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ON A QUESTION OF JAIKIN-ZAPIRAIN ABOUT THE AVERAGE ORDER ELEMENTS OF FINITE GROUPS

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Abstract. For a finite group G, the average order $o(G)$ is defined to be the average of all order elements in G, that is $o(G) = \frac{1}{|G|}\sum_{x\in G} o(x)$, where $o(x)$ is the order of element x in G. Jaikin-Zapirain in [On the number of conjugacy classes of finite nilpotent groups, Advances in Mathematics, **227** (2011) 1129-1143] asked the following question: if G is a finite (p-) group and N is a normal (abelian) subgroup of G, is it true that $o(N)^{\frac{1}{2}} \leq o(G)$? We say that G satisfies the average condition if $o(H) \leq o(G)$, for all subgroups H of G. In this paer we show that every finite abelian group satisfies the average condition. This result confirms and improves the question of Jaikin-Zapirain for finite abelian groups.

1. Introduction

Let G be a finite group. For a non-empty subset S of G, let $\psi(S)$ be the sum of the orders of all elements of S, i.e.,

$$\psi(S) = \sum_{x\in S} o(x),$$

where $o(x)$ denotes the order of the element x of G. Amiri, Jafarian Amiri and Isaacs [1] defined the function ψ and proved that the maximum value of ψ on the set of groups of order n occurs at the cyclic group C_n of order n.

Theorem 1.1. [1] If G is a non-cyclic finite group of order n, then $\psi(G) < \psi(C_n)$.

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Jaikin-Zapirain [6] defined the average of the order $o(G) = \frac{\psi(G)}{|G|}$, of the elements of G, and used it to determine a lower bound for the number of conjugacy classes of a finite nilpotent group. He also put forward the following question:

Question 1. Let G be a finite (p-) group and N a normal (abelian) subgroup of G. Is it true that $o(N)^{\frac{1}{2}} \leq o(G)$?

Ten years later, Khokhro, Moreto and Zarrin [7] gave a strong negative answer to this question. They proved that "If c is a real number and $p \geq \frac{3}{c}$ is a prime, then there exists a finite p-group with a normal abelian subgroup N such that $o(G) < o(N)^c$".

Now, given the incorrectness of Question 1 in general, one can ask which groups satisfy Jaikin-Zapirain's question. In this paper we consider a stronger condition and ask the following question:

Question 2. For which finite group G, $o(H) \leq o(G)$, for all (normal) subgroups H of G.

We say that G satisfies the average condition if it satisfies the Question 2. It is obvious that the cyclic group C_p of order p, where p is a prime, satisfies the average condition. Note that $o(S_3) = \frac{13}{6}$ and $o(A_3) = \frac{14}{6}$ and so S_3 does not satisfy the average condition. In this paper we show that every finite abelian group satisfies the average condition, which confirms and improves the question of Jaikin-Zapirain for abelian groups.

Theorem A. Let G be a finite abelian group. Then G satisfies the average condition.

All groups in this paper are assumed to be finite. Our notation and terminology are standard and taken mainly from [4]. In particular, the size of a finite group G is shown by $|G|$. The cyclic group of order n, the quaternion group of order 8, and the elementary abelian p-group of order p^m, are denoted by C_n, Q_8, and C_p^m, respectively.

Note that by Theorem 1.1, C_n has the maximum value of o in the class of groups of order n. Hence if G is a finite group of order n, then $o(G) \leq o(C_n)$. Also the function o is multiplicative. This is a direct consequence of the following lemma.

Lemma 1.2. [2, Lemma 2.1] If G and H are finite groups, then $\psi(G \times H) \leq \psi(G)\psi(H)$. Also $\psi(G \times H) = \psi(G)\psi(H)$ if and only if $\gcd(|G|, |H|) = 1$.

