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QUANTITATIVE CHARACTERIZATION OF FINITE SIMPLE GROUPS: A COMPLEMENT

WUJIE SHI^{OR}

ABSTRACT. In this paper, we summarize the research on the characterization of finite simple groups and the study of finite groups based on their “set of element orders” and “two orders” (the order of the group and the set of element orders). We also discuss some related topics, their applications, and generalizations. The original version of this paper was published in Chinese in [Scientia Sinica Mathematica, **53** (2023), no. 7, 931–952]. This updated version corrects several errors and includes additional content. It particularly emphasizes the applications of this work in mathematics and computational complexity theory.

1. Introduction

Group theory is a significant branch of mathematics with important applications in various fields, including mathematics, physics, and chemistry. One of the most remarkable achievements in mathematics during the 20th century is the classification theorem for finite simple groups, which was completed in 2004. This accomplishment is unprecedented in the history of mathematics in terms of the number of researchers involved, the quantity of published papers, and the total length of the proof, which exceeds 15,000 pages. The first paper to classify an infinite family of finite simple groups was published by W. Burnside in 1899, building on a hypothesis regarding the structure of certain proper subgroups.

In the history of group theory, Burnside conjectured in 1911 that every nonabelian simple group has an even order. In 1954, R. Brauer suggested classifying simple groups by examining the centralizers of

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involutions within them. Brauer, along with K. A. Fowler, demonstrated that if G is a simple group of even order and τ is an involution in G , then the order of G satisfies $|G| \leq (|C_G(\tau)|^2)!$. This means that for a given centralizer $C_G(\tau)$, there are only finitely many simple groups with centralizers of involutions isomorphic to $C_G(\tau)$. This approach is known as the Brauer program. The Brauer-Fowler theorem further established that if the centralizer of an involution in a simple group is known, the simple group can be uniquely determined (see [26]). It is also well-known that a group of odd order cannot contain an element of order 2. Thus, to utilize the Brauer program effectively, we must first demonstrate that the orders of nonabelian simple groups are even. This is equivalent to proving that groups with odd orders are solvable, a result later proven by W. Feit and J. G. Thompson.

In 1957, M. Suzuki began to address Burnside's conjecture. He studied CA groups, which are defined as groups where the centralizer of every nontrivial element is abelian. In his groundbreaking paper, he demonstrated that all CA groups of odd order are solvable. He then went on to classify all simple CA groups and, more broadly, to classify simple groups where the centralizers of involutions have a normal Sylow 2-subgroup. During this work, he discovered a family of simple groups of Lie type known as the Suzuki simple groups. In 1960, Feit, Thompson, and M. Hall extended Suzuki's research to study CN groups, which are defined as groups where the centralizer of every nontrivial element is nilpotent. They proved that every CN group of odd order is also solvable.

To prove the theorem that odd-order groups are solvable, Feit and Thompson first established that no nonabelian simple group of odd order exists in which every subgroup is solvable. This result indicates that the minimal counterexample to the solvability of odd-order groups would be a simple group where every subgroup is solvable. Although the overall approach of their proof is similar to the methods used in the CA and CN theorems, the details are significantly more complex, resulting in a final paper that spans 254 pages.

Theorem 1.1 (Feit-Thompson Theorem (1963)). *Let G be a finite group. If the order of G is odd, then G is a solvable group.*

The odd-order theorem states that numerical quantities associated with a group can determine its properties.

Recently, many significant topics in group representation theory have been successfully reduced to cases involving simple groups and quasi-simple groups. Notable examples include the McKay Conjecture and the Alperin Conjecture (see [164]). Therefore, there is an urgent need to gain a deeper understanding of the properties of simple groups. The renowned researcher R. Solomon, a prominent figure in group theory, noted that “the ongoing project to publish a series of more than 12 volumes presenting a complete proof of this theorem is expected to be completed by 2025” (see the email Solomon sent to the author on November 2, 2020, or [205]).

The odd-order theorem offers valuable insights into the relationship between numerical quantities and the structure of a group. In the context of finite groups, both the “order of the group” and the

“orders of the elements” are fundamental quantities. The author first characterized the alternating group A_5 : a finite group G is isomorphic to A_5 if and only if the set of element orders in G , denoted by $\pi_e(G)$, is equal to $\{1, 2, 3, 5\}$. To generalize this result, it is necessary to include the condition related to the “order of the group.” In 1987, after characterizing all sporadic simple groups based on the “order of the group” and the “set of element orders,” the author proposed the following conjecture:

Conjecture 1.2. [184] *Let G be a finite group and S a finite simple group. Then $G \cong S$ if and only if the following two conditions are satisfied:*

- (1) $\pi_e(G) = \pi_e(S)$;
- (2) $|G| = |S|$.

In other words, every finite simple group can be uniquely characterized by its order and the orders of its elements, this can be referred to as the “two orders”.

In 1987, the author presented the conjecture above to Thompson, who responded with encouragement and full affirmation. In his reply, Thompson expressed, “Good luck with your conjecture about simple groups. I hope you continue to work on it”, as well as, “I like your arguments”, and “This would certainly be a nice theorem”. Additionally, in two letters dated 1987 and 1988, Thompson introduced a problem and conjecture (see [112, Problem 12.37-12.39]).

Definition 1.3. *Let G be a finite group and let d be a positive integer. We define the set $G(d) = \{x \in G \mid x^d = 1\}$. We say that two groups G_1 and G_2 are of the same order type if and only if $|G_1(d)| = |G_2(d)|$ for every positive integer d .*

Thompson Problem (1987)¹ Let G_1 and G_2 be groups of the same order type. If G_1 is solvable, is G_2 also necessarily solvable?

In Thompson’s letter, he pointed out that: “The problem initially arose in the study of algebraic number fields and is of considerable interest.”

Let G be a finite group, and define $N(G) = \{n \in \mathbb{Z}^+ \mid G \text{ has conjugacy class } C \text{ with } |C| = n\}$. This set represents all lengths of the conjugacy classes of G .

Thompson Conjecture (1988)² Let G and M be two finite groups. If $N(G) = N(M)$, M is a non-abelian simple group, and the center of G is trivial, then G and M are isomorphic.

This paper presents an overview of the proof for Conjecture 1.2, known as the Thompson Conjecture. It discusses the current status of solutions for the Thompson Problem and examines related issues

¹On April 22, 1987, J. G. Thompson sent a private letter to the author regarding this problem. It is also mentioned in the book by E. I. Khukhro and V. D. Mazurov, titled “Unsolved Problems in Group Theory”, which can be found in The Kourovka Notebook, No. 20, 2022, Problem 12.37, on page 58.

²A private letter from J. G. Thompson, dated January 4, 1988, posed a conjecture to the author. This conjecture is also mentioned in E. I. Khukhro and V. D. Mazurov’s book, “Unsolved Problems in Group Theory”, found in The Kourovka Notebook, No. 20, 2022, Problem 12.38, page 59.

concerning quantities and groups. Additionally, it explores their applications and generalizations, and includes several unsolved problems.

Conjecture 1.2 originated from the author's master's thesis, which focused on finite groups characterized by elements of prime order, excluding the identity element (referred to as element prime order groups, or EPO-groups) (see [203]). Additionally, the thesis explored finite groups composed of elements with prime power orders (known as element prime power order groups, or EPPO-groups) (see [204]). Furthermore, the following theorem was established (see [182]):

Theorem 1.4. *Let G be a finite group. Then $G \cong A_5$ if and only if $\pi_e(G) = \{1, 2, 3, 5\}$.*

After learning about this result, Xuefu Duan wrote in a letter to Zhongmu Chen: "Very interesting! I will also think about it when I have time". The encouragement from both Xuefu Duan and Zhongmu Chen motivated the author to further characterize simple groups using the "set of element orders".

However, when working with simple groups, such as A_6 , which could not be characterized only by the "set of element orders", it became natural to consider incorporating the "order of the group". Consequently, the conjecture 1.2 was proposed.

Paper [203] was initially published in Chinese. Five years later, a similar article [48] was released, but it contained some errors in its main theorem. In response to these errors, the author and co-authors published paper [45] as a corrigendum to address the issues in paper [48]. Paper [204], which was also published in Chinese, has recently gained attention, leading the authors to translate it into English and submit it to arXiv. Furthermore, paper [168] generalized the findings of paper [203] to explore finite groups where all elements, except those in a normal subgroup, have prime order. The recently published paper [119] serves as a generalization of paper [204].

2. The characterization of finite simple groups by spectrum

Before characterizing the alternating group A_5 using the set of element orders, the author explored the number $|\pi(G)|$ of prime factors of $|G|$ and imposed certain restrictive conditions on the "set of element orders" to characterize the simple group $\text{PSL}_2(7)$ (see [179]) and some other simple groups (see [178, 180, 181, 183]). Furthermore, it is not difficult to deduce that the previously mentioned constraints arise from the "set of element orders". For example, in [179], the author proved the following result:

Theorem 2.1. *Let G be a finite group that satisfies the following conditions:*

- (1) *The order of G contains at least three different prime factors, meaning that $|\pi(G)| \geq 3$.*
- (2) *the order of every nontrivial element in G is either a power of 2 or a prime different from 5.*

Then G is isomorphic to $\text{PSL}_2(7)$.

It is clear that conditions (1) and (2) hold for a group G with spectrum $\pi_e(G) = \{1, 2, 3, 4, 7\}$, which matches $\pi_e(\text{PSL}_2(7))$.

The alternating group A_5 has been characterized using the “set of element orders” as a defining condition (see [182]). This characterization is proven using only elementary concepts from group theory. Additionally, in [172], an elementary method is employed to demonstrate that a finite group G is isomorphic to A_5 if and only if $\pi_e(G) = \{1, p, q, r\}$, where p , q , and r are distinct prime numbers.

The concept of “the set of element orders in a group” is significant enough that it is referred to as the “spectrum” in some subsequent papers.

V. D. Mazurov (see [146]) notes: “This result opened up a wide way for investigations of recognizability of groups”.

It is clear that the set $\pi_e(G)$ of element orders in a group G is a subset of the positive integers $\mathbb{Z}^+$. However, conversely, the following question presents a quite challenging.

Problem 2.2. *What types of sets of numbers can represent the set of element orders $\pi_e(G)$ of a finite group G ?*

Zh. Wang, A. V. Vasil’ev, M. A. Grechkoseeva, and A. Kh. Zhurtov proved the following theorem about the spectrum $\pi_e(G)$, which serves as a useful criterion for determining the nonsolvability of finite groups (see [221, Theorem 1]).

Theorem 2.3. *Let G be a finite group. Suppose that there exists a subset $\sigma(G)$ of the set of prime factors $\pi(G)$, which contains at least four elements and satisfies the following conditions:*

- (1) $pq \in \pi_e(G)$ for all distinct primes $p, q \in \sigma(G)$.
- (2) $pqr \notin \pi_e(G)$ for all pairwise distinct primes $p, q, r \in \sigma(G)$.

