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ON FINITE C-TIDY GROUPS

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Abstract. A group G is said to be a C-tidy group if for every element $x \in G \setminus K(G)$, the set $Cyc(x) = \{y \in G \mid \langle x, y \rangle \text{ is cyclic}\}$ is a cyclic subgroup of G, where $K(G) = \bigcap_{x \in G} Cyc(x)$. In this short note we determine the structure of finite C-tidy groups.

1. Introduction

Let G be a group and $x \in G$. The centralizer of x denoted by $C(x)$ can be defined by

$$C(x) = \{y \in G \mid \langle x, y \rangle \text{ is abelian}\}.$$

In [4], D. Patrick and E. Wepsic replaced the word "abelian" of the above definition with the word "cyclic" and introduced the notion of cyclicizer of an element. To be explicit, define the cyclicizer of x denoted by $Cyc(x)$, by

$$Cyc(x) = \{y \in G \mid \langle x, y \rangle \text{ is cyclic}\}.$$

The cycel of G denoted by $K(G)$ is defined by $K(G) = \bigcap_{x \in G} Cyc(x)$. The group G is called a tidy group if $Cyc(x)$ is a subgroup for all $x \in G$ and G is called a C-tidy group if $Cyc(x)$ is a cyclic subgroup for all $x \in G \setminus K(G)$.

Throughout this paper, all groups are finite and all notations are usual. $PGL(n, q)$ and $PSL(n, q)$ denote the projective general linear and the projective special linear group of degree n over the field of size q respectively. For a group G, 1 and $Z(G)$ denote the identity element and the center respectively.

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A group G is said to be a Frobenius group if it contains a subgroup H such that $\{1\} \neq H \neq G$ and $H \cap H^g = \{1\}$ for all $g \in G \setminus H$. A subgroup with these properties is called a Frobenius complement of G. The Frobenius kernel of G, with respect to H, is defined by $K = (G \setminus \underset{g \in G}{\cup} H^g) \cup \{1\}$.

Let G be a group and p be a prime. We recall that the Hughes subgroup denoted by $H_p(G)$ is defined to be the subgroup generated by all the elements of G whose order is not p. The group G is said to be a group of Hughes-Thompson type if it is not a p-group and $H_p(G) \neq G$ for some prime p.

A finite group G is a CA-group if $C(x)$ is abelian for all $x \in G \setminus Z(G)$. The structure of finite CA-groups were determined by R. Schmidt in [6] (see also Theorem 9.3.12 of [7]).

Similarly, one can think of an analogues study for C-tidy groups. In [2], we classified finite C-tidy groups with $K(G) = \{1\}$. In this paper, we continue with this problem and determine the structure of finite C-tidy groups.

2. Structure theorem for C-tidy groups

We begin with the following proposition which will be used in proving the main result of this section. In this section G is a non-cyclic group.

Proposition 2.1. *Let G be a finite group. Then G is C-tidy if and only if $\frac{G}{K(G)}$ is C-tidy.*

Proof. Let G be a finite C-tidy group and $x \in G \setminus K(G)$. By Lemma 2.3 of [1], we have $\frac{Cyc(x)}{K(G)} = Cyc(xK(G))$ and $xK(G) \in \frac{G}{K(G)} \setminus K(\frac{G}{K(G)})$. Since $Cyc(x)$ is cyclic, therefore $\frac{Cyc(x)}{K(G)}$ is cyclic and hence $\frac{G}{K(G)}$ is C-tidy.

