z \rangle$-central. The equality $C_1 = \bigcup_{k \in \mathbb{N}} \Omega_{R,k}(C_1)$ shows that C_1 is $\langle z \rangle$-hypercentral. This is true for each element $z \in \zeta(G)$. Since the subgroup $\zeta(G)$ is finitely generated, by Lemma 3.8 we obtain that C_1 is $\zeta(G)$-hypercentral. Now we use induction on the nilpotency class of G, $ncl(G)$. If G is abelian,

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then $G = \zeta(G)$, and we have the result. Suppose now that $ncl(G) > 1$. We have proved that C_1 has an upper $\zeta(G)$-central series:

$$\langle 0 \rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_\alpha \leq Z_{\alpha+1} \leq \cdots \leq Z_\gamma = C_1.$$

Consider a factor $Z_{\alpha+1}/Z_\alpha$ of this series. Since $\zeta(G) \leq C_G(Z_{\alpha+1}/Z_\alpha)$, we can consider this factor as a $\mathbb{Z}(G/\zeta(G))$-module. By induction the factor $Z_{\alpha+1}/Z_\alpha$ is $\mathbb{Z}(G/\zeta(G))$-hypercentral. The choice of this series implies that every $\mathbb{Z}(G/\zeta(G))$-central factor of $Z_{\alpha+1}/Z_\alpha$ is also $\mathbb{Z}G$-central. Since this is true for every ordinal $\alpha < \gamma$, then we can construct a refinement of this series $\{Z_\alpha \mid \alpha \leq \gamma\}$, whose factors are G-central. Write $C_k = \{a \mid |a| \leq p^k\}$, $k \in \mathbb{N}$. The map $\phi : C_2 \to C_1$, defined by the rule $\phi(a) = pa$, $a \in C_2$, is a $\mathbb{Z}G$-endomorphism. We have $Ker(\phi) = C_1$, $Im(\phi) \leq C_1$, therefore C_2/C_1 is isomorphic to some $\mathbb{Z}G$-submodule of C_1. Then C_2/C_1 is G-hypercentral. Then the submodule C_2 is G-hypercentral, and, by induction, we obtain that C is G-hypercentral.

Suppose now that C_1 is not J_g-periodic. Then, by Lemma 3.7, A contains a $\mathbb{Z}(G)$-submodule U, such that $A/U = (A/U)(g-1)$. Using the above arguments, we obtain a contradiction, which proves the result. □

Corollary 3.10. *Let G be a group, and let A be a normal abelian subgroup of G, such that G/A is a finitely generated soluble-by-finite group. Suppose that p is a prime, and let P be the Sylow p-subgroup of A. If G is contranormal-free, then P is contained in the hypercenter of G.*

Corollary 3.11. *Let G be a group, and let A be a normal abelian subgroup of G, such that G/A is a finitely generated soluble-by-finite group. Write T the periodic part of A. If G is contranormal-free, then T is contained in the hypercenter of G.*

Corollary 3.12. *Let G be a nilpotent finitely generated group, and let A be a finitely generated $\mathbb{Z}G$-module, whose additive group is torsion-free. If the factor A/pA is G-hypercentral for each prime p, then A is G-nilpotent.*