Then G is a nonsolvable group. In particular, there are no groups of odd order that meet these conditions.

Let $\lambda(G)$ denote the maximum number of pairwise non-adjacent vertices in the prime graph $\text{GK}(G)$ (refer to Definition 2.18). It is evident that a direct square $L \times L$ satisfies those two conditions of Theorem 2.3 if the group L satisfies the condition $\lambda(L) \geq 4$. Consequently, the following result can be derived from Theorem 2.3 (see [221, Corollary 2]).

Corollary 2.4. *Let L be a finite group and $\lambda(L) \geq 4$. If G is a finite group such that $\pi_e(G) = \pi_e(L \times L)$, then G is nonsolvable.*

In October 1996, the author presented the following definition during a “Group Theory Seminar” held at Princeton University. For a subset Ω of positive integers, the function h is defined as follows: $h(\Omega)$ represents the number of isomorphism classes of groups G for which $\pi_e(G) = \Omega$.

Definition 2.5. [194] *For any group G , we have $h(\pi_e(G)) \geq 1$. A group G is called recognizable if $h(\pi_e(G)) = 1$. A group G is called almost recognizable (or k -recognizable) if $h(\pi_e(G)) = k$ is finite; otherwise G is called unrecognizable.*

Solvable groups are unrecognizable (see [194, Theorem 4]). In particular, the following result was obtained (see [76, 142, 153, 189, 196]).

Theorem 2.6. *If G has a minimal normal subgroup that is solvable, or if G is isomorphic to one of the following simple groups: A_6 , A_{10} , $L_3(3)$, $U_3(3)$, $U_3(5)$, $U_3(7)$, $U_4(2)$, $U_5(2)$, J_2 , $S_4(q)$ where $q \neq 3^{2m+1}$ and $m > 0$, then G is unrecognizable.*

In April 2010, during the Ischia Group Theory Conference held in Italy, the author presented the following theorem concerning recognizable groups (see [196]).

Theorem 2.7. *Let G be one of the following simple groups:*

- (1) *An alternating group A_n , with the exception of A_6 and A_{10} ;*
- (2) *One of the 26 sporadic simple groups, with the exception of J_2 ;*
- (3) *A simple group of Lie type:*
 - $L_2(q)$ (with $q \neq 9$); $L_3(2^m)$ (for $m \geq 1$); $L_3(q)$ (where $3 < q \equiv 2 \pmod{5}$ and $(6, \frac{q-1}{2}) = 1$);
 - $U_3(2^m)$ (where $m \geq 2$); $L_4(2^m)$ (where $m \geq 1$); $U_4(2^m)$ (where $m \geq 2$);
 - $Sz(2^{2m+1})$ (with $m \geq 1$); $R(3^{2m+1})$ (with $m \geq 1$); ${}^2F_4(2^{2m+1})$ (where $m \geq 1$); $S_4(3^{2m+1})$ (where $m \geq 0$);
 - $B_p(3)$ (where $p > 3$ is odd prime); $C_p(3)$ (where p is odd prime); $D_p(5)$ (where p is odd prime);
 - $D_n(q)$ (where $q = 2, 3$ or 5 for some n); $G_2(3^m)$;
 - ${}^2F_4(2)'$, $L_3(7)$, $L_4(3)$; $L_n(2)$ (where $n \geq 3$); $L_5(3)$, $U_3(9)$, $U_3(11)$, $U_4(3)$, $U_6(2)$, $G_2(3)$;
 - $G_2(4)$, $S_6(3)$, $O_8^-(2)$, $O_{10}^-(2)$, $F_4(2)$, ${}^2E_6(2)$.

Then G is recognizable.

Corollary 2.8. *Let G be a finite group and let M be the Monster group. Then $G \cong M$ if and only if $\pi_e(G) = \{1, 2, \dots, 36, 38, \dots, 42, 44, \dots, 48, 50, 51, 52, 54, \dots, 57, 59, 60, 62, 66, 68, \dots, 71, 78, 84, 87, 88, 92, \dots, 95, 104, 105, 110, 119\}$.*

The preface of the book *Moonshine, the Monster, and Related Topics* [53] highlights an important legacy of the classification of finite simple groups: the existence of the Monster group. It was the exploration of this group that initially suggested intriguing connections between finite groups and certain elliptic modular functions. This possibility, articulated through the Conway-Norton conjectures, led to what can be described as the first iteration of “Moonshine,” which involves the study of class functions on groups with values in a ring of modular functions.

Does Corollary 2.8 have relevance for the ongoing research on the Monster group and related topics?

In the case where $h(\pi_e(G)) = 2$, Buturlakin (see [30]) proved that if G and H are finite simple groups such that $\pi_e(H) = \pi_e(G)$ and H is not isomorphic to G , then the pairs of groups must be either

$\{G, H\} = \{S_6(2), O_8^+(2)\}$ or $\{G, H\} = \{O_7(3), O_8^+(3)\}$. Furthermore, the author of [30] established that there are no instances where $h(\pi_e(G)) = 3$ for three distinct finite simple groups G .

The research on the recognizable groups mentioned earlier has been comprehensively summarized; see [76, Table 1-9]. Recognizable simple groups are those finite simple groups for which the function h equals 1 in these tables. Additionally, $h(G)$ is known for all finite simple groups G , with the exception of some classical groups in dimensions ranging from 5 to 36 over fields of odd characteristic. Furthermore, the following theorem can be found in [76, Table 10].

Theorem 2.9. *Let L be one of the following simple groups: A_6 , A_{10} , J_2 , ${}^3D_4(2)$, $L_3(3)$, $U_3(3)$, $U_3(5)$, $U_3(q)$ (where $q \geq 7$ and q is a special Mersenne prime), $U_5(2)$, $S_4(3)$, $S_4(2^m)$, $S_4(3^m)$ (for even m), $S_4(q)$ (where $q = p^m$ and $p \geq 5$), $S_8(q)$ (for $q = p^m$ and $p \neq 2, 7$), and $O_9(q)$ (where $q = p^m$). Then L is an unrecognizable simple group.*

In recent years, some researchers have characterized the direct square of finite simple groups based on their spectrum and have obtained some interesting results. We define two groups, G and H , as isospectral if $\pi_e(G) = \pi_e(H)$. Wang, Vasil'ev, Grechkoseeva, and Zhurtoev established the recognizability of the direct square of the Suzuki groups, as noted in Corollary 2.4. They also proved the following two theorems (see [221, Theorems 3, 4]):

Theorem 2.10. *If $q \geq 8$ and $q \neq 32$, then the group $Sz(q) \times Sz(q)$ is recognizable by spectrum.*

Theorem 2.11. *Let $L = Sz(32)$. The finite groups isospectral to $L \times L$ are exactly the groups of the form $(L \times L) \rtimes \langle \varphi \rangle$, where either $\varphi = 1$, or φ normalizes each direct factor and induces a field automorphism of order 5 on it. In particular, up to isomorphism, there are four finite groups isospectral to $L \times L$.*

A nonsolvable simple group with abelian Sylow 2-subgroups can be classified into three types: it is either a linear group of the form $\text{PSL}_2(q)$, a small Ree group $R(q) = {}^2G_2(q)$ where $q = 3^a$ with $a \geq 3$ being odd, or the sporadic Janko group J_1 . R. Brandl and Wujie Shi [24] have shown that every finite nonabelian simple group with abelian Sylow 2-subgroups is recognizable.

Li, Moghaddamfar, Vasil'ev and Wang [122] examined the recognizability of the direct square of the finite nonsolvable simple groups with abelian Sylow 2-subgroups and established the following result (see [122, Corollary 1.3]).

Theorem 2.12. *Suppose that L is a simple group with abelian Sylow 2-subgroups. Then the direct square $L \times L$ of L is recognizable by spectrum if L is either a small Ree group $R(q)$ or the sporadic Janko group J_1 , and $L \times L$ is unrecognizable otherwise.*

Mazurov et al., relaxed the previous condition by removing the restriction that the group must be "finite", leading to a series of results (see [80, 96, 130–135, 143–145, 147–150, 152, 229, 245–248]). A typical result is presented as follows (see [129]).

Theorem 2.13. *If the spectrum of a group G is $\{1, 2, 3, 4, 7\}$, then $G \cong \text{PSL}_2(7)$.*

It is important to note that if the group G is assumed to be finite, then the conclusion of the theorem mentioned above was published in the literature [179] in 1984. In contrast, without the “finite” condition, a paper with the same conclusion was published in 2007, after a span of 23 years.

In previous research, particularly in [167] and [112, Problem 12.84], it was conjectured that the function h satisfies $h(\Omega) \in \{0, 1, \infty\}$. However, the authors of [46, 139, 140, 155, 192, 202, 211] later provided counterexamples to this conjecture and presented instances where h takes the value 2. Additionally, a question was raised in [192]: For an arbitrary subset Ω of the natural numbers, does there exist a positive integer k such that $h(\Omega)$ falls within the set $\{0, 1, 2, \dots, k, \infty\}$? If so, what is the value of such k ?

A. V. Zavarnitsine [235] provided the following example: For any $r > 0$, the function $h(\pi_e(L_3(7^n))) = r + 1$ where $n = 3^r$. This indicates that there are $r + 1$ groups, all of the form $L_3(7^n)\langle\rho\rangle$, where ρ is a field automorphism of $L_3(7^n)$. This serves as an example of the function h such that for any positive integer k , there exist k -recognizable groups.

Additionally, while considering an example of a group for which the function h equals 2 (see [189, Theorem 5.4]), the author raised the question of whether, for any k , there exist two finite k -recognizable isospectral groups that are section-free? It means they are not simple sections of each other (see [196, Problem 5.2]). Grechkoseeva [75] pointed out the existence of such groups, specifically $G_1 = L_{15}(2^{60}).3$ and $G_2 = L_{15}(2^{60}).5$ (see [112, Problem A:16.106]). It is noteworthy that although G_1 and G_2 are section-free, both share the same normal subgroup $L_{15}(2^{60})$.

The spectrum of a finite group consists of a set of positive integers that includes the number 1. When the integer set $\pi_e(G)$ is divided into three parts: the number 1, the set of prime numbers $\pi'_e(G)$, and the set of composite numbers $\pi''_e(G)$, the following result can be established:

Theorem 2.14. [51] *Suppose that G is a finite group, then $|\pi'_e(G)| \leq |\pi''_e(G)| + 3$. Moreover if $|\pi'_e(G)| = |\pi''_e(G)| + 3$, then G is simple and G can be characterized by its spectrum. Furthermore, G is one of the following simple groups.*

- (I) $A_5, L_2(11), L_2(13), L_2(16), L_3(4)$ and J_1 .
- (II) $Sz(q)$, where $q = 2^{2n+1}$ satisfies that each of $q - 1$, $q - \sqrt{2q} + 1$, and $q + \sqrt{2q} + 1$ is either a prime number or a product of two distinct prime numbers.
- (III) $L_2(2^n)$, where n ($n \geq 5$) is an odd prime and satisfies both $\frac{2^n+1}{3}$ is prime and $2^n - 1$ is either a prime number or a product of two distinct prime numbers.
- (IV) $L_2(3^n)$ where n is an odd prime and satisfies both $\frac{3^n+1}{4}$ is a prime and $\frac{3^n-1}{2}$ is either prime or a product of two distinct prime numbers.
- (V) $L_2(5^n)$, where n is an odd prime satisfying both $\frac{5^n-1}{4}$ and $\frac{5^n+1}{6}$ are primes.
- (VI) $L_2(p)$, where p is a prime greater than 13 and one of the following holds:
 - (i) $\frac{p-1}{4}$ and $\frac{p+1}{6}$ are primes.
 - (ii) $\frac{p-1}{6}$ and $\frac{p+1}{4}$ are primes.