2. Proof of Theorem A

Let G be a finite abelian group. Tărnăuceanu and Fodor [5] obtained an explicit formula for computing $\psi(G)$. Chew, Chin, and Lim [3] obtained another formula for computing $\psi(G)$. This formula is recursive and has been built upon the sum of element orders of finite abelian p-groups of lower rank. For positive integers p, i and n we define

$$f(p,i,n) = p^i - 1 + \sum_{j=1}^{i-1} (p^i - p^j)\big(p^{j(n-1)} - p^{(j-1)(n-1)}\big).$$

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Theorem 2.1. [3, Corollary 2.9] Let $G = C_{p^{r_1}} \times C_{p^{r_2}} \times \cdots \times C_{p^{r_n}}$ be a finite abelian p-group, where p is a prime, $1 \leq r_1 \leq r_2 \leq \cdots \leq r_n$ and r_i is a positive integer. Then

(a) If $r_1 = 1$, then $\psi(G) = p\psi(C_{p^{r_2}} \times \cdots \times C_{p^{r_n}}) + (p-1)^2$.

(b) If $r_1 > 1$, then $\psi(G) = p^{r_1}\psi(C_{p^{r_2}} \times \cdots \times C_{p^{r_n}}) + (p-1)^2 + \sum_{i=2}^{r_1}(p^i - p^{i-1})f(p,i,n)$.

We may restrict the study of the average condition of a group to its maximal subgroups by the following easy result.

Proposition 2.2. Let G be a finite group. If every maximal subgroup M of G satisfies the average condition and $o(M) \leq o(G)$, then G satisfies the average condition.

Proof. Suppose that H is a proper subgroup of G. Then there exists a maximal subgroup M of G such that $H \leq M$. By assumption $o(M) \leq o(G)$. Also since M satisfies the average condition, $o(H) \leq o(M)$. Therefore, $o(H) \leq o(G)$ and the result follows. □

In the sequel, we use Theorem 2.1 and find an explicit formula for $\psi(G)$, where G is a finite abelian group. First we need to prove some computational and easy results.

Lemma 2.3. Let p be a prime and $i, n \geq 2$ be positive integers. Then $f(p,i,n) = \frac{(p-1)(p^{ni}-1)}{p^n-1}$.

Proof. Since

$$\sum_{j=1}^{i-1}\left(x^j - x^{j-1}\right) = \sum_{j=1}^{i-1} x^{j-1}(x-1) = (x-1)\frac{x^{i-1}-1}{x-1} = x^{i-1}-1$$

we have

$$p^i - 1 + \sum_{j=1}^{i-1} p^i\left(x^j - x^{j-1}\right) = p^i - 1 + p^i(x^{i-1}-1) = p^i x^{i-1} - 1.$$

Also, we have

$$\sum_{j=1}^{i-1} p^j\left(x^j - x^{j-1}\right) = \sum_{j=1}^{i-1} p^j x^{j-1}(x-1) = p(x-1)\frac{(px)^{i-1}-1}{px-1},$$

and therefore

$$\begin{aligned} p^i - 1 + \sum_{j=1}^{i-1}(p^i - p^j)(x^j - x^{j-1}) &= p^i x^{i-1} - 1 - p(x-1)\frac{(px)^{i-1}-1}{px-1} \\ &= \frac{(px-1)(p^i x^{i-1}-1) - (px-p)((px)^{i-1}-1)}{px-1}. \end{aligned}$$

Thus putting $x := p^{n-1}$ we obtain that

$$\begin{aligned} f(p,i,n) &= \frac{(p^n-1)(p^{ni-n+1}-1) - (p^n-p)(p^{ni-n}-1)}{p^n-1} \\ &= \frac{p^{ni+1} - p^n - p^{ni-n+1} + 1 - p^{ni} + p^n + p^{ni-n+1} - p}{p^n-1} \\ &= \frac{(p^{ni}-1)(p-1)}{p^n-1} \end{aligned}$$

and the result follows. □

Lemma 2.4. If $n \geq 2$, $r \geq 2$ are positive integers, then

$$\sum_{i=2}^{r}(p^i - p^{i-1})(p^{ni}-1) = \frac{p^{2n+1}(p-1)\big(p^{(n+1)(r-1)}-1\big) - \big(p^{n+1}-1\big)(p^r-p)}{p^{n+1}-1}.$$