Proof. Since G is a finitely generated nilpotent group, A contains a free abelian subgroup E, such that the additive group of A/E is periodic, and the set $\Pi(A/E)$ is finite (see, for example [8, Theorem 1.7]). Let $\pi = \Pi(A/E)$ and let $p \notin \pi$. Write $S = pE$. Since E is a free abelian group, then $S \neq E$. The choice of the prime p implies that E/S is the p-component of A/S. Then $A/S = E/S \times B/S$, where B/S is the p'-component of A/S. Hence $p(A/S) \leq B/S$, in particular, $A/S \neq p(A/S)$. We have $p(A/S) = p(A/pE) = (pA + pE)/pE = pA/pE$, therefore $A \neq pA$. Since the factor A/pA is G-hypercentral and G/A is nilpotent, then G/pA is hypercentral. Since A is finitely generated as $\mathbb{Z}G$-module and G/A is a finitely generated group, then G/pA is a finitely generated group. Being hypercentral, G/pA is nilpotent. Then its periodic subgroup A/pA is finite. This is true for all primes $p \in \pi$. Since the set π contains almost all primes, it follows that A has finite 0-rank, say r (see, for example [18, Theorem 3.1]). Then the factor A/pA has finite order less or equal to p^r. Since G/pA is nilpotent, we obtain that $[A,_n G] \leq pA$. This is true for each $p \in \pi$, therefore $[A,_n G] \leq \bigcap_{p\in\pi} pA = C$. We have proved that $pA \cap E = pE$, hence $C \cap E = (\bigcap_{p\in\pi} pA) \cap E = \bigcap_{p\in\pi}(pA \cap E) = \bigcap_{p\in\pi} pE$. The choice of the set π implies that $\bigcap_{p\in\pi} pE = \langle 0\rangle$. Therefore we have $C \simeq C/(C \cap E) \simeq CE/E$. On the other hand, A/E is periodic and A is torsion-free, hence $C = \langle 0\rangle$. Hence $[A,_n G] = \langle 0\rangle$, and A is G-nilpotent. □

Lemma 3.13. *Let G be a group, and let A be a normal abelian subgroup of G such that G/A is a finitely generated soluble-by-finite group. Write T the periodic part of A. If G is contranormal-free, then G/T is hypercentral.*

Proof. Without loss of generality, we may assume that T is trivial. Then G/A is nilpotent, arguing as in Lemma 3.9. Let M be an arbitrary finite subset of A, and write $D = \langle M\rangle^G$. If p is a prime, then D^p is a G-invariant subgroup of D. By Lemma 3.9, D/D^p is G-hypercentral. Then, by Lemma 3.12, there exists a positive number $n(p)$ such that $D/D^p \leq \zeta_{n(p)}(G/D^p)$. It follows that $A \leq \zeta_\omega(G)$. Then G is hypercentral, since G/A is nilpotent. □

From Lemma 3.13 the following result follows, yet proved in the paper [1].

Corollary 3.14. *Let G be a group, and let A be a normal abelian subgroup of G, such that G/A is a finitely generated soluble-by-finite group. If G is contranormal-free, then G is hypercentral.*

Proof. As in Lemma 3.9 we can prove that G/A is nilpotent. Let T be the periodic part of A. Then, by Corollary 3.11, the hypercenter of G contains T. Furthermore, G/T is hypercentral by Lemma 3.13, therefore G is hypercentral, as required. □

Let J be a principal ideal domain, and let A be a simple J-module. Then $A \simeq J/P$, for some maximal ideal P. It is not difficult to prove that J/P^k and P/P^{k+1} are isomorphic as J-modules, for every positive integer k. In particular, the J-module D/P^k is embedded in the J-module J/P^{k+1}, $k \in \mathbb{N}$. Therefore we can consider the injective family of J-modules $\{J/P^k \mid k \in \mathbb{N}\}$. Denote by C_{P^∞} the injective limit of this family. The J-module C_P^∞ is called the **Prüfer P-module**.

If $P = Jy$ then the Prüfer P-module C has a set $\{a_k \mid k \in \mathbb{N}\}$ of generators such that $a_1y = 0$, $a_2y = a_1$, $a_{k+1}y = a_k$, for every $k \in \mathbb{N}$. Moreover, every proper submodule of C is equal to a_mJ, for some positive integer m and $Cy = C$ (see, for example [13, Chapter 5]).

