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GROUPS WHOSE SAME-ORDER TYPES ARE ARITHMETIC PROGRESSIONS

RULIN SHEN* AND YUTAO JIN

Abstract. For any group G, define $g \sim h$ if $g, h \in G$ have the same order. The set of sizes of the equivalent classes with respect to this relation is called the same-order type of G. In this short note we prove that there is no finite group whose same-order type is an arithmetic progression of length 4. This answered an open problem posed by Lazorec and Tărnăuceanu.

1. Introduction

The size of the set of elements of the same order is an important number in finite groups theory. It plays an important role in the structure of finite group. Let G be a finite group, and denote the same-order type of G by $\tau_e(G)$, as defined in [1]. Lazorec and Tărnăuceanu [2] proved that if G is a finite group such that its same-order type $\tau_e(G)$ is an arithmetic progression, then $|\tau_e(G)| \leqslant 4$. Also, they provided some results concerning the classification and existence of finite groups whose same-order types are arithmetic progressions formed of 3 or 4 elements. Moreover, Lazorec and Tărnăuceanu showed that if G is a nilpotent group such that $|\tau_e(G)| = 4$ then $\tau_e(G)$ is not an arithmetic progression. At the same article, they posed the following open problem:

Open Problem 1.1. *Is there any finite non-nilpotent group whose same-order type is an arithmetic progression of length* 4*?*

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*Corresponding author.

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In this paper we give an answer of it as follows.

Theorem 1.2. *Let G be a finite group such that $|\tau_e(G)| = 4$. Then $\tau_e(G)$ is not an arithmetic progression.*

In [2], Lazorec and Tărnăuceanu also proved that if G is not a finite 2-group having more than one cyclic subgroup of order 2 and $|\tau_e(G)| = 3$, then $\tau_e(G)$ is an arithmetic progression if and only if $G \cong S_3$. Also, in [1], Shen proved if $|\tau_e(G)| \leq 2$ then G is nilpotent. We have the following corollary.

Corollary 1.3. *Let G be a finite non-nilpotent group such that $\tau_e(G)$ is an arithmetic progression. Then G is isomorphic to S_3.*

2. Some Lemmas

In this section, we give some lemmas. Let $\pi(G)$ be the set of prime divisors of $|G|$. We denote by s_n and c_n the number of elements of order n and the number of cyclic subgroups of order n of G, respectively. Also, Euler's totient function is denoted by ϕ, and $f(n) = \sum_{m|n} s_m$.

Lemma 2.1. [3, Section 2] *Let G be a finite group and n be a positive integer dividing $|G|$. Then $n|f(n)$.*

Lemma 2.2. [1, Lemma 1.3] *Let G be a finite group and k be a positive integer. If s_k is finite, then $\phi(k)|s_k$.*

Lemma 2.3. [4, Theorem 3] *Let G be a finite group n be a positive integer. Then the number of elements whose orders are multiples of n is either zero, or a multiple of the greatest divisor of $|G|$ that is prime to n.*

Lemma 2.4. [2, Lemma 2.4] *Let G be a finite group of odd order such that $|\tau_e(G)| = 4$. Then $\tau_e(G)$ is not an arithmetic progression.*

Lemma 2.5. [2, Theorem 2.6] *Let G be a finite nilpotent group such that $|\tau_e(G)| = 4$. Then $\tau_e(G)$ is not an arithmetic progression.*

3. Proof of Theorem 1.2

Lazorec and Tărnăuceanu [2] proved that the same-order type of odd order group and of a nilpotent group, respectively, cannot form an arithmetic progression if the length of its same-order type is 4 (See Lemmas 2.4 and 2.5). Before proving Theorem 1.2, we first establish the following Claim, which plays a crucial role in the proof of Theorem 1.2.