Proof. Since

$$\sum_{i=2}^{r} x^i = \frac{x^{r+1}-x^2}{x-1} = x^2\,\frac{x^{r-1}-1}{x-1}$$

we have

$$\sum_{i=2}^{r}\left(p^{(n+1)i} - p^{(n+1)i-1}\right) = \frac{p-1}{p}\sum_{i=2}^{r} p^{(n+1)i} = \frac{p-1}{p}p^{2(n+1)}\frac{p^{(n+1)(r-1)}-1}{p^{n+1}-1}.$$

Now since

$$\sum_{i=2}^{r}(p^i - p^{i-1}) = p^r - p$$

we have

$$\begin{aligned}\sum_{i=2}^{r}\left(p^i - p^{i-1}\right)\left(p^{ni}-1\right) &= \sum_{i=2}^{r}\left(p^{(n+1)i} - p^{(n+1)i-1}\right) - \sum_{i=2}^{r}\left(p^i - p^{i-1}\right)\\ &= \frac{p-1}{p}p^{2(n+1)}\frac{p^{(n+1)(r-1)}-1}{p^{n+1}-1} - (p^r-p)\\ &= \frac{p^{2n+1}(p-1)\big(p^{(n+1)(r-1)}-1\big) - \big(p^{n+1}-1\big)(p^r-p)}{p^{n+1}-1}\end{aligned}$$

and the result follows. □

Let $n, r \geq 2$ be positive integers. We put $A_{1,n} = (p-1)^2$ and

$$A_{r,n} = (p-1)^2 + \sum_{i=2}^{r}\left(p^i - p^{i-1}\right)f(p,i,n).$$

As a consequence of Lemmas 2.3 and 2.4 we have the following corollary:

Corollary 2.5. For every positive integers $r \geq 1$, $n \geq 2$, we have

$$A_{r,n} = (p-1)\frac{p^{nr+n+r+1} - p^{nr+n+r} - p^{n+r+1} + p^n + p^r - 1}{(p^{n+1}-1)(p^n-1)}.$$

Proof. If $r = 1$, then by simple computation the result follows. Now let $r \geq 2$, from Lemma 2.3, we have

$$\begin{aligned}A_{r,n} &= (p-1)^2 + \sum_{i=2}^{r}(p^i - p^{i-1})f(p,i,n)\\ &= (p-1)^2 + \sum_{i=2}^{r}(p^i - p^{i-1})\ \frac{(p-1)(p^{ni}-1)}{p^n-1}\\ &= (p-1)^2 + \frac{p-1}{p^n-1}\sum_{i=2}^{r}(p^i - p^{i-1})(p^{ni}-1).\end{aligned}$$

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Now using Lemma 2.4, we obtain that

$$\begin{aligned} A_{r,n} &= (p-1)^2 + \frac{p-1}{p^n-1}\,\frac{p^{2n+1}(p-1)\big(p^{(n+1)(r-1)}-1\big)-\big(p^{n+1}-1\big)(p^r-p)}{p^{n+1}-1} \\ &= (p-1)\frac{(p^{n+1}-1)(p-1)(p^n-1)+p^{2n+1}(p-1)\big(p^{(n+1)(r-1)}-1\big)-\big(p^{n+1}-1\big)(p^r-p)}{(p^n-1)(p^{n+1}-1)} \\ &= (p-1)\,\frac{p^{(n+1)(r+1)}-p^{(n+1)(r+1)-1}-p^{n+r+1}+p^n+p^r-1}{(p^n-1)(p^{n+1}-1)} \\ &= (p-1)\frac{p^{nr+n+r+1}-p^{nr+n+r}-p^{n+r+1}+p^n+p^r-1}{(p^{n+1}-1)(p^n-1)} \end{aligned}$$