Lemma 3.15. *Let $\langle g\rangle$ be an infinite cyclic group, and let A be a J-module where $J = F_p\langle g\rangle$. Suppose that A is J-periodic, and A is an S-module where $S = (g-1)J$. If $Ann_J(A) = \langle 0\rangle$, then A contains a proper J-submodule B such that $(A/B) = (A/B)(g-1)$.*

Proof. By [13, Theorem 9.4] of the book, A contains a J-submodule D, satisfying the following conditions:

D is a direct sum of cyclic submodules;

$Dx = B \cap Ax$, for every element $x \in S$, in particular, $D(g-1) = D \cap A(g-1)$;

$A/D = (A/D)(g-1)$.

If $D \neq A$, write $B = D$. Suppose that $A = D$, then A is a direct sum of cyclic submodules: $A = \bigoplus_{\lambda \in \Lambda} C_\lambda$, where C_λ is a cyclic J-submodule for every $\lambda \in \Lambda$. Fix some index λ_1 and choose in C_{λ_1} an element v_1 such that $Ann_J(v_1) = S$. Write $V_1 = v_1 J$. Since $Ann_J(A) = \langle 0 \rangle$, there exists an index λ_2 such that $Ann_J(C_{\lambda_2}) = S^n$, where $n \geq 2$. Choose in C_{λ_2} an element v_2 such that $Ann_J(v_2) = S^2$. Put $V_2 = v_2 J$. Using the same arguments, we construct, for every $n \in \mathbb{N}$, a J-submodule $V = \oplus_{n \in \mathbb{N}} V_n$, where V_n is a cyclic J-submodule, $V_n = v_n J$, such that $Ann_J(v_n) = S^n$. Put $u_2 = v_2(g-1) - v_1$, $u_3 = v_3(g-1) - v_2, \dots, u_{n+1} = v_n + 1(g-1) - v_n$, $n \in \mathbb{N}$. Now write $U = \sum_{n \geq 2} u_n J$. Then it is easy to prove that $U = \oplus_{n \geq 2} u_n J$. Then we have

$$V_1 \cap U = \langle 0 \rangle, \ (v_2 + U)(g-1) = v_2(g-1) + U = v_1 + U,$$

$$\dots, \ (v_{n+1} + U)(g-1) = v_n + U,$$

for every $n \in \mathbb{N}$. Hence the factor V/U is a Prüfer $(g-1)J$-module. In particular, $(V/U)(g-1) = V/U$. Then A/U contains a J-submodule B/U such that $A/U = (B/U) \oplus (V/U)$ (see, for example [13, Theorem 5.13 and Lemma 5.20]). Hence $A/B \simeq (A/U)/(B/U) \simeq V/U$, in particular, $(A/B)(g-1) = A/B$, as required. □

Corollary 3.16. *Let $\langle g \rangle$ be an infinite cyclic group, and let A be a $\mathbb{Z}\langle g \rangle$-module whose additive group is an abelian p-group, where p is a prime. Suppose that $\Omega_{pZ,1}(A) = L$ is an S-module where $S = (g-1)F_p\langle g \rangle$. If $Ann_J(L) = \langle 0 \rangle$, then A contains a $\langle g \rangle$-submodule V such that $A/V = A/V(g-1)$.*

Proof. Let $J = F_p\langle g \rangle$. By Lemma 3.15 L contains a J-submodule B such that $(L/B)(g-1) = L/B$. Then L/B is a direct sum of Prüfer S-modules (see, for example [13, Theorem 5.26]). Therefore, without loss of generality, we may assume that L/B is a Prüfer S-module. Set

$$\mathcal{M} = \{X \mid X \ is \ a \ \mathbb{Z}\langle g \rangle - submodule \ of \ A \ such \ that \ X \cap L = B\}.$$

Let D be a maximal element of the family $\mathcal{M}$, ordered by inclusion. Put $A_k/D = \Omega_{\mathbb{Z},k}(A/D)$, $k \in \mathbb{N}$. Suppose that $A_1/D \neq (L+D)/D$. Then $A_1/D = (L+D)/D \oplus C/D$, for some non-zero $\mathbb{Z}\langle g \rangle$-submodule C/D (see, for example [13, Theorem 5.13]). Then $C \cap L = B$, and we obtain a contradiction with the choice of D. This contradiction shows that $A_1/D = (L+D)/D \simeq L/(D \cap L) = L/B$, in particular, A_1/D is a Prüfer S-module.