Claim 3.1. *Let G be a finite group such that $\tau_e(G)$ an arithmetic progression and $|\tau_e(G)| = 4$. Then $|\pi(G)| \leqslant 3$. In particular, if the numbers of elements of all odd prime divisors of $|G|$ are equal, then $|\pi(G)| \leqslant 2$.*

Proof. In the light of Lemma 2.4, $|G|$ is even. It follows that $s_2 \neq 0$. Moreover, $s_2 \neq 1$. Otherwise, all numbers that more than 1 in $\tau_e(G)$ are even. Then $\tau_e(G)$ is an arithmetic progression implies its common difference is 1 or an even number, and so it forces to $|\tau_e(G)| = 2$, which contradicts our hypothesis $|\tau_e(G)| = 4$. By Lemma 2.5, we can assume that G is non-nilpotent. We prove the result by contradiction. Assume $|\pi(G)| \geqslant 4$. Let $\pi(G) = \{2, p_1, p_2, \ldots, p_k\}$, where $k \geqslant 3$ and p_i is odd for $i = 1, 2, \ldots, k$. Since there are only two odd integers s_1 and s_2 in $\tau_e(G)$ and $\tau_e(G)$ is an arithmetic progression formed 4 elements, it implies that $s_2 \neq 1$. Write $\tau_e(G) = \{1, m, s_2, n\}$. Since 1 and s_2 are odd, each of $s_{p_1}, s_{p_2}, \ldots, s_{p_k}$ is either m or n.

If not all $s_{p_1}, s_{p_2}, \ldots, s_{p_k}$ are equal, then we may assume that $s_{p_1} = s_{p_2} = m$ and $s_{p_3} = n$. Thus $f(p_1) = s_1 + s_{p_1} = 1 + m$. Lemma 2.1 implies that $p_1 | f(p_1)$, thus $m \equiv -1 (\mathrm{mod}\ \ p_1)$. Moreover p_1 divides $f(p_1p_2)$ by Lemma 2.1. Therefore $f(p_1p_2) = 1 + 2m + s_{p_1p_2}$ is divisible by p_1. It follows that $s_{p_1p_2} \equiv 1 (\mathrm{mod}\ \ p_1)$. Thus $s_{p_1p_2}$ must be equal to n. Since $p_1 | f(p_1p_3)$, p_1 divides $1 + m + n + s_{p_1p_3}$. With the help of the fact that $m \equiv -1 (\mathrm{mod}\ \ p_1)$ and $n \equiv 1 (\mathrm{mod}\ \ p_1)$ we have $s_{p_1p_3} \equiv -1 (\mathrm{mod}\ \ p_1)$. Therefore, $s_{p_1p_3} = m$. Similarly, we can get $s_{p_2p_3} = m$. Note that p_3 divides both $f(p_1p_3)$ and $f(p_3)$, so $1 + 2m + n$ and $1 + n$ are divisible by p_3, implying that p_3 divides $2m$. Since p_3 is odd, we can deduce that p_3 divides m. Observe that $f(p_1p_2p_3) = 1 + 4m + 2n + s_{p_1p_2p_3}$ and it is divisible by p_1. We have $s_{p_1p_2p_3} \equiv 1 (\mathrm{mod}\ \ p_1)$, so that $s_{p_1p_2p_3} = n$. On the other hand,

$$f(p_1p_2p_3) = 1 + 4m + 2n + s_{p_1p_2p_3} = 1 + 4m + 3n$$

is also divisible by p_3 and $p_3 | m$. Hence $3n \equiv -1 (\mathrm{mod}\ \ p_3)$, contradicting our assumption that $n \equiv -1 (\mathrm{mod}\ \ p_3)$. Therefore, $s_{p_3} = 0$. Hence $k \leqslant 2$, and then $|\pi(G)| \leqslant 3$.