Problem 2.15. *In the cases (II)-(VI) of the aforementioned theorem, is the number of simple groups satisfying the respective conditions finite or infinite?*

Let $\iota(G)$ denote the set of numbers of the form pq (with $p \neq q$) in the set $\pi_e(G)$. The following result holds:

Theorem 2.16. [50] *For a finite group G , we have $|\pi(G)| \leq |\iota(G)| + 4$. If $|\pi(G)| = |\iota(G)| + 4$, then G is a simple group and can be characterized by its spectrum.*

According to Theorem 2.14 (Theorem 2.16), the number of prime divisors of the order of a group G is limited by $|\pi_e''(G)|$ (or $|\iota(G)|$).

If the “order of a group” is square-free, then that group is classified as a metacyclic group, which implies that it is solvable. Taking this conclusion into account, along with the property of A_5 as stated in Theorem 1.4, the author has posed the following question:

Problem 2.17. *Examine finite groups in which the orders of the elements are square-free.*

Without any additional conditions, what properties can the spectrum of a finite group have? This is Problem 2.2: What types of sets can the element orders $\pi_e(G)$ represent?

After the publication of the literature [179], Brandl wrote to the author and posed two questions, one of which is: If the orders of elements in a finite group G are consecutive integers, that is, $\pi_e(G) = \{1, 2, 3, \dots, n\}$, then what is the largest possible value of n that can continue consecutively? This question imposes a “consecutive” condition on the spectrum of the group. To solve this question, the author considered prime graph components of finite groups and found the key reference [222] to resolve this problem (see [94, 114] for the relevant literature and [117] for the most comprehensive description of the prime graph components).

Definition 2.18. [222] *Let G be a finite group. The simple graph $\Gamma(G)$ associated with G is constructed based on the set $\pi_e(G)$ as follows: The vertex set of this graph is $\pi(G)$, and two vertices p and q are adjacent if and only if $pq \in \pi_e(G)$, which we denote as $p \sim q$. Let $t(G)$ represent the number of connected components of the graph $\Gamma(G)$. We denote the i -th connected component of $\Gamma(G)$ as $\pi_i = \pi_i(G)$ for $i = 1, 2, \dots, t(G)$. If the order of G is even, we specifically denote the component containing the prime 2 as $\pi_1(G)$. The graph $\Gamma(G)$ is commonly referred to as the prime graph of G . This concept was proposed by K. W. Gruenberg and O. H. Kegel in 1975, which is why it is also known as the Gruenberg-Kegel graph of the group G (abbreviated as the prime graph $\text{GK}(G)$).*

The definition of the prime graph indicates that the set $\pi_e(G)$ determines the graph $\Gamma(G)$. However, the same graph $\Gamma(G)$ can correspond to different sets $\pi_e(G)$. Consequently, using $\Gamma(G)$ as a tool to study finite groups, particularly finite simple groups, has emerged as a new area of research.

Theorem 2.19. [23] *Let G be a finite group. If the elements of the spectrum of G are consecutive integers, i.e., $\pi_e(G) = \{1, 2, 3, \dots, n\}$ (such groups are referred to as finite OC_n -groups), then $n \leq 8$. Furthermore, these groups have been classified as follows:*

- (I) $n \leq 2$ and G is an elementary abelian 2-group.
- (II) $n = 3$ and $G = [N]Q$ is a Frobenius group where either $N \cong Z_3^t$, $Q \cong Z_2$ or $N \cong Z_2^{2t}$, $Q \cong Z_3$.
- (III) $n = 4$ and $G = [N]Q$ and one of the following holds:
 - (i) N has exponent 4 and class ≤ 2 and $Q \cong Z_3$.
 - (ii) $N \cong Z_2^{2t}$ and $Q \cong \Sigma_3$.
 - (iii) $N \cong Z_3^{2t}$ and $Q \cong Z_4$ or Q_8 and G is a Frobenius group.
- (IV) $n = 5$ and $G \cong A_6$ or $G = [N]Q$, where $Q \cong A_5$ and N is an elementary abelian 2-group and a direct sum of natural $\mathrm{SL}(2, 4)$ -modules.
- (V) $n = 6$ and G is one of the following types:
 - (i) $G = [P_5]Q$ is a Frobenius group, where $Q \cong [Z_3]Z_4$ or $Q \cong \mathrm{SL}(2, 3)$ and $P_5 \cong Z_5^{2t}$.
 - (ii) $G/O_2(G) \cong A_5$ and $O_2(G)$ is elementary abelian and a direct sum of natural and orthogonal $\mathrm{SL}(2, 4)$ -modules.
 - (iii) $G = \Sigma_5$ or $G \cong \Sigma_6$.
 - (iv) $n = 7$ and $G \cong A_7$.
 - (v) $n = 8$ and $G = [\mathrm{PSL}(3, 4)]\langle\beta\rangle$, β is an unitary automorphism of $\mathrm{PSL}(3, 4)$.

where $[A]B$ denotes the split extension of its normal subgroup A by a complement B .

Corollary 2.20. *Let G be a finite group, then $\pi_e(G) = \{1, 2, 3, \dots, 7\}$ if and only if $G \cong A_7$.*

For the infinite OC_7 groups, A. S. Mamontov and E. Jabara [136] demonstrated that these groups are locally finite. This means that any finite set of elements within G generates a finite subgroup. From this, they concluded that G is isomorphic to A_7 (see [112, Problem A:19.80]).

It is clear that Z_2^∞ and $[Z_3^\infty]Z_2$ represent the infinite OC_2 groups and OC_3 groups, respectively. Furthermore, for $4 \leq n \leq 6$, infinite OC_n groups also exist.

Problem 2.21. *What is the maximum value of n for the infinite OC_n group? The infinite OC_n group is periodic, which means that the orders of elements in the group are finite. However, is it locally finite? This is a special case of Burnside's problem (see [82]).*

The literature [121, 223] discuss finite groups whose elements have orders that are consecutive odd integers, as well as finite groups whose elements have orders that are consecutive integers, except for certain primes.

Problem 2.22. *By drawing an analogy to Theorem 2.19, we investigate finite groups whose spectra consist of segmented consecutive integers. For example, the groups A_5 and $\mathrm{PSL}_2(7)$, with spectra $\pi_e(A_5) = \{1, 2, 3, 5\}$ (see [203]) and $\pi_e(\mathrm{PSL}_2(7)) = \{1, 2, 3, 4, 7\}$ (see [179]), both represent finite simple groups with spectra composed of two segments of consecutive integers. Furthermore, we aim to determine the maximum length of consecutive integers for finite groups exhibiting this characteristic of segmented consecutiveness. For instance, in the case of an OC_n group, when the maximum length is 8.*

It is important to note that the order of an element in a group corresponds to the order of the cyclic subgroup generated by that element. Yanquan Feng [59] and the author [190] have provided examples of finite groups in which the orders of abelian subgroups and the orders of proper subgroups are consecutive integers, respectively. Additionally, since the order of an element in a group is invariant under conjugation, Guohua Qian [169] has explored finite groups with consecutive nonlinear character degrees.

3. Characterizing finite simple groups by their order and element orders

According to the classification theorem for finite simple groups (abbreviated as CFSG), every finite simple group is isomorphic to one of the following types (see [63]):

- (i) Groups of prime order;
- (ii) Alternating groups of degree at least 5;
- (iii) Simple classical groups;
- (iv) Simple exceptional groups of Lie type;
- (v) The 26 sporadic groups.

From 1987 to 2003, the author and his coauthors demonstrated that Conjecture 1.2 is true for all finite simple groups, except the families $B_n(q)$, $C_n(q)$, and $D_n(q)$ (where n is even) (see [37, 184, 191, 199–201, 226]). In 2009, this conjecture was also proved for the families $B_n(q)$, $C_n(q)$, and $D_n(q)$ (where n is even) (see [215]). Consequently, the conjecture has been established as a theorem stating that all finite simple groups can be characterized by two properties: the “order of the group” and the “set of element orders,” collectively referred to as the “two orders”. This theorem is highly beneficial in the field of computational complexity theory (see [21, 78]).

The 26 sporadic simple groups exhibit “two orders”, which can be analyzed through the prime graph components of finite groups, as detailed in [222]. In fact, all of these 26 sporadic simple groups, with the exception of J_2 , can be uniquely identified by their “set of element orders” (refer to Theorem 2.7(2)).

For alternating groups of degree 5 or greater, a series of lemmas are established in [201] based on the order of the group A_n , which is $n!/2$. Notably, the Stirling formula from number theory is employed to reach conclusions without relying on the prime graph components of finite groups, as discussed in [222]. In fact, all alternating groups of degree at least 5, with the exceptions of A_6 and A_{10} , can be characterized only by their “set of element orders” (see Theorem 2.7(1)).

For the classical simple groups and the exceptional simple groups of Lie type, the author initially applied the classification theorem for simple groups to solve a problem posed by E. Artin in 1955. This problem involves determining all finite simple groups whose Sylow subgroup orders exceed $|G|^{1/3}$ (see [14]). As a result, two lemmas were established (see [187]):

Lemma 3.1. *Let G be a finite simple group. If $|G| = p^k m$, where p does not divide m , p is an odd prime, and $|G| < p^{3k}$, then G is one of the following groups:*

- (1) A simple group of Lie type in characteristic p ;
- (2) The alternating groups A_5 , A_6 and A_9 ;
- (3) $L_2(p-1)$ where p is a Fermat prime, $L_2(8)$ and $U_5(2)$.

Lemma 3.2. *Let G be a finite simple group. If $|G| = 2^k m$ with m odd, and $|G| < 2^{3k}$, then G is one of the following groups:*

- (1) A simple group of Lie type in characteristic 2;
- (2) $L_2(r)$ where r is a Fermat prime or a Mersenne prime;
- (3) A_6 , $U_3(3)$, A_9 , M_{12} , $U_3(4)$, A_{10} , M_{22} , J_2 , HS , M_{24} , Suz , Ru , Fi_{22} , Co_2 , Co_1 and B .