The map $\phi : A_2/D \to A_1/D$, defined by the rule $\phi(a+D) = p(a+D)$, $a \in A_2$, is a $\mathbb{Z}G$-endomorphism. We have $Ker(\phi) = A_1/D$, $Im(\phi) \leq A_1/D$, thus A_2/A_1 is isomorphic to some $\mathbb{Z}\langle g \rangle$-submodule of A_1/D. It follows that either $Im(\phi) = A_1/D$ (in this case A_2/A_1 is a Prüfer S-module) or $Im(\phi)$ is a proper $\mathbb{Z}\langle g \rangle$-submodule of A_1/D. In the first case

$$A_2/A_1 = (A_2/A_1)(g-1) = (A_2(g-1) + A_1)/A_1.$$

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The last equality, together with $A_1(g-1) = A_1$, implies that $(A_2/D)(g-1) = A_2/D$. In the second case the lower layer of A/A_1 is finite. Then either A/A_1 is finite or A/A_1 is an infinite Chernikov group. If A/A_1 is finite, then it is not difficult to prove that A/D contains a finite $\mathbb{Z}\langle g\rangle$-submodule V/D such that $A/D = (A_1/D) + (V/D)$. Then $(A/V)(g-1) = A/V$. If A/A_1 is an infinite Chernikov group, then A/D contains a divisible Chernikov subgroup U/D such that $A/(A_1+U)$ is finite, and we are in the previous case. More generally, either A_{k+1}/A_k is a Prüfer S-module for every $k \in \mathbb{N}$, or there exists a positive integer m such that A_{k+1}/A_k is a Prüfer S-module whenever $0 \le k \le m$–1, and A_{m+1}/A_m is finite (here $A_0 = D$). In the first case, by induction, we obtain that $(A_k/D)(g-1) = A_k/D$, for every positive integer k. Then $A/D = \bigcup_{k\in\mathbb{N}} A_k/D = \bigcup_{k\in\mathbb{N}}(A_k/D)(g-1) = (\bigcup_{k\in\mathbb{N}} A_k/D)(g-1) = (A/D)(g-1)$. In this case, we choose $V = D$. In the second case, using again the above arguments, we get a $\mathbb{Z}\langle g\rangle$-submodule V such that $(A/V)(g-1) = A/V$. □

Corollary 3.17. *Let G be a group, let A be a normal abelian p-subgroup of G, p a prime, and let g be an element of infinite order. If the subgroup $A\langle g\rangle$ is contranormal-free, then $\Omega_{p\mathbb{Z},1}(A) = L$ is $\langle g\rangle$-nilpotent.*

Proof. Let $J = F_p\langle g\rangle$. By Corollary 3.14, the subgroup $C = A\langle g\rangle$ is hypercentral. It follows that L is an S-module, where $S = J(g-1)$. If we suppose that L is not $\langle g\rangle$-nilpotent, then $Ann_J(A) = \langle 0\rangle$. Then, by Corollary 3.16, A contains a $\langle g\rangle$-invariant subgroup V such that $(A/V)(g-1) = A/V$. By Corollary 2.4, $\langle g\rangle V/V$ is contranormal in C/V. Then, by Lemma 2.1, the subgroup $\langle g\rangle V$ is contranormal in C, and we obtain a contradiction. □

Corollary 3.18. *Let G be a group, let A be a normal abelian p-subgroup of G, p a prime, and let g be an element of infinite order. If every subnormal subgroup of $S = A\langle g\rangle$ is contranormal-free and the $\mathbb{Z}\langle g\rangle$-module A is monolithic, then S is nilpotent.*