and the result follows. □

Let $G_n = C_{p^{r_1}} \times C_{p^{r_2}} \times \cdots \times C_{p^{r_n}}$ be a finite abelian p-group of order p^m, where p is a prime, $1 \le r_1 \le r_2 \le \cdots \le r_n$ and r_i is a positive integer. Every maximal subgroup of G_n is of the form $H_i = C_{p^{r_1}} \times \cdots \times C_{p^{r_{i-1}}} \times C_{p^{r_i-1}} \times C_{p^{r_{i+1}}} \times \cdots \times C_{p^{r_n}}$, $1 \le i \le n$. We put $G_{n-i} = C_{p^{r_{i+1}}} \times C_{p^{r_{i+2}}} \times \cdots \times C_{p^{r_n}}$, which obtained from G_n, by omitting the first i direct factors. In Theorems 2.6, 2.10 and Lemmas 2.7, 2.9 below, we will use the above notation. We can restate Theorem 2.1 as follows:

Theorem 2.6. $\psi(G_n) = A_{r_1,n} + p^{r_1}\psi(G_{n-1})$.

Now, we prove the following basic lemma:

Lemma 2.7. For all $i = 1, 2, \ldots, n-1$, we have $\psi((H_i)_{n-(i-1)}) \ge \psi((H_{i+1})_{n-(i-1)})$.

Proof. If $i = 1$, then by using the recursive formula in the Theorem 2.6, we have

$$\begin{aligned} \psi(H_1) = \psi((H_1)_n) &= A_{r_1-1,n} + p^{r_1-1}\psi(C_{p^{r_2}} \times C_{p^{r_3}} \times \cdots \times C_{p^{r_n}}) \\ &= A_{r_1-1,n} + p^{r_1-1}\Big(A_{r_2,n-1} + p^{r_2}\psi(C_{p^{r_3}} \times \cdots \times C_{p^{r_n}})\Big) \\ &= A_{r_1-1,n} + p^{r_1-1}A_{r_2,n-1} + p^{r_1+r_2-1}\psi(G_{n-2}). \end{aligned}$$

Similarly, for H_2 we have

$$\psi(H_2) = \psi((H_2)_n) = A_{r_1,n} + p^{r_1}A_{r_2-1,n-1} + p^{r_1+r_2-1}\psi(G_{n-2}).$$

Since

$$A_{r_1,n} - A_{r_1-1,n} = \frac{(p-1)^2 p^{r_1-1}(p^{nr_1}-1)}{p^n-1}$$

and

$$p^{r_1}A_{r_2-1,n-1} - p^{r_1-1}A_{r_2,n-1} = -\frac{(p-1)^2 p^{r_1-1}(p^{nr_2}-1)}{p^n-1}$$

we obtain that

$$\psi(H_2) - \psi(H_1) = \frac{(p-1)^2 p^{r_1-1}(p^{nr_1}-p^{nr_2})}{p^n-1}.$$

Now, since $1 \le r_1 \le r_2$, it follows that $\psi(H_2) - \psi(H_1) \le 0$. So the result is true for $i = 1$. Note that we can use Mathematica software [8] to avoid some tedious computations, see the Remark 2.8 below.

Now let $i = 2$. we must show that $\psi((H_2)_{n-1}) \geq \psi((H_3)_{n-1})$. By using the recursive formula in Theorem 2.6, we have

$$\begin{aligned}\psi((H_2)_{n-1}) &= A_{r_2-1,n-1} + p^{r_2-1}\psi(C_{p^{r_3}} \times C_{p^{r_4}} \times \cdots \times C_{p^{r_n}}) \\ &= A_{r_2-1,n-1} + p^{r_2-1}\Big(A_{r_3,n-2} + p^{r_3}\psi(C_{p^{r_4}} \times \cdots \times C_{p^{r_n}})\Big) \\ &= A_{r_2-1,n-1} + p^{r_2-1}A_{r_3,n-2} + p^{r_2+r_3-1}\psi(G_{n-3}).\end{aligned}$$