The previously mentioned lemmas and the group order limit the scope of the research to simple groups of Lie type. Moreover, after further analysis, it has been established that Conjecture 1.2 is valid for all cases, with the exception of the groups $B_n(q)$, $C_n(q)$, and $D_n(q)$ when n is even.

For exceptional simple groups of Lie type, several groups can be characterized only by their “set of element orders.” These groups include:

- Suzuki-Ree groups: ${}^2B_2(2^{2n+1})$ for $n \geq 1$ (see [188]),
- ${}^2G_2(3^{2n+1})$ for $n \geq 1$ (see [24]),
- ${}^2F_4(2^{2n+1})$ for $n \geq 1$ (see [52]),
- $G_2(q)$ (see [216]),
- $E_8(q)$ (see [115]),
- $F_4(2^m)$ (see [36]),

and others, as summarized in [76, Table 8].

For the classical groups $B_n(q)$ and $C_n(q)$, which have the same order, we investigate whether there exist values of n and a prime power q such that their spectra are also identical. This would provide a counterexample to Conjecture 1.2. Shi [195] and Grechkoseeva [74] addressed this question almost simultaneously and proved that for all n and prime power q , the elementary partition spectra satisfy $\pi_e(B_n(q)) \neq \pi_e(C_n(q))$. Therefore, such a counterexample does not exist.

To ultimately prove Conjecture 1.2, it has been shown that the restrictions on the nonabelian composition factors of finite groups have the same spectrum as $B_n(q)$, $C_n(q)$, and $D_n(q)$ (where n is even) as stated in [214]. Moreover, Conjecture 1.2 was successfully proved in [215] by applying the three theorems from [214]. As a result, Conjecture 1.2, which characterizes finite simple groups using “two orders”, has been proved and established as a theorem.

Numbers and sets of numbers play a significant role in mathematics. The concept of “two orders” was first introduced by the author and has since been extensively studied in the field of group theory. Simple groups are inherently complex, and the idea of “two orders” represents one of the simplest concepts within this area. The validation of Conjecture 1.2 establishes a connection between the terms “simple” and “complex.” Notably, the proof relied on the classification theorem for finite simple groups.

Except for all finite simple groups, we can characterize the following groups using only “two orders”. Jianxing Bi proved the following [18]:

Theorem 3.3. *Let G be a finite group and S_n be a symmetry group. Then $G \cong S_n$ if and only if (1) $\pi_e(G) = \pi_e(S_n)$; (2) $|G| = |S_n|$.*

And Yanwei Gao and Hongping Cao [61] proved the following theorem:

Theorem 3.4. *Let G be a finite group and $\text{Aut}(G)$ be the automorphism group of a sporadic simple group S . Then $G \cong \text{Aut}(S)$ if and only if (1) $\pi_e(G) = \pi_e(\text{Aut}(S))$; (2) $|G| = |\text{Aut}(S)|$.*

In fact, in [61], the condition (1) is weakened to $m_i(G) = m_i(\text{Aut}(S))$ for $i = 1, 2, 3$. Here, $m_1(G)$, $m_2(G)$, and $m_3(G)$ denote the largest, second largest, and third largest element orders of G , respectively.

All finite simple groups can be characterized by their “two orders”, but this characterization fails for some groups with small orders. For instance, the dihedral group D_8 , which has 8 elements, and the quaternion group Q_8 both satisfy $\pi_e(D_8) = \pi_e(Q_8) = \{1, 2, 4\}$. However, D_8 and Q_8 are not isomorphic.

Problem 3.5. *Other than all finite simple groups and certain automorphism groups of these groups, what types of finite groups can be characterized by their “two orders”?*

Next, we will present another application of Conjecture 1.2 as follows:

Let G be a finite group, and let $B(G)$ be the Burnside ring of G . T. Yoshida posed an open problem: If $B(G) \cong B(H)$ for two finite groups G and H , does it imply that $G \cong H$? This problem can be found in [232, Page 340, Problem 2]. It has been proved in [113, Corollary 5.2] that the spectrum of a finite group is determined by its Burnside ring. Therefore, we obtain the following application:

Corollary 3.6. *Let G be a finite group, and S a finite simple group. If $B(G) \cong B(S)$, then $G \cong S$.*

Simple groups can be characterized using their Burnside ring, making them recognizable.

Problem 3.7. *Is it possible to prove Conjecture 1.2 without using the classification theorem for finite simple groups?*

In the case of a few simple groups with small orders, it is possible to provide a proof without using the classification theorem for simple groups, such as with A_5 (see [182]). However, it appears to be impossible to provide a proof for all finite simple groups without using the classification theorem.

Problem 3.8. *Is it possible to characterize all finite simple groups by weakening the condition of “two orders”? In other words, can we describe these groups using either the “group order” and the “set of orders of certain elements” or the “orders of Hall subgroups within a group” along with the “set of orders of elements” for all simple groups?*

For a prime power $q = p^e$, we write $ch(q) = p$, and for a Lie type simple group G defined over $GF(q)$, the characteristic of G denoted by $ch(G) = p$. W. M. Kantor and Á. Seress proved the following theorem (see [103, Theorem 1.2]):

Theorem 3.9. *Let G and H be simple groups of Lie type of odd characteristic. If $m_i(G) = m_i(H)$ for $i = 1, 2, 3$, then the characteristics of G and H must also be the same, where $m_1(G)$, $m_2(G)$ and $m_3(G)$ represent the largest, second largest, and third largest element orders of G , respectively.*

This raises the question: Can simple groups be characterized by their “group order” and the “set of orders of certain elements”? In fact, the following theorem has been proven in [85, 86, 120, 233]:

Theorem 3.10. *Let G be a group, and let S be one of the following simple groups:*

- (1) *Sporadic simple groups* (see [86]);
- (2) $L_2(p)$, *where $p \neq 7$ is a prime* (see [120]);
- (3) *Simple K_4 -groups of type $L_2(q)$, where q is a prime power* (see [85]);
- (4) *Simple K_5 -groups of type $L_3(p)$, where p is a prime number and $(3, p-1) = 1$* (see [233]).

Then $G \cong S$ if and only if $|G| = |S|$ and $m_1(G) = m_1(S)$, where $m_1(G)$ denotes the largest element order of G .

If we consider the second largest or third largest element order of a group G , it can also be used to characterize more simple groups (see [84]).

Let G be a finite group. We define the set $\text{Van}(G) = \{g \in G \mid \text{there exists } \chi \text{ such that } \chi(g) = 0\}$, where χ is an irreducible complex character of G . Let $\text{Vo}(G)$ denote the set of element orders of $\text{Van}(G)$. It is clear that $\text{Vo}(G)$ is a subset of the set of element orders $\pi_e(G)$ of G . The following conjecture was proposed in [112, Problem 19.30]:

Conjecture 3.11. *Let G be a finite group and S a finite simple group. Then $G \cong S$ if and only if (1) $\text{Vo}(G) = \text{Vo}(S)$; (2) $|G| = |S|$.*

The conjecture mentioned above as a question was also put forward in Jinshan Zhang’s PhD thesis (see [238]). We refer to the literature [177] and footnote³ for its recent studies.

The group order, element order, the number of elements of the same order, the lengths of conjugacy classes, the degrees of characters, and other related metrics are known as conjugation invariants. Additionally, important quantities for studying groups include the order of the normalizer of a Sylow subgroup, the order of the largest abelian (or solvable) subgroup, the index of the largest subgroup, and the number of Sylow subgroups, among others. Analyzing the structure of groups, particularly finite simple groups, using these significant quantities is a vast and meaningful area of study. For instance, one might explore the structure of finite groups where the lengths of all conjugacy classes are prime powers.

Qian [170] and Qian, Wang and Wei [173] defined the codegree of the character and established a connection between the element order and the degree of the character.

Next, we will examine how simple groups can be characterized using the concepts of “the order of Hall subgroups of a group” and “the set of element orders.” First, we utilize “the order of a Sylow

³Q. F. Yan, Z. C. Shen, J. S. Zhang et al, A new characteristic of sporadic simple groups J_1 and J_4 . Submitted

2-subgroup” of a group G (denoted by $|G|_2$) along with the “set of element orders” to characterize alternating and sporadic simple groups. The following theorems are established:

Theorem 3.12. *Let G be a finite group and S an alternating simple group. Then $G \cong S$ if and only if (1) $|G|_2 = |S|_2$; (2) $\pi_e(G) = \pi_e(S)$.*

Theorem 3.13. *Let G be a finite group and S a sporadic simple group. Then $G \cong S$ if and only if (1) $|G|_2 = |S|_2$; (2) $\pi_e(G) = \pi_e(S)$.*

Lemma 3.14. [65] *Suppose G is a finite group with $\pi_e(G) = \pi_e(S)$, where S is a simple alternating group A_n for $n \neq 6, 10$. Then $G \cong S$.*

The authors of [140, 167, 182] previously examined the characterization of alternating groups and symmetric groups using the set of element orders. In subsequent studies [65, 117, 234], the authors continued to investigate the characterization of alternating groups through the set of element orders. Additionally, it was noted in [206] that the groups A_6 and A_{10} are unrecognizable.

Lemma 3.15. [23] *Let G be a finite group with $\pi_e(G) = \pi_e(A_6) = \{1, 2, 3, 4, 5\}$. Then $G \cong A_6$ or $G = [N]Q$, where $Q \cong A_5$ and N is an elementary abelian 2-group and a direct sum of natural $\text{SL}(2, 4)$ -modules.*

Lemma 3.16. [206] *Let G be a finite group with $\pi_e(G) = \pi_e(A_{10}) = \{1, 2, \dots, 10, 12, 15, 21\}$. Then $G \cong A_{10}$, or $G = [A]C$ where A is an abelian $\{3, 7\}$ -group, $C = C_G(t) = [\langle t \rangle]S_5$, where t is an involution and $a^t = a^{-1}$, $a \in A$ and the Sylow 2-subgroup of C is a generalized quaternion group Q_{16} with 16 elements.*

Lemma 3.17. [151, 189] *Let G be a finite group such that $\pi_e(G) = \pi_e(M)$, where M is a sporadic simple group distinct from J_2 . Then, $G \cong M$.*

It was proven in [189] that, with the exception of Co_2 and J_2 , all of the remaining 24 sporadic simple groups can be characterized by the set of their element orders. Mazurov and Shi [151] demonstrated that Co_2 can also be characterized using the “set of element orders.” However, if $\pi_e(G) = \pi_e(J_2) = \{1, 2, \dots, 8, 10, 12, 15\}$, then G is isomorphic to either J_2 , S_8 , or G is isomorphic to the split extension of a 2-group N of order 2^{6t} (where $t = 1, 2, \dots$) by A_8 .

From the arguments above and Lemma 3.14, we can conclude that Theorems 3.12 and 3.13 are valid.