Proof. The subgroup $S = A\langle g\rangle$ is hypercentral, by Corollary 3.14. Let X be an arbitrary finite subset of A, then the subgroup $\langle X, g\rangle$ is nilpotent (see, for example, [15, p. 50]). Obviously, the periodic part of the finitely generated nilpotent group $\langle X, g\rangle$ is finite, and then it is contained in some term with finite index of the upper central series of S. Hence some term of the $\langle g\rangle$-central series of A of finite index contains X. Therefore the length of the upper $\langle g\rangle$-central series of A is at most ω. Since S is hypercentral, then the $\mathbb{Z}\langle g\rangle$-monolith M lies in the center of S and has order p. Then, using [9, Proposition 1.8], we obtain that A satisfies the minimal condition for the $\langle g\rangle$-invariant subgroups. It follows that there exists a positive integer n such that $[A,_n g] = [A,_{n+1} g] = D$. Suppose that $D \neq \langle 1\rangle$. Then S/D is nilpotent. It follows that the subgroup $\langle g\rangle D/D$ is subnormal in S/D, hence $\langle g\rangle D$ is subnormal in S. Hence $\langle g\rangle D$ is contranormal-free. On the other hand $[g, D] = D$ and, by Corollary 2.4, the subgroup $\langle g\rangle$ is contranormal in $\langle g, D\rangle$. This contradiction proves that $D = \langle 1\rangle$. Hence S is nilpotent. □

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Corollary 3.19. *Let G be a group, let A be a normal abelian Chernikov subgroup of G, and let g be an element of G of infinite order. If every subnormal subgroup of $S = A\langle g\rangle$ is contranormal-free, then S is nilpotent.*

Proof. Since A is a Chernikov group, then $Soc(A)$ is finite. Let $Soc(A) = Dr_{1\leq j\leq n}\langle a_j\rangle$. For every j, $1 \leq j \leq n$, let

$$\mathcal{M}_j = \{X \mid X \ is\ a\ \langle g\rangle - invariant\ subgroup\ of\ A\ such\ that\ a_j \notin X\}.$$

Let D_j be a maximal element of the family $\mathcal{M}_j$, ordered by inclusion. Then A/D_j is a monolithic $\mathbb{Z}\langle g\rangle$-module. Then, by Lemma 3.18, S/D_j is nilpotent. The equality $\langle 1\rangle = \bigcap_{1\leq j\leq n} D_j$, together with Remak's theorem, implies that S can be imbedded in the group $Dr_{1\leq j\leq n}S/D_j$. The last group is nilpotent, therefore S is nilpotent, as required. □

Proposition 3.20. *Let G be a group, let A be a normal abelian p-subgroup of G, p a prime, and let g be an element of infinite order. If every subnormal subgroup of $S = A\langle g\rangle$ is contranormal-free, then S is nilpotent.*

Proof. Let $J = F_p\langle g\rangle$. Then, by Corollary 3.14, the subgroup $C = A\langle g\rangle$ is hypercentral. Then $\Omega_{p\mathbb{Z},1}(A) = L$ is $\langle g\rangle$- nilpotent, by Corollary 3.17. Hence L is an S-module where $S = (g-1)J$, moreover $Ann_J(L) = S^n$, for some positive integer n. Then $L = \bigoplus_{\lambda\in\Lambda} C_\lambda$, where C_λ is a cyclic J-submodule for all $\lambda \in \Lambda$ (see, for example, [13, Theorem 6.5]). Suppose that there exists an element $a \in \zeta_{\langle g\rangle}(A)$ such that there is a subset $\{a_k \mid k \in \mathbb{N}\}$ with $a_1(g-1) = a, a_{k+1}(g-1) = a_k$, $k \in \mathbb{N}$. Let

$$\mathcal{M} = \{X \mid X \ is\ a\ \langle g\rangle - invariant\ subgroup\ of\ A\ s.t.\ a \notin X\}.$$