Conversely, suppose $\frac{G}{K(G)}$ is a finite C-tidy group. Let $xK(G) \in \frac{G}{K(G)} \setminus K(\frac{G}{K(G)})$. Then $x \in G \setminus K(G)$. By Lemma 2.3 of [1], we have $\frac{Cyc(x)}{K(G)} = Cyc(xK(G))$ and so G is a tidy group. Again, since $\frac{Cyc(x)}{K(G)}$ is cyclic, therefore $Cyc(x)$ is an abelian group. Hence $Cyc(x)$ is an abelian tidy group. Suppose $Cyc(x) = X_{p_1} \times X_{p_2} \times \cdots \times X_{p_n}$, where n is a positive integer, p_i's are distinct primes and X_{p_i}, $1 \leq i \leq n$ is a Sylow p_i-subgroup of $Cyc(x)$. By Lemma 2.12 of [3], we have $K(Cyc(x)) = K(X_{p_1}) \times K(X_{p_2}) \times \cdots \times K(X_{p_n})$. Now, suppose $Cyc(x)$ is not cyclic. Then X_{p_i} is not cyclic for some i. By Lemma 10 of [5], we have X_{p_i} is a tidy group. Again, by Theorem 11 of [5], X_{p_i} is elementary abelian and hence $K(X_{p_i}) = \{1\}$. Therefore $|\frac{Cyc(x)}{K(Cyc(x))}| = p_i^{e_i} m$, where $|X_{p_i}| = p_i^{e_i} > p_i$ and e_i, m are positive integers. Again, since $K(G) \subsetneq K(Cyc(x))$, therefore $|\frac{Cyc(x)}{K(G)}| = p_i^{e_i} k$, where k is a positive integer. Now, since $\frac{Cyc(x)}{K(G)}$ is cyclic, therefore $\frac{Cyc(x)}{K(G)}$ has an element of order $p_i^{e_i}$, say $yK(G)$. But $o(yK(G)) \mid o(y)$, which is a contradiction since $y \in Cyc(x)$ and $Cyc(x)$ has no element of order $p_i^{e_i}$. Therefore $Cyc(x)$ is cyclic. □

Now, we state the structure theorem for C-tidy groups.

Theorem 2.2. *Let G be a finite group. Then G is C-tidy if and only if $\frac{G}{K(G)}$ satisfies one of the following conditions:*

(1) *$\frac{G}{K(G)}$ is a non-cyclic p-group with a cyclic normal subgroup $\frac{H}{K(G)}$ such that $o(xK(G)) = p$ for all $xK(G) \in \frac{G}{K(G)} \setminus \frac{H}{K(G)}$, p being a prime,*

(2) $\frac{G}{K(G)}$ *is a Frobenius group in which the Frobenius kernel* $\frac{K}{K(G)}$ *is cyclic or* $\frac{K}{K(G)}$ *is a* p*-group with a cyclic normal subgroup* $\frac{N}{K(G)}$ *such that* $o(xK(G)) = p$ *for all* $xK(G) \in \frac{K}{K(G)} \setminus \frac{N}{K(G)}$ *and* $C(yK(G))$ *is cyclic for all* $yK(G) \in \frac{G}{K(G)} \setminus \frac{K}{K(G)}$*,* p *being a prime,*

(3) $\frac{G}{K(G)}$ *is of Hughes-Thompson type in which* $H_p(\frac{G}{K(G)})$ *is cyclic normal or* $H_p(\frac{G}{K(G)})$ *is a normal* q*-group with a cyclic normal subgroup* $\frac{N}{K(G)}$ *such that* $o(hK(G)) = q$ *for all* $hK(G) \in H_p(\frac{G}{K(G)}) \setminus \frac{N}{K(G)}$ *and* $|Cyc(xK(G))| = p$ *for all* $xK(G) \in \frac{G}{K(G)} \setminus H_p(\frac{G}{K(G)})$*,* p, q *being distinct primes,*

(4) $\frac{G}{K(G)} \cong PGL(2, p^h)$*,* p *being an odd prime,* $h \geq 1$*,*

(5) $\frac{G}{K(G)} \cong PSL(2, p^h)$*,* p *being a prime,* $h \geq 1$*.*

Proof. Let G be a finite C-tidy group. By Theorem 4 of [5], we have $K(\frac{G}{K(G)}) = \{1\}$. Therefore by Proposition 2.1, $\frac{G}{K(G)}$ is a C-tidy group with $K(\frac{G}{K(G)}) = \{1\}$. Now, using Theorem 3.13 of [2], we get the result.

Conversely, if $\frac{G}{K(G)}$ satisfies one of the above conditions, then by Theorem 3.13 of [2], $\frac{G}{K(G)}$ is C-tidy. Therefore by Proposition 2.1, G is C-tidy. □

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