Similarly, for $(H_3)_{n-1}$ we have

$$\psi((H_3)_{n-1}) = A_{r_2,n-1} + p^{r_2}A_{r_3-1,n-2} + p^{r_2+r_3-1}\psi(G_{n-3}).$$

Since

$$A_{r_2,n-1} - A_{r_2-1,n-1} = \frac{(p-1)^2 p^{r_2-1}(p^{(n-1)r_2} - 1)}{p^{n-1} - 1}$$

and

$$p^{r_2}A_{r_3-1,n-2} - p^{r_2-1}A_{r_3,n-2} = -\frac{(p-1)^2 p^{r_2-1}(p^{(n-1)r_3} - 1)}{p^{n-1} - 1}$$

we obtain that

$$\begin{aligned}\psi((H_3)_{n-1}) - \psi((H_2)_{n-1}) &= A_{r_2,n-1} + p^{r_2}A_{r_3-1,n-2} - A_{r_2-1,n-1} - p^{r_2-1}A_{r_3,n-2} \\ &= \frac{(p-1)^2 p^{r_2-1}(p^{(n-1)r_2} - p^{(n-1)r_3})}{p^{n-1} - 1}.\end{aligned}$$

Since $r_2 \leq r_3$ and $n-1 \geq 2$, it follows that $\psi((H_3)_{n-1}) - \psi((H_2)_{n-1}) \leq 0$. Therefore the result is true for $i = 2$. We continue in this fashion, using Mathematica [8] or tedious computations, and obtain that

$$\begin{aligned}\psi((H_{i+1})_{n-(i-1)}) - \psi((H_i)_{n-(i-1)}) &= A_{r_i,n-(i-1)} - A_{r_i-1,n-(i-1)} + p^{r_i}A_{r_{i+1}-1,n-i} - p^{r_i-1}A_{r_{i+1},n-i} \\ &= \frac{(p-1)^2 p^{r_i-1}(p^{(n-(i-1))r_i} - p^{(n-(i-1))r_{i+1}})}{p^{n-(i-1)} - 1}.\end{aligned}$$

Since $r_i \leq r_{i+1}$ and $n-i \geq 1$, it follows that $\psi((H_{i+1})_{n-(i-1)}) - \psi((H_i)_{n-(i-1)}) \leq 0$ for all i, $1 \leq i \leq n-1$ and the proof is complete. □

Remark 2.8. We can use the Mathematica software [8] to avoid some tedious computations. For example in the proof of the case $i = 1$ in the above Lemma, if we use the following commands

```
f[x_,i_,n_]:=x^i-1+Sum[(x^i-x^j)*(x^(j(n-1))-x^((j-1)(n-1))),{j,1,i-1}]//Simplify
A[p_,r_,n_]:=(p-1)^2+Sum[(p^i-p^(i-1))*f[p,i,n],{i,2,r}]//Simplify
f[p,i,n]
Simplify[A[p,r,n]-A[p,r-1,n],Assumptions->Element[{r1,n},Integers]]
Simplify[p^(r1)*A[p,r2-1,n-1]-p^(r1-1)*A[p,r2,n-1],
                              Assumptions->Element[{r1,r2,n},Integers]]
Simplify[A[p,r1,n]+p^(r1)*A[p,r2-1,n-1]-A[p,r1-1,n]-p^(r1-1)*A[p,r2,n-1],
                             Assumptions->Element[{r1,r2,n},Integers]]
```

we obtain that

$$\frac{(p-1)\left(p^{in}-1\right)}{p^n-1}$$

$$\frac{(p-1)^2p^{r-1}\left(p^{n\,r}-1\right)}{p^n-1}$$

$$-\frac{(p-1)^2p^{r_1-1}\left(p^{n\,r_2}-1\right)}{p^n-1}$$

$$\frac{(p-1)^2p^{r_1-1}\left(p^{n\,r_1}-p^{n\,r_2}\right)}{p^n-1}$$

Lemma 2.9. $o(H_1) \leq o(G_n)$.