Problem 3.18. *Let G be a finite group and S a finite simple group. If (1) $|G|_\pi = |S|_\pi$, where $|G|_\pi$ denotes the order of π -Hall subgroup of G , $\pi \neq \pi(G)$; and (2) $\pi_e(G) = \pi_e(S)$, can we conclude that $G \cong S$?*

In 2007, Mazurov presented the following conjecture at the International Algebraic Conference held in St. Petersburg: Let G be a finite group, and let L be either an alternating group of sufficiently large

degree or a simple group of Lie type with sufficiently large Lie rank. If G is isospectral to L , then G is an almost simple group with a socle isomorphic to L . In fact, if L is a finite simple group and G is isospectral to L , then G must either be solvable, in which case L could be one of the following simple groups: $L_3(3)$, $U_3(3)$, or $S_4(3)$; or G has exactly one nonabelian composition factor (see [212], [217, Corollary 7.3], and [65, Theorem 2]). This conjecture proposed by Mazurov was ultimately proven to be valid in [77]. By utilizing these results and comparing $|G|_\pi$ and $|M|_\pi$, we can answer Problem 3.18.

4. Thompson Conjecture and Thompson Problem

4.1. The origin and early work of the Thompson Conjecture. As mentioned earlier, the author posed Conjecture 1.2 to Thompson in 1987, and Thompson proposed the following conjecture in his response letter in 1988 (Thompson Conjecture (1988)): Let G and M be finite groups, set $N(G) = \{n \in \mathbb{Z}^+ \mid G \text{ has conjugacy class } C, |C| = n\}$. Suppose M is a non-abelian simple group, and the center of G is trivial. If $N(G) = N(M)$, then G and M are isomorphic.

Guiyun Chen [38] proved that if M is a sporadic simple group then Thompson conjecture is correct. Subsequently, he finished his PhD thesis titled “On Thompson Conjecture”. The author (see [39]) based on the foundation of prime graph components in [222], a definition of the order components was provided as follows:

Definition 4.1. Let G be a finite group, $\pi_1, \pi_2, \dots, \pi_t$ are prime graph components of G , where $t = t(G)$ is the number of prime graph components of $\Gamma(G)$. Assume $|G| = m_1 m_2 \cdots m_t$ where $\pi_i = \pi(m_i)$. Then $m_1, m_2, \dots, m_t$ are called order components of G . Set $\text{OC}(G) = \{m_1, m_2, \dots, m_{t(G)}\}$, the set of order components of G .

Thompson Conjecture was proved to be true for all sporadic simple groups (see [38, 39]). Its validity was also proved for finite group having at least three prime graph components (see [40, 41]). For the case of two prime graph components, we can use the same method to prove it. For instance, it was proved that Thompson Conjecture is correct for the group ${}^3D_4(q)$ (see [42]).

4.2. The case of groups with connected prime graph. In 2009, Vasil’ev proved that Thompson Conjecture is valid for the simple groups A_{10} and $L_4(4)$ with connected prime graph (see [213]). Subsequently, N. Ahanjideh proved that Thompson Conjecture is also correct for simple groups of Lie type $A_n(q)$ (see [5]), $B_n(q)$ (see [8]), $C_n(q)$ (see [6]), $D_n(q)$ where $n \neq 4, 8$ (see [7]), ${}^2A_n(q)$ (see [9]) and ${}^2D_n(q)$ (see [10]) in a series of articles. For other exceptional simple groups of Lie type, Thompson Conjecture was established the validity in the literature [71, 95, 109, 110, 227]. The authors [13, 64, 66–69, 126, 225] proved that Thompson Conjecture is true for alternating simple groups. Finally, I. B. Gorshkov [70] proved the validity of Thompson Conjecture for the remaining cases $D_4(q)$ and $D_8(q)$. Therefore, Thompson Conjecture was completely proved and was became a theorem.

4.3. Thompson Problem. Thompson problem posed by Thompson in a letter to the author in 1987. Let G_1 and G_2 be groups of the same order type (as defined in Definition 1.3). Suppose G_1 is solvable, is it true that G_2 is also necessarily solvable? For groups G of even order, we can not determine the solvability of G by using the order of G (see [58]). But we may judge it using the same order type of G if the answer of Thompson Conjecture is positive. In particular, Thompson wrote in the letter: I have talked with several mathematicians concerning groups of the same order type. The problem arose initially in the study of algebraic number fields, and is of considerable interest. At the same letter, Thompson provided the following example of nonsolvable groups with the same order type in the letter:

$$G_1 = 2^4 : A_7, \quad G_2 = L_3(4) : 2_2.$$

Both of them are maximal subgroups of M_{23} , where “:” denotes semidirect product, we refer to [47, page 72 and page 23] for the maximal subgroups of M_{23} and the automorphism group of $L_3(4)$, respectively.

The author gave a lecture titled “Thompson Problem and Thompson Conjecture” at the international conference “Algebra and Mathematical Logic” held in Russia in September 2011, commemorating the 100th anniversary of V. V. Morozov’s birthday. During the conference, the author presented an overview of the aforementioned problem and conjecture.

4.4. Order equation. Let $M_G(n)$ be the set of elements of G with order n , the elements of G can be partitioned into the elements of same order, that is, $G = \bigcup M_G(n)$ where $n \in \pi_e(G)$. Since every element of order n must be contained in some cyclic subgroup of order n , we have $|M_G(n)| = V_n(G)\phi(n)$, where $V_n(G)$ is the number of cyclic subgroups of order n , $\phi(n)$ is Euler totient function. Hence we can obtain the following equation, $|G| = \sum V_n(G)\phi(n)$, $n \in \pi_e(G)$. Here we denote the “order equation” of G as $Ord(G)$. That is, if two order equations of finite groups G and H are same, then $\pi_e(G) = \pi_e(H)$ and $V_d(G) = V_d(H)$ for any $d \in \pi_e(G)$. It is easy to see that same order equation is equivalent to the same order type of Thompson Problem.

Let G and H are two groups of same order type, it is not hard to prove that if H is nilpotent then G is also nilpotent and if H is supersolvable then G is solvable (see [224]). Of course, if H is a finite nonabelian simple group, then $G \cong H$ is nonsolvable because Conjecture 1.2 has been proved. It was proved that if H is solvable and the prime graph of H is disconnected, then G is solvable (see [175, 176]). The following results were proved in papers [175, 176]:

Lemma 4.2. *Let G be a finite group and H a Frobenius (resp. 2-Frobenius) group. If G and H have same order equation, then G is a Frobenius (resp. 2-Frobenius) group. Moreover, if H is solvable, then G is also solvable.*

From Lemma 4.2 we deduce the following theorem:

Theorem 4.3. *Let G and H be any two finite groups of the same order type. If H is solvable and its prime graph is disconnected, then G is solvable.*

Note that orders of two elements in a group are equal if they are conjugate. Consequently, “same order class” is the union of several “conjugacy classes”. Conversely, two elements of the “same order” might not be “conjugate”. The series of books *Unsolved Problems in Group Theory* records the following question: Suppose that, in a finite group G , each two elements of the same order are conjugate. Is then $|G| \leq 6$ (see [112, Problem 7.48])? It was proved in [57, 60, 210, 236] successively using the classification for finite simple groups (CFSG).

Problem 4.4. *Without using classification for finite simple groups, can we prove that if each two elements of the same order are conjugate in finite group G , then $|G| \leq 6$?*

Thompson Problem investigates the structure of groups from the “same order” perspective, while Thompson Conjecture studies groups from the “conjugacy” perspective. These two have both distinctions and connections. It is well-known that conjugacy classes play a significant role in the study of finite group structure. It was introduced in [35] that the influence of the conjugacy class size or the number of conjugacy classes on finite group structure. A_5 was characterized by using the number of elements of the same order (see [174]), that is, the following theorem was proved:

Theorem 4.5. *A group G is isomorphic to A_5 if and only if G has the same order type as A_5 (the order type of A_5 is $\{1, 15, 20, 24\}$).*

Now denote $nse(G) = \{m_k | k \in \pi_e(G)\}$, where m_k denotes the number of elements of order k in G , then the above theorem can be stated as follows: A group G is isomorphic to A_5 if and only if $nse(G) = nse(A_5)$. For some alternating simple groups and linear simple groups, there is a lot of literature about characterizing these simple groups using $nse(G)$, see [11, 15].

The following result was obtained in [174]:

Proposition 4.6. *Let G be a group (not assume finite) containing more than two elements. If the number of elements with the same order in G is finite, and the maximal number is s , then $|G| \leq s(s^2 - 1)$.*

Thompson Problem investigates that using “same order” to judge the solvability of groups. The condition “same order” cannot be changed to “conjugacy” here. In fact, we have the following conclusion:

Theorem 4.7. [163] *There exist two finite groups G and H such that $N(G) = N(H)$, i.e., the set of their conjugacy class sizes are same, but G is solvable and H is nonsolvable.*

A related work to Thompson Problem is the study of the solvability of groups using the number of elements of maximal order or the type of this number (see [43, 56, 87, 98–102, 208, 228]).

Next, we continue discussing Thompson problem and give the following definitions.

For each finite group G and each integer $d \geq 1$, let $G(d) = \{x \in G | x^d = 1\}$.

Definition 4.8. *G_1 and G_2 are of the same two orders type if and only if $|G_1| = |G_2|$ and $\pi_e(G_1) = \pi_e(G_2)$.*

Definition 4.9. G_1 and G_2 are of the same Burnside ring type if and only if $B(G_1) \cong B(G_2)$.

The authors [124] proved that if G_1 and G_2 are of the same Burnside ring type, then G_1 and G_2 are of the same order type (see Definition 1.3). Thus G_1 and G_2 are of the same two orders type immediately. But the reverse side is not correct. We have the following examples:

Example 4.10. Let $G_1 = L_3(4) : 2_2$ and $G_2 = 2^4 : A_7$. Then G_1 and G_2 are of the same order type, but $B(G_1) \not\cong B(G_2)$.

Example 4.11. Let $G_1 = A_5 \times A_5 \times A_5$, G_1 is not solvable and $|G_1| = 60^3$ and $\pi_e(G_1) = \{1, 2, 3, 5, 6, 10, 15, 30\}$ since $\pi_e(A_5) = \{1, 2, 3, 5\}$. Let $G_2 = C_{30} \times C_{30} \times C_{30} \times C_2 \times C_2 \times C_2$. Then $|G_2| = 60^3$ and $\pi_e(G_2) = \{1, 2, 3, 5, 6, 10, 15, 30\}$. G_1 and G_2 are of the same two orders type, but G_2 is solvable. Moreover, G_1 and G_2 are of not the same order type.

Noting Thompson problem, which states that if G_1 and G_2 are of the same order type, and G_1 is a solvable group, is G_2 also solvable? If we strengthen “same order type” to “same Burnside ring type”, then the answer is positive (see [55]). However, if we weaken “same order type” to “same two orders type”, then the above Example 4.11 is a counterexample.

Recently, the author of [166] constructs two finite groups of size $2^{365} \cdot 3^{105} \cdot 7^{104}$, a solvable group G and a non-solvable group H , such that for every integer n the groups have the same number of elements of order n . It gives a negative answer to Thompson problem. As a much smaller counterexample to Thompson problem, reference [162] constructed two same order type groups of order $2^{13} \cdot 3^4 \cdot 7^3$, one solvable and the other non-solvable.