Let D be a maximal element of the family $\mathcal{M}$, ordered by inclusion. Then A/D is a monolithic $\mathbb{Z}\langle g\rangle$-module. Thus S/D is nilpotent, by Lemma 3.18. But the choice of the elements a_k, $k \in \mathbb{N}$, shows that this group can not be nilpotent. This contradiction proves that for each element $a \in \zeta_{\langle g\rangle}(A)$, the subset of all elements a_k such that $a_1(g-1) = a, a_{k+1}(g-1) = a_k$, $k \in \mathbb{N}$, must be finite. Suppose that A is not $\langle g\rangle$-nilpotent. Let $b_1 \in \zeta_{\langle g\rangle}(A)$. Choose the elements b_{1j} such that $b_{11}(g-1) = b_1, b_{1j+1}(g-1) = b_{1j}$, $1 \leq j \leq t_1$. Let

$$\mathcal{M}_1 = \{X \mid X \ is\ a\ \mathbb{Z}\langle g\rangle - submodule\ of\ A\ s.t.\ \ b_1 \notin X\}.$$

Let D_1 be a maximal element of the family $\mathcal{M}_1$, ordered by inclusion. Then A/D_1 is a monolithic $\mathbb{Z}\langle g\rangle$-module. Then, by Lemma 3.18, S/D_1 is nilpotent. Then D_1 is not $\langle g\rangle$-nilpotent. Therefore D_1 contains an element $b_2 \in \zeta_{\langle g\rangle}(A)$ and elements b_{2j} such that $b_{21}(g-1) = b_2, b_{2j+1}(g-1) = b_{2j}$, $1 \leq j \leq t_2$ and $t_2 > t_1$. By induction, we can choose an infinite subset $\{b_n \mid n \in \mathbb{N}\}$ such that $b_n \in \zeta_{\langle g\rangle}(A)$ and for each element b_n there exists a subset $\{b_{nj} \mid 1 \leq j \leq t_n\}$ such that $b_{n1}(g-1) =$

$b_n, b_{nj+1}(g-1) = b_{nj}$, $1 \le j \le t_n$, and $t_{n+1} > t_n$, for all $n \in \mathbb{N}$. Since $b_n \in \zeta_{\langle g\rangle}(A)$, $n \in \mathbb{N}$, the subgroup $U = \langle b_{n+1}b_n^{-1} \mid n \in \mathbb{N}\rangle$ is $\langle g\rangle$-invariant. Let

$$\mathcal{M}_2 = \{X \mid X \ is\ a\ \langle g\rangle - invariant\ subgroup\ of\ A\ s.t.\ U \le X,\ b_1 \notin X\}.$$

Let D_2 be a maximal element of the family $\mathcal{M}_2$, ordered by inclusion. Then A/D_2 is a monolithic $\mathbb{Z}\langle g\rangle$-module. Then S/D_2 is nilpotent, by Lemma 3.18. But the choice of the elements b_n shows that S/D_2 can not be nilpotent, and we obtain a contradiction. This contradiction shows that A is $\langle g\rangle$-nilpotent and hence S is nilpotent. □

Corollary 3.21. *Let G be a group, let A be a normal abelian periodic subgroup of G and g be an element of infinite order. If the set $\Pi(A)$ is finite and every subnormal subgroup of $S = A\langle g\rangle$ is contranormal-free, then S is nilpotent.*

Proof. We have $A = Dr_{p\in\Pi(A)}A_p$, where A_p is the Sylow p-subgroup of A, $p \in \Pi(A)$. Write $C_p = Dr_{q\in\Pi(A),q\neq p}A_q$. Then every subnormal subgroup of S/C_p is contranormal-free, by Lemma 2.1, and, by Proposition 3.20, S/C_p is nilpotent. Since $\langle 1\rangle = \bigcap_{p\in\Pi(A),q\neq p} Cp$, by Remak's theorem, S can be embedded in the group $Dr_{p\in\Pi(A)}S/C_p$, and this group is nilpotent, since $\Pi(A)$ is finite. Hence S is nilpotent, as required. □