Proof. This inequality is equivalent to $\psi(H_1)|G_n : H_1| \leq \psi(G_n)$. Since $|G_n : H_1| = p$, it is enough to show $p\psi(H_1) \leq \psi(G_n)$. By Theorem 2.6, we have $\psi(G_n) = p^{r_1}\psi(G_{n-1}) + A_{r_1,n}$ and $\psi((H_1)_n) = p^{r_1-1}\psi(G_{n-1}) + A_{r_1-1,n}$. It follows that,

$$\begin{aligned} o(H_1) < o(G_n) &\Longleftrightarrow p^{r_1}\psi(G_{n-1}) + pA_{r_1-1,n} < p^{r_1}\psi(G_{n-1}) + A_{r_1,n} \\ &\Longleftrightarrow A_{r_1,n} - pA_{r_1-1,n} > 0. \end{aligned}$$

Now, by Corollary 2.5 and Mathematica [8], the denominator of $A_{r,n} - pA_{r-1,n}$ is equal to $(p^n - 1)(p^{1+n} - 1)$, which is positive, and its numerator is

$$1 - 2p + p^2 - p^n + 2p^{1+n} - p^{2+n} - p^{r+nr} + 2p^{1+r+nr} - p^{2+r+nr} + p^{n+r+nr} - 2p^{(1+n)(1+r)} + p^{2+n+r+nr},$$

which is also positive, as it is equal to

$$\begin{aligned} (p-1)^2 - p^n(1-2p+p^2) - p^{r+nr}(1-2p+p^2) + p^{n+r+nr}(1-2p+p^2) \\ = (p-1)^2(1 - p^n - p^{r+nr} + p^{n+r+nr}) \\ = (p-1)^2(p^n-1)(p^{r+nr}-1) \end{aligned}$$

which is positive. Hence $A_{r_1,n} - pA_{r_1-1,n} > 0$ and the result follows. □

To prove the Theorem A, we first consider abelian p-groups.

Theorem 2.10. G_n satisfies the average condition.

Proof. Our proof is by induction on m. When $m = 1$, $G_n = C_p$ has no proper subgroup, and so G_n satisfies the average condition. Assume that the result is true for all abelian p-groups of order p^{m-1}. We show that the result holds for G_n. Since every maximal subgroup H_i of G_n is of order p^{m-1}, by induction hypothesis H_i satisfies the average condition for all $i = 1, 2, \ldots, n$. Hence by Proposition

2.2, it is enough to show that $o(H_i) \leq o(G)$, $1 \leq i \leq n$.
We claim that $\psi(H_{i+1}) \leq \psi(H_i)$ for all i, $1 \leq i \leq n-1$. By recursive formula of Theorem 2.6, we have

$$\begin{aligned}
\psi(H_i) = \psi((H_i)_n) &= A_{r_1,n} + p^{r_1}\psi((H_i)_{n-1}) \\
&= A_{r_1,n} + p^{r_1}\big(A_{r_2,n-1} + p^{r_2}\psi((H_i)_{n-2})\big) \\
&= A_{r_1,n} + p^{r_1}A_{r_2,n-1} + p^{r_1+r_2}\big(A_{r_3,n-2} + p^{r_3}\psi((H_i)_{n-3})\big) \\
&\ \ \vdots \\
&= A_{r_1,n} + p^{r_1}A_{r_2,n-1} + p^{r_1+r_2}A_{r_3,n-2} + \cdots + p^{r_1+r_2+\cdots+r_{i-2}}A_{r_{i-1},n-(i-2)} \\
&\qquad\qquad + p^{r_1+r_2+\cdots+r_{i-1}}\psi\big((H_i)_{n-(i-1)}\big).
\end{aligned}$$