5. Problems related to quantitative characterization

5.1. Width of order and width of spectrum in finite groups. The fundamental theorem of arithmetic states that every integer n can be represented as $n = p^a q^b \cdots r^c$, where $p, q, \dots, r$ are distinct prime numbers.

The prime factorization of numbers is one of the most important results in mathematics. In the following section, we will study the influence of the number of prime divisors on the structure of groups.

Note that both the order $|G|$ of a finite group G and the order $|g|$ of an element g are numbers in G , let $|\pi(G)|$ and $|\pi(g)|$ denote the number of prime factors of $|G|$ and $|g|$, respectively. These are respectively called the width $\omega_0(G)$ of the order of finite group G and the width of an element g .

Definition 5.1. Let G be a finite group, we define $\omega_0(G) = |\pi(G)|$ the width of order of G and $\omega_s(G) = \max\{|\pi(g)| \mid g \in \pi_e(G)\}$ the width of spectrum of G .

For other sets of conjugation invariants of group G , the corresponding width can also be defined.

In August 2020, at the online conference “Ural Workshop on Group Theory and Combinatorics” held in Russia, the author gave the first presentation and introduced the aforementioned two definitions of

width, and discussed the structure of finite groups with a small width, we refer to [198] for some results about the “width of order”. This paper points out that groups with $\omega_0(G) = 1$, finite p -groups, although the orders of these groups have only one prime factor, their structures are quite complex. Recent Chinese literature can be found in [242, 243]. For the especially “simple” condition, $\pi_e(G) = \{1, 3\}$, but its structure is quite complex, and currently, there is no a detailed classification.

When $\omega_0(G) = 2$, that is, $|G| = p^a q^b$, it is a special class of solvable groups. In 1904, Burnside [29] provided the first proof of its solvability using representation theory. Later, D. M. Goldschmidt [62] provided a purely group theoretical proof for this theorem when both p and q are odd primes. Then, H. Bender [17] and H. Matsuyama [138], in 1972 and 1973 respectively, proved the remaining cases, thereby providing an abstract group proof for the $p^a q^b$ theorem.

If $\omega_0(G) = 3$, then G can be a nonsolvable group, and its chief factors are finite simple groups with $\omega_0(G) = 3$. M. Herzog [90] gave all 8 finite simple groups (simple K_3 -groups) with $\omega_0(G) = 3$, they are A_5 , $L_2(7)$, $L_2(8)$, A_6 , $L_2(17)$, $L_3(3)$, $U_3(3)$, and $U_4(2)$.

Using classification for finite simple groups, simple groups with $\omega_0(G) = 4$ (simple K_4 -groups) were successively given in the papers [93, 186, 220]⁴. The further investigation of the simple group with $\omega_0(G) = 4$ (simple K_4 -group) has been discussed in [28, 244], it is one of the following groups:

- (I) A_n ($n = 7, 8, 9, 10$), M_{11} , M_{12} , J_2 ; $L_2(q)$ ($q = 16, 25, 49, 81, 97, 243, 577$), $L_3(q)$ ($q = 4, 5, 7, 8, 17$), $L_4(3)$; $O_5(q)$ ($q = 4, 5, 7, 9$), $O_7(2)$, $O_8^+(2)$, $G_2(3)$; $U_3(q)$ ($q = 4, 5, 7, 8, 9$), $U_4(3)$, $U_5(2)$; ${}^3D_4(2)$, ${}^2F_4(2)'$, $Sz(8)$, $Sz(32)$;
- (II) $L_2(r)$, where r is prime and satisfies the following equation: $r^2 - 1 = 2^a 3^b u$ for $a \geq 1$, $b \geq 1$, $u > 3$ is prime;
- (III) $L_2(2^m)$ and the following equations hold: $2^m - 1 = u$, $2^m + 1 = 3t$ where $m \geq 1$, u and t are prime numbers and $t > 3$;
- (IV) $L_2(3^m)$ and the following equations hold: $3^m + 1 = 4t$, $3^m - 1 = 2u$ where $m \geq 1$, u and t are odd primes.

Every group of order $p^a q^b$ is solvable, so the number of simple K_2 -group is zero. However, the number of simple K_3 -groups is 8. The following is a conjecture about the number of simple K_4 -groups.

Conjecture 5.2. *There are infinitely simple K_4 -groups.*

In particular, the author conjectured that there are infinite prime number r such that $\omega_o(L_2(r)) = 4$. However, it is difficult to prove this result because it is a special case of the unsolved Dickson conjecture (see [54]). In the following pairs of numbers

⁴Reference [220] contains conclusions without specific proofs, and there are omissions (such as $L_3(8)$ and $U_3(7)$) and repetitions (for instance $O_7(2)$ and $S_6(2)$ are the same group) in some places, furthermore, Suzuki simple groups can be completely determined.

$$(x, 3x - 2), \quad (x, 2x + 1), \quad (x, 4x + 1), \quad (x, 6x - 1),$$

$$(x, 6x + 1), \quad (x, 2x - 1), \quad (x, 4x - 1), \quad (x, 8x - 1),$$

if it can be proved that for prime number x such that there are infinitely many prime pairs appearing in any of the aforementioned pairs, then the number of simple K_4 -groups is infinite. Conversely, if for infinite prime numbers x , the second position of the above pairs can only appear finite prime numbers, can a contradiction be deduced from this?

Simple groups with $\omega_0(G) = 5$ and $\omega_0(G) = 6$ were studied in [97, 116].

The following results hold for the width $\omega_0(G)$ of order of G : $\omega_0(G) \leq |\pi_e''(G)| + 3$, and if $\omega_0(G) = |\pi_e''(G)| + 3$ then G is simple where $\pi_e''(G)$ denotes the set of composite numbers in the set $\pi_e(G)$ (see Theorem 2.14); $\omega_0(G) \leq |\iota(G)| + 4$, and if $\omega_0(G) = |\iota(G)| + 4$ then G is simple, where $\iota(G)$ denotes the set of pq ($p \neq q$) type numbers in $\pi_e(G)$ (see Theorem 2.16).

Similar to Conjecture 5.2, there also arise the following number theory problems from the classification theorem for finite simple groups:

Problem 5.3. *Are there infinitely simple groups whose order being a square number?*

In fact, the arising of the above problem is also originated from the classification of simple groups. In the historical process of researching the classification of simple groups, the work of studying simple groups whose order precisely contains a prime number to the first power played an important role. Brauer and Duan have made outstanding contributions to this topic. Brauer [25] pointed out that among simple groups with orders less than 10^9 , only one simple group, namely the order of the classical group $B_2(7)$, does not precisely contain a prime number to the first power. However, determining which simple groups do not have orders precisely containing a prime number to the first power is a challenging problem, and it seems difficult to determine whether the number of such simple groups is finite or infinite. Zhongmu Chen [44] proved the following theorem.

Theorem 5.4. *The order of a finite simple group G is not a k -th power, where $k \geq 3$. The necessary and sufficient condition for the order of a group G to be a perfect square is that G is a simple group $B_2(p)$ of Lie type, where p is a prime number satisfying*

$$p = 1 + \binom{2n+1}{2}2 + \binom{2n+1}{4}2^2 + \cdots + \binom{2n+1}{2n}2^n, n \geq 1.$$

The order of $B_2(p)$ is $p^4(p^2-1)(p^4-1)/2$. For example, if $n = 1$, then $p = 7$, and $|B_2(7)| = 2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$.

However, in the above expression for the summation, the number of n for which the sum is a prime number, remains an unresolved problem.

5.2. Finite groups with a small spectrum width. If we consider a group as a whole, its elements can be seen as local, forming an inseparable connection between the local and the global. Our consideration of the element orders is a way to understand the whole from a local perspective. As previously mentioned, for any finite group G and any element g in G , the order $|g|$ of g divides the order $|G|$ of G . During the author's graduate studies, the author considered such a question: the study of finite groups in which the orders of elements, excluding the identity element, are all prime numbers. This idea has received full approval from the author's advisor, Zhongmu Chen. Furthermore, this work was extended to study groups in which the orders of elements are prime powers, i.e., groups whose spectrum width is 1. These are the topics of the author's master's thesis and they were studied in [203, 204]. Now we say a group G is called prime power order group (briefly, EPPO-group) if $\omega_s(G) = 1$. Moreover, a group G is called prime element group (briefly, EPO-group) if its every nontrivial element has prime order.

After completing the master's thesis, the authors [204] found that G. Higman had previously studied such groups and proved the following theorem (see [91, Theorem 1]).

Theorem 5.5. *Let G be a soluble EPPO-group. Let p be the prime such that G has a normal p -subgroup greater than 1, and let P be the greatest normal p -subgroup of G . Then G/P is either (i) a cyclic group whose order is a power of a prime other than p ; or (ii) a generalized quaternion group, p being odd; or (iii) a group of order $p^a q^b$ with cyclic Sylow subgroups, q being a prime of the form $kp^a + 1$. Thus G has order divisible by at most two primes, and G/P is metabelian.*

For solvable EPPO-groups, the authors of [204] provided a more detailed structure of groups than Theorem 5.5.

Theorem 5.6. [204, Theorem 2.4] *Let G be a solvable EPPO-group such that $|\pi(G)| > 1$ and Q be a maximal normal q -subgroup of G for some $q \in \pi(G)$. Then*

- (1) *Suppose that G_2 is not a generalized quaternion group, where G_2 is a Sylow 2-subgroup of G . Then G/Q is a metacyclic group. Let $|G| = p^\alpha q^\beta$. Then G/Q is of order $p^\alpha q^\gamma$ with $q^\gamma | (p-1)$ and the chief series of G is as follows:*

$$\underbrace{q, \dots, q}_\gamma; \underbrace{p, \dots, p}_\alpha; q^{b_1}, \dots, q^{b_k}; \quad b | b_i, i = 1, \dots, k, \quad \gamma < b,$$

where $p^\alpha | (q^b - 1)$. If Q is the Sylow q -subgroup of G , then the chief series of G is as follows:

$$\underbrace{p, \dots, p}_\alpha; \underbrace{q^b, \dots, q^b}_k, \quad \beta = kb,$$

and the length of nilpotency class of Q is bounded by k .

- (2) *If the Sylow 2-subgroups of G are generalized quaternion groups, then G has the following chief series*

$$\underbrace{2, \dots, 2}_\alpha; q^{b_1}, \dots, q^{b_k},$$

where $b_i > 1$, $b|b_i$ for $i = 1, \dots, k$, where b is the exponent of $q \pmod{2^{\alpha-1}}$.

For nonsolvable EPPO-groups, based on the result of Suzuki [207], Brandl [22] and the authors of [204] obtained the following theorem (see [22, Theorem 3], the Mathieu group M_{10} is omitted, and [204, Theorem 3.1]).