Proposition 3.22. *Let G be a group, A be a normal abelian torsion-free subgroup of G and g be an element of infinite order. If A has finite 0-rank and every subnormal subgroup of $S = A\langle g\rangle$ is contranormal-free, then S is nilpotent.*

Proof. Since A has finite 0-rank, there exists a free abelian subgroup B of A such that A/B is periodic. Write C the $\mathbb{Z}\langle g\rangle$-submodule of A generated by B. Then, being a finitely generated module over an infinite cyclic group, C contains a free abelian submodule E such that the additive group of C/E is periodic and the set $\Pi(C/E)$ is finite (see, for example, [8, Theorem 1.7]). Since A has finite 0-rank, then E is a minimax subgroup. Moreover A/E is periodic. Let p be a prime and write $D = E^p$. Denote by P/D the p-component of A/D. Then $A/D = P/D \times Q/D$ where Q/D is the p'-component of A/D. By Lemma 2.1 every subnormal subgroup of S/Q is contranormal-free. Since A has finite 0-rank, then S/Q is a Chernikov p-group. Then, by Corollary 3.19, S/Q is nilpotent. In particular, the $\mathbb{Z}\langle g\rangle$-module A/Q is $\langle g\rangle$-nilpotent. It follows that P/D also is $\langle g\rangle$-nilpotent. Therefore its submodule E/D is $\langle g\rangle$-nilpotent. By Proposition 3.12, E is $\langle g\rangle$-nilpotent. Finally, since A/E is periodic, also A is $\langle g\rangle$-nilpotent. Therefore S is nilpotent, as required. □

Corollary 3.23. *Let G be a group, A be a normal abelian subgroup of G and let g be an element of infinite order. Suppose that the set $\Pi(A)$ is finite and A has finite 0-rank. If every subnormal subgroup of $S = A\langle g\rangle$ is contranormal-free, then S is nilpotent.*

Proof. Let T be the periodic part of A. By Lemma 2.1, every subnormal subgroup of S/T is contranormal-free. Then Proposition 3.22 implies that S/T is nilpotent. Hence the subgroup $T\langle g\rangle$ is subnormal in S. By Corollary 3.19, $T\langle g\rangle$ is nilpotent. Therefore the $\mathbb{Z}\langle g\rangle$-submodule T is G-nilpotent. Then T is contained in the nth term of the upper central series of S, for some positive integer n. Since S/T is nilpotent, we obtain that S is nilpotent. □

Corollary 3.24. *Let G be a group, A be a normal nilpotent subgroup of G and let g be an element of infinite order. Suppose that A has a finite 0-rank and the set $\Pi(A)$ is finite. If every subnormal subgroup of $S = A\langle g\rangle$ is contranormal-free, then S is nilpotent.*

Proof. Let $D = [A, A]$. By Lemma 2.1, every subnormal subgroup of S/D is contranormal-free. By Corollary 3.23, S/D is nilpotent. Then S is nilpotent by [3]. □

Theorem 3.25. *Let G be a group, A be a normal nilpotent subgroup of G such that G/A is a Baer group. Suppose that A has finite 0-rank and the set $\Pi(A)$is finite. If every subnormal subgroup of G is contranormal-free, then G is generated by subnormal nilpotent subgroups. In particular, G is a Baer group.*

Proof. Let g be an element of G such that $g \notin A$. Since G/A is a Baer group, then the subgroup $\langle g, A\rangle$ is subnormal in G. By Corollary 3.24 the subgroup $\langle g, A\rangle$ is nilpotent. □

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