Similarly, for H_{i+1} we have

$$\begin{aligned}
\psi(H_{i+1}) = &A_{r_1,n} + p^{r_1}A_{r_2,n-1} + p^{r_1+r_2}A_{r_3,n-2} + \cdots + p^{r_1+r_2+\cdots+r_{i-2}}A_{r_{i-1},n-(i-2)} \\
&+ p^{r_1+r_2+\cdots+r_{i-1}}\psi\big((H_{i+1})_{n-(i-1)}\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\psi(H_{i+1}) - \psi(H_i) &= p^{r_1+r_2+\cdots+r_{i-1}}\psi\big((H_{i+1})_{n-(i-1)}\big) - p^{r_1+r_2+\cdots+r_{i-1}}\psi\big((H_i)_{n-(i-1)}\big) \\
&= p^{r_1+r_2+\cdots+r_{i-1}}\Big(\psi\big((H_{i+1})_{n-(i-1)}\big) - \psi\big((H_i)_{n-(i-1)}\big)\Big).
\end{aligned}$$

Since $p^{r_1+r_2+\cdots+r_{i-1}} > 1$ and, by Lemma 2.7, $\psi\big((H_{i+1})_{n-(i-1)}\big) - \psi\big((H_i)_{n-(i-1)}\big) \leq 0$, we conclude that $\psi(H_{i+1}) - \psi(H_i) \leq 0$. So our claim is true and we can infer that

$$\psi(H_1) \geq \psi(H_2) \geq \cdots \geq \psi(H_n)$$

and so

$$p\psi(H_1) \geq p\psi(H_2) \geq \cdots \geq p\psi(H_n). \tag{2.1}$$

Since $|G_n : H_i| = p$ for all i, $1 \leq i \leq n$, it follows that $o(H_i) \leq o(G_n)$ if and only if $p\psi(H_i) \leq \psi(G_n)$. By Lemma 2.9, $p\psi(H_1) < \psi(G_n)$. So by (2.1), $p\psi(H_i) < \psi(G_n)$ for all i, $1 \leq i \leq n$. Hence G_n satisfies the average condition and the proof is complete. □

Now we are in the position to prove Theorem A.

Proof of theorem A: . Let G be an abelian group of order $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, p_i's are distinct primes. Then $G = G_{p_1} \times G_{p_2} \times \cdots \times G_{p_k}$ such that G_{p_i}, $1 \leq i \leq k$, is a Sylow p_i-subgroup of G. Suppose that H is a subgroup of G. Then $H = H_1 \times H_2 \times \cdots \times H_k$, where $H_i \leq G_{p_i}$ for all i, $1 \leq i \leq k$. By Theorem 2.10, G_{p_i}, $1 \leq i \leq k$, satisfies the average condition, and hence $o(H_i) \leq o(G_{p_i})$, for all

$1 \leq i \leq k$. Since o is multiplicative, we infer that

$$\begin{aligned} o(H) &= o(H_1 \times H_2 \times \cdots \times H_k) \\ &= o(H_1)o(H_2)\cdots o(H_k) \\ &\leq o(G_{p_1})o(G_{p_2})\cdots o(G_{p_k}) \\ &= o(G_{p_1} \times G_{p_2} \times \cdots \times G_{p_k}) = o(G). \end{aligned}$$

Therefore $o(H) \leq o(G)$, for all subgroup H of G. This completes the proof. □

Note that, in the proof of Theorem A, if H is a maximal subgroup of G, then H_i is a maximal subgroup of G_{p_i}, for some i. So by the proof of Theorem 2.10, $o(H_i) < o(G_{p_i})$ and hence $o(H) < o(G)$. Thus we have the following corollary

Corollary 2.11. *Let G be a finite abelian group and H be a maximal subgroup of G. Then $o(H) < o(G)$.*

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