Theorem 5.7. *Suppose that G is a nonsolvable EPPO-group. Then one of the following holds.*

- (1) G is isomorphic to $\text{PSL}_2(q)$ for $q = 5, 7, 9$ or $\text{PSL}_3(4)$ or M_{10} .
- (2) There exists a normal 2-subgroup T , T is an elementary abelian group such that G/T is isomorphic to
 - (i) $\text{PSL}_2(q)$ for $q = 5, 8, 17$. In this case, the class length of T is not greater than 2 and $|T| - 1$ is divisible by $3 \cdot 5$, $3^2 \cdot 7$, $3^2 \cdot 17$, respectively.
 - (ii) $\text{Sz}(2^3)$ with $5 \cdot 7 \cdot 13 \mid (|T| - 1)$.
 - (iii) $\text{Sz}(2^5)$ with $5^2 \cdot 31 \cdot 41 \mid (|T| - 1)$.

From the above Theorem 5.7 we can get the following result:

Corollary 5.8. *Let G be a finite group. Then $G \cong M_{10}$ if and only if $\pi_e(G) = \{1, 2, 3, 4, 5, 8\}$.*

About Theorem 5.7 the authors of [34] have summeraized in their paper all known results about non-solvable EPPO-groups in the form “if and only if” (see [34, Theorem 1.7]).

We also obtained the following interesting result (see [204, Theorem 2.1]):

Theorem 5.9. *A solvable EPPO-group G is an M -group. That is, every irreducible representation of G can be induced from a linear representation of a subgroup of G , i.e., a monomial representation.*

Note that the abstract group definition of an M -group is an unresolved question.

The following result was proved for EPPO-group (see [204, Theorem 1.4]):

Property 5.10. *Let G be an EPPO-group and H be a subgroup of G such that $(|H|, d) = 1$ for a natural number $d > 1$. Then $|H|$ divides the number of elements of order d in G .*

The following result shows that the aforementioned property is a characteristic property of EPPO-groups.

Theorem 5.11. [31] *Let G be a finite group and let $H < G$ such that $(|H|, d) = 1$ where $1 \neq d \in \pi_e(G)$. Then G is a EPPO-group if and only if $|H|$ divides the number of elements of order d in G .*

The study of finite EPPO-groups was extended to the infinite case in [89, 230].

Problem 5.12. *Study the structure of finite groups G with $\omega_s(G) = 2$ and $\omega_s(G) = 3$.*

T. M. Keller and A. Moretó [108] studied finite groups G with $\omega_s(G) = 2$, that is, there exists $pq \in \pi_e(G)$ such that $pqr \notin \pi_e(G)$ where p, q, r are different primes. For the special cases, i.e., solvable $\{p, q, r\}$ -groups, the following result holds (see [108, Theorem C]):

Theorem 5.13. *Let G be a solvable $\{p, q, r\}$ -group, p , q , and r divide $|G|$. If $pqr \notin \pi_e(G)$ (i. e., $\omega_s(G) \leq 2$), then $h(G) \leq 21$. Furthermore, if $|G|$ is odd, then $h(G) \leq 15$, where $h(G)$ is the Fitting height of G .*

Theorem C of [108] is a recently obtained result, it can be seen that study the structure of finite groups with $\omega_s(G) = 2$ is a rather challenging problem. Qian [171] has recently provided the structure of solvable groups whose spectrum width is 3. Note that, the classification of finite groups with $\omega_s(G) = 1$ (EPPO-group) was accomplished based on the work in [207]. The classification for simple groups might be used to determine the structure of finite groups satisfying $\omega_s(G) = 2$ and $\omega_s(G) = 3$.

5.3. Generalizations of related problems. Using spectrum to characterize the finite alternating group A_5 was a study conducted three to four decades ago. Here are several generalizations from this work.

“Spectrum” is the set of element orders, i.e., it is the set of orders of cyclic subgroups. We can consider the influence of sets of numbers on group structures, as mentioned after Conjecture 3.11: the set of orders of abelian subgroups, the set of orders of nilpotent subgroups, the set of orders of solvable subgroups (see [20, 49]), the set of orders of the normalizer of Sylow subgroups (see [19]), the set of indices of maximal subgroups (see [123]), etc. The order of element, the number of elements of the same order, the length of conjugacy classes, and the degree of character, and others, are conjugation invariants in G . Different sets of numbers can be constructed from these conjugation invariants.

“Characterization” can be “isomorphic” or “homomorphic”. From the perspective of the “function h ”, it can be $h(\Omega) = 1$ or $h(\Omega) = k$ where k is finite, otherwise, it is called unrecognizable.

“Finite simple group” can be a “simple group” or a nonsolvable group, such as the automorphism groups of some simple groups, it can also be the direct product of simple groups, for example $Sz(2^7) \times Sz(2^7)$ (see [141]) and $J_4 \times J_4$ (see [73]).

“Spectrum” represents the research conditions, “characterization” represents the research methodology, and “finite simple groups” represent the research subject. Many questions can be raised from various perspectives to conduct research on these three aspects.

5.4. The quantity relationship between the width of order and the width of spectrum.

In 1989, the author gave a lecture at The 3th National Conference on Algebra in China, and the main content was published in [185]. In this lecture, the author summarized the early research on characterizing simple groups by using the “order” and introduced Thompson Problem and Thompson Conjecture in [185] (see Sections 4.1 and 4.4 of this article). The author [185] posed the following question (see [185, Question 4.3]):

Given a positive integer k , denote by $|\pi(k)|$ the number of distinct prime divisors of k . Let G be a finite group, $n = \max\{|\pi(k)| \mid k \in \pi_e(G)\}$, does there exist a function f such that $|\pi(G)| \leq f(n)$? In other words, for any finite group, is the width of its order bound by the width of its spectrum?

If $\omega_s(G) = n = 1$, then G is a finite EPPO-group, from [204, Theorem 2.4, 3.1 and 3.2] we deduce that $\omega_0(G) = |\pi(G)| \leq 4$. T. M. Keller [105] proved that if G is solvable and $\omega_s(G) = 3$ then $\omega_0(G) \leq 8$.

For the above question, Jiping Zhang [237], C. Bellotti, T. M. Keller and T. S. Trudgian [16] proved that if G is solvable, then the width $\omega_0(G)$ of the order is bounded by a quadratic function and linear function of the spectrum width $\omega_s(G)$, respectively. That is,

$$\begin{aligned}\omega_0(G) &\leq \frac{\omega_s(G)(\omega_s(G) + 3)}{2} \text{ (see [237])}, \\ \omega_0(G) &\leq 5\omega_s(G) \text{ (see [16])}.\end{aligned}$$

For general group, $\omega_0(G)$ is bounded by a super-exponential function of $\omega_s(G)$. The above result was subsequently improved in the literature [106, 107, 161]. Recently, the following theorem was proved in [92]:

Theorem 5.14. *Let G be a finite group, then $\omega_0(G) < 210\omega_s(G)^4$.*

An analogous result regarding the order of a group and its co-degrees can be found in [231].

It is easy to get that $\omega_s(G) \leq \omega_0(G)$ by the definitions of the order width and the spectrum width. The following result holds for all finite simple groups:

Theorem 5.15. [198] *Let G be a finite simple group, then $\omega_s(G) < \omega_0(G)$.*

Naturally, the author of this article put forward a question to investigate the difference between two widths, denoted as $d = \omega_0(G) - \omega_s(G)$. For example, discuss the classification of finite simple groups with $d = 1$.

6. Group, quantity, and graph

6.1. Group and quantity. Groups and their corresponding numerical quantities have a strong connection. The order of a group can reveal important properties about it. For example, groups of order 15 are cyclic, groups of order p^2 for some prime p are abelian, and groups of order 135 are nilpotent. Additionally, many well-known results regarding the solvability of groups have been established throughout the history of group theory. This includes the Feit-Thompson odd order theorem (see [58]) and Thompson's classification theorem for minimal simple groups (see [209]). From these results, we can derive the following theorem:

Theorem 6.1. *Let G be a finite group. If $(|G|, 2) = 1$ or $(|G|, 15) = 1$, then G is solvable.*

The numbers 2 and 15 mentioned in the theorem above are referred to as “solvable coprime numbers” in lecture [83]. This means that a group whose order is coprime to 2 or 15 is necessarily a solvable group. Additionally, it has been established in [83] that all solvable coprime numbers must be multiples of either 2 or 15. In contrast to the study of finite groups based on the order of the group, we characterize finite simple groups using the set $\pi_e(G)$, which consists of the orders of the elements. Moreover, the set $\pi_e(G)$ can also be utilized to determine the solvability of the group G .

Definition 6.2. Let G be a finite group, and let $\pi_e(G)$ denote the set of orders of elements in G . The set T of numbers is called the set of “intersection empty solvability” if $\pi_e(G) \cap T = \emptyset$ implies that G is a solvable group.

Theorem 6.3. [197] Let G be a finite group, and let $\pi_e(G)$ denote the set of orders of elements in G . If $\pi_e(G) \cap T = \emptyset$, where $T = \{2\}$, $\{3, 4\}$ or $\{3, 5\}$, then G is solvable. Furthermore, when using the set T for determining the solvability of G , the only relevant cases are those three: $T = \{2\}$, $\{3, 4\}$, or $\{3, 5\}$.

Problem 6.4. Let G be a finite group. Can we use $\pi_e(G)$ to establish sufficient conditions for G to be cyclic, abelian, nilpotent, supersolvable, or an M -group, etc.?

In addition to the quantity problem concerning the “order of a group,” similar issues can arise when considering “element orders.” For instance, Lagrange’s theorem states that in a finite group G , the order of every subgroup of G divides the order of G . However, it is possible that a subgroup of a particular order does not exist for every divisor of $|G|$. In other words, the converse of Lagrange’s theorem is not necessarily true.

Definition 6.5. A finite group G is called CLT group if and only if the converse of Lagrange’s theorem holds for G .

Definition 6.6. A subset M of a set of positive integers is called a closed subset if, for all $k \in M$, all divisors of k also belong to M .

Let G be a group, and let $\pi_e(G)$ denote the set of element orders in G . Not all closed subsets of $\pi_e(G)$ can correspond to the set of element orders of a subgroup of G .

In comparison to the CLT group, we introduce the following definition:

Definition 6.7. A finite group G is called a COE group if for every closed subset M of $\pi_e(G)$, there exists a subgroup H such that $\pi_e(H) = M$.

Theorem 6.8. [193] Suppose G is a COE group. Then, $|\pi(G)| \leq 3$, and one of the following conditions holds:

- (1) G is a p -group;
- (2) $|G| = p^a q^b$, where p and q are distinct primes, a and b are both non-zero, and if the exponent of G is $p^m q^n$, then G contains an EPPO-subgroup with this same exponent.
- (3) $|G| = 2^a 3^b 5^c$, where a , b , and c are all non-zero, and the exponent of G is either 30, 60 or 120. Additionally, G has a proper subgroup that is isomorphic to A_5 , and G is not an EPPO-group.

6.2. Groups and graphs are closely interconnected. Gruenberg and Kegel⁵ were the first to introduce the definition of the prime graph $\Gamma(G)$ of a group G (see Definition 2.18) and its classification.

⁵K. W. Gruenberg, O. H. Kegel. Unpublished manuscript, 1975

Following that, the literature [94, 114, 117, 222] provided explicit descriptions of the connected components of the prime graphs of finite simple groups. M. S. Lucido [127] examined the diameter of the prime graph, as follows:

Definition 6.9. *The diameter of the prime graph of a group G is defined as follows:*

$$\text{diam}(\Gamma(G)) = \max\{d(p, q) \mid p, q \text{ in the same connected component of } \Gamma(G)\}.$$

In this definition, $d(p, q)$ represents the distance between the elements p and q when both are located in the same connected component of $\Gamma(G)$.

If G is a finite solvable group, then $\text{diam}(\Gamma(G)) \leq 3$. Lucido [127] demonstrated that for all finite groups G , the diameter $\text{diam}(\Gamma(G))$ is at most 5 and provided a characterization of almost simple groups for which $\text{diam}(\Gamma(G)) = 5$. An almost simple group B is defined as one for which there exists a nonabelian simple group A such that $A \leq B \leq \text{Aut}(A)$.

In [72], finite groups with $\text{diam}(\Gamma(G)) = 5$ were found. Note that $\text{diam}(\Gamma(G)) = 0$ if and only if G is an EPPO-group (see [204]). Therefore, we pose the following question:

Problem 6.10. *Examine finite groups G for which the diameter $\text{diam}(\Gamma(G))$, is equal to 1, 2, 3, or 4.*

In the study of solvable groups, the literature [108, 171] investigated finite solvable groups G with $\text{diam}(\Gamma(G)) = 2$ and $\text{diam}(\Gamma(G)) = 3$, respectively.

Additionally, Lucido [128] examined the case in which a prime graph forms a tree, based on the work in reference [127]. A tree is defined as a connected graph without loops. The following result was obtained:

Theorem 6.11. *If the prime graph of a finite group G is a tree, then $|\pi(G)| \leq 8$.*

Lucido [128] also provided an example where $|\pi(G)| = 8$.

Compared to the prime graph $\Gamma(G)$ of G , the definition of solvable graph was introduced in [4] as follows:

Definition 6.12. *Let G be a finite group. The solvable graph $\Gamma_s(G)$ of G is defined as follows: its vertices are the primes dividing the order of G , and two distinct vertices p and q are joined by an edge if and only if there exists a solvable subgroup of G whose order is divisible by pq .*

M. Hagie [81] proved that $\text{diam}(\Gamma_s(G)) \leq 4$.

There is a lot of work in study groups using graphs, P. Erdős defined non-commuting graph in 1976 (see [165]).

Definition 6.13. *Let G be a finite group and let $Z(G)$ be the center of G . The non-commuting graph $\nabla(G)$ of G is defined as follows: Take $G \setminus Z(G)$ as the vertex set of $\nabla(G)$ and join two distinct vertices x and y by an edge if and only if $xy \neq yx$.*

The following problem was conjectured in [1, 157]: *Let G and H be two nonabelian finite groups such that $\nabla(G) = \nabla(H)$. Then $|G| = |H|$.* A. R. Moghaddamfar [154] gave some counterexamples to this conjecture. Moreover, the following conjecture was posed in [1], which is called AAM (Abdollahi-Akbari-Maimani) conjecture:

Conjecture 6.14. *Let S be a nonabelian simple group and G a group. If $\nabla(G) = \nabla(S)$, then $G \cong S$.*

It was proved in [219, 241] that for the alternating simple group A_{10} and the simple group $L_4(4)$, whose prime graphs are connected, Conjecture 6.14 is valid. In addition, the following result was also proved in [219]:

Proposition 6.15. *Let S be a nonabelian simple group and let G be a group with $Z(G) = 1$. If $\nabla(G) = \nabla(S)$, then $N(G) = N(S)$.*

Because Thompson Conjecture has been proved (see sections 4.1 and 4.2), consequently, AAM Conjecture is correct.

Definition 6.16. *Let G be a finite group and $|G| = p^a q^b \cdots r^c$, where $p < q < \cdots < r$ are primes, and $a, b, \dots, c$ are positive integers. For $s \in \pi(G)$, let $\deg(s) = |\{t \in \pi(G) \mid t \sim s\}|$, which we call the degree of s in $\Gamma(G)$. We also define $D(G) = (\deg(p), \deg(q), \dots, \deg(r))$, which is called the degree pattern of G . A group M is called k -fold OD-characterizable if there exist exactly k non-isomorphic groups G such that $|G| = |M|$ and $D(G) = D(M)$. Moreover, a 1-fold OD-characterizable group is simply called an OD-characterizable group.*

An extensive research has been conducted by A. R. Moghaddamfar, Liangcai Zhang, et al., on OD-characterizable (k -fold OD-characterizable) simple groups (almost simple groups) (see [12, 158–160, 239, 240]).

6.3. The adjacency criterion for the vertices of the prime graph of a finite simple group.

Both the characterization of finite simple groups using the set of element orders and the aforementioned OD-characterization of finite simple groups are inseparable from the study of the prime graph of groups (see Definition 2.18). Consequently, examining the adjacency of the vertices in prime graphs is a crucial area of research. The literature [217, 218] provides criteria for determining the adjacency of the vertices in the prime graphs associated with each nonabelian simple group. This is particularly relevant for understanding vertex adjacency to the vertex “2” and for analyzing “independent sets,” where vertices are not adjacent to one another. Additionally, references [217, 218] contain numerous graphs and detailed information for all simple groups presented in tabular form. Reference [218] serves as a continuation of reference [217] and includes corrections to some of the content in the earlier reference.

6.4. Unrecognizable finite group by prime graph. Finite simple groups such as $E_6(2)$, $E_6(3)$, and ${}^2E_6(3)$ have been characterized by their prime graph $\Gamma(G)$ as discussed in [79, 111]. However, there are many simple groups that cannot be characterized only using prime graphs. To address this, P. J. Cameron and N. V. Maslova [34] explored criteria for finite groups that do not possess unique characterizations based on their prime graphs. They aimed to investigate the following questions:

- (1) Which groups G are uniquely determined by their prime graph $\Gamma(G)$?
- (2) For which groups are there only finitely many other groups that share the same prime graph $\Gamma(G)$ as G ?
- (3) Which groups G are uniquely determined by the isomorphism type of their prime graph?

In line with the previous discussion regarding the function h (see Definition 2.5), the authors in [34] provided the following definition (see [34], page 188).

Definition 6.17. *A group G is characterized based on its prime graph in the following ways:*

- Recognizable (by prime graph): *A group G is said to be recognizable if, for every group H , the condition $\Gamma(H) = \Gamma(G)$ implies that $G \cong H$.*
- k -recognizable (by prime graph): *A group G is k -recognizable if there are exactly k pairwise non-isomorphic groups that share the same prime graph as G , where k is a positive integer.*
- Almost recognizable (by prime graph): *A group G is almost recognizable if it is k -recognizable (by prime graph) for some positive integer k .*
- Unrecognizable (by prime graph): *A group G is unrecognizable if there are infinitely many pairwise non-isomorphic groups that share the same prime graph as G .*

The following theorem can be found in [137].

Theorem 6.18. *Every finite simple exceptional group of Lie type that is not isomorphic to either ${}^2B_2(2^{2n+1})$ (for $n \geq 1$) or $G_2(3)$, and has a prime graph with at least three connected components, is almost recognizable by prime graph. Moreover, the groups ${}^2B_2(2^{2n+1})$ (where $n \geq 1$) and $G_2(3)$ are unrecognizable by prime graph.*

There are differences between having the same prime graph and possessing isomorphic prime graphs. For instance, $\Gamma(A_{10}) \neq \Gamma(\text{Aut}(J_2))$, but $\Gamma(A_{10})$ and $\Gamma(\text{Aut}(J_2))$ are isomorphic as abstract graphs.

Problem 6.19. *Which simple groups can be characterized by their prime graphs and orders?*

The question above generalizes the research that characterizes all simple groups using “two orders”. Consider the classical groups $B_n(q)$ and $C_n(q)$ (see [195]). For $n \geq 3$ and when q is odd, these groups share the same prime graphs and orders. Consequently, not all simple groups can be uniquely characterized by their prime graphs and orders.

6.5. Graphs defined on groups. There are various types of graphs defined on a group, but the one most closely associated with quantitative characterization is known as the prime graph (or GK graph). Cameron [32] introduced several types of graphs, where the vertex sets correspond to finite groups G . The edges of these graphs reflect the underlying structure of the groups in specific ways, such that the automorphism group of G relates to the automorphisms of the graph. These graphs include:

1. *Commuting Graph*: Two vertices x and y are joined if and only if $xy = yx$. This graph was first introduced in 1955 [26].
2. *Generating Graph*: Two vertices x and y are joined if and only if $\langle x, y \rangle = G$. The generating graph was first introduced in 1996 [27, 125].
3. *Power Graph*: Two vertices x and y are joined if and only if $\langle x \rangle \subseteq \langle y \rangle$ or $\langle y \rangle \subseteq \langle x \rangle$. This graph was first studied in 2000 [104, 118, 156].
4. *Enhanced Power Graph*: Two vertices x and y are joined if the group generated by x and y , denoted $\langle x, y \rangle$, is not cyclic. This graph was first studied in 2007 [2, 3].
5. *Deep Commuting Graph*: Two vertices x and y are joined if their pre-images, $\bar{x}$ and $\bar{y}$, commute in every central extension of G . In other words, for each group H with a central subgroup Z such that $H/Z \cong G$, $\bar{x}$ and $\bar{y}$ must commute in H [33].

These graphs serve as important tools for understanding the relationships and structures within finite groups.

This paper mainly discusses on the quantitative characterization of finite simple groups, which is an essential component of quantitative properties of groups and a significant topic in group theory research. The author was the first to propose and investigate the study of finite groups using the set of element orders, specifically characterizing finite simple groups by “two orders”. This approach has greatly influenced the field of group theory both nationally and internationally.

Since the 1980s, the author has successively worked at Southwest Normal University (including a concurrent doctoral supervisor position at Sichuan University), Suzhou University, and Chongqing University of Arts and Sciences. The author has also received funding for 11 projects from the National Natural Science Foundation of China, which has facilitated ongoing progress in this area of research.

Our intention in writing this article is not only to systematically summarize our work, distill new viewpoints, ideas, and methods to solve related problems, but also to further delve into this work and find more significant applications.

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Wujie Shi

School of Mathematical Sciences, Suzhou University Suzhou, Jiangsu, P. R. China

and

School of Mathematics and Big Data, Chongqing University of Arts and Sciences, Chongqing, P. R. China

Email: wjshi@suda.edu.cn

Email: shiwujie@outlook.com