Next suppose that $s_{p_1}, s_{p_2}, \ldots, s_{p_k}$ are equal to each other and $\tau_e(G) = \{1, 1 + r, 1 + 2r, 1 + 3r\}$ is an arithmetic progression with a common difference of r. Obviously, r must be odd and $s_2 = 1 + 2r$ because $\tau_e(G)$ has exactly two odd integers. We claim that $k \leqslant 1$. If this was not the case, then $s_{p_1} = s_{p_2} = \cdots = s_{p_k} \neq 0$. Assume $s_{p_1} = s_{p_2} = \cdots = s_{p_k} = 1 + r$. Since $f(p_1) = 2 + r$ and $f(p_1p_2) = 3 + 2r + s_{p_1p_2}$, by Lemma 2.1, we have $2 + r$ and $3 + 2r + s_{p_1p_2}$ are divisible by p_1. If $s_{p_1p_2} = 0$, then $p_1 | (3 + 2r)$. Together $p_1 | (2 + r)$, these imply that $p_1 = 1$. If $s_{p_1p_2} = 1 + r$, then $p_1 | (4 + 3r)$, and so $p_1 = 2$. These cases are impossible. Thus, $s_{p_1p_2} = 1 + 3r$, which implies that $f(p_1p_2) = 4 + 5r$ and $p_1 | (4 + 5r)$. Since $p_1 | (2 + r)$, we have $p_1 | 6$. Thus $p_1 = 3$ because p_1 is odd. Note that $f(p_2) = 2 + r$ and $f(p_1p_2) = 4 + 5r$ are divisible by p_2, so $p_2 = 3$, contradicting the fact that $p_1 \neq p_2$. Therefore, $s_{p_2} = 0$. Thus $k \leqslant 1$ and hence $|\pi(G)| \leqslant 2$. Now, we assume that $s_{p_1} = s_{p_2} = \cdots = s_{p_k} = 1 + 3r$. Note that $f(p_1) = 2 + 3r$ and $f(p_1p_2) = 3 + 6r + s_{p_1p_2}$. With the help of Lemma 2.1, we have p_1 divides both $f(p_1)$ and $f(p_1p_2)$. If $s_{p_1p_2} = 0$ then $p_1 | (3 + 6r)$. It follows that

$p_1 = 1$. If $s_{p_1p_2} = 1 + r$, then $p_1|(4 + 7r)$, and hence $p_1 = 1$ since p_1 is odd. Finally, if $s_{p_1p_2} = 1 + 3r$, then $p_1|(4 + 9r)$ and this also leads to $p_1 = 1$. These cases are impossible. Thus $s_{p_2} = 0$, and then $k \leqslant 1$ and $|\pi(G)| \leqslant 2$. □

In the sequel, we will prove Theorem 1.2. Assume, by contradiction, that G is a finite group such that $\tau_e(G)$ is an arithmetic progressions of length 4. In the light of Lemma 2.4, the order of G is even and this implies that $s_2 \neq 0$. Let $\tau_e(G) = \{1, 1 + r, 1 + 2r, 1 + 3r\}$ be an arithmetic progression with a common difference of r. As explained above, $s_2 = 1 + 2r$ and r is odd. Since $\tau_e(G)$ is an arithmetic progression with $|\tau_e(G)| = 4$, $|\pi(G)| \leqslant 3$ by Claim 3.1. Since G is non-nilpotent by Lemma 2.5, we also have $|\pi(G)| \geq 2$. Hence $|\pi(G)| \in \{2, 3\}$ and we can divide our proof in two cases.

Case 1. Assume $\pi(G) = \{2, p, q\}$, where p and q are different odd prime numbers. Note that $s_p \neq s_q$ because, otherwise, $|\pi(G)| \leqslant 2$ by Claim 3.1. It would follow that $p = q$, a contradiction. Now we may assume that $s_p = 1 + r$ and $s_q = 1 + 3r$. Since $f(pq) = 1 + s_p + s_q + s_{pq}$ is divisible by q, we have $q|(3 + 4r + s_{pq})$. Note that $s_{pq} \neq 0$ because, otherwise, $q|(3 + 4r)$, and since $f(q) = 2 + 3r$ and $q|f(q)$, we get $q = 1$, a contradiction. If $s_{pq} \in \{1 + r, 1 + 3r\}$, then by the facts that $q|(3 + 4r + s_{pq})$ and $q|(2 + 3r)$ we can deduce that $q = 1$. These cases are impossible.

Case 2. Assume $\pi(G) = \{2, p\}$ where p is odd. We denote by $\pi_e(G)$ the set of element orders of G. Set the exponent of G as $2^{i_1}p^{i_2}$, with $i_1, i_2 \geqslant 0$. Then $\pi_e(G) \subseteq \{1, 2, \ldots, 2^{i_1}, p, \ldots, p^{i_2}, 2p, \ldots, 2^{i_1}p^{i_2}\}$.

We first suppose that $s_p = 1 + r$. Then $p|f(p)$ implies that $p|(2 + r)$. Moreover, $f(2p) = s_1 + s_2 + s_p + s_{2p} = 3 + 3r + s_{2p}$ and it is divisible by p. If $s_{2p} = 1 + r$, then $p|(4 + 4r)$, and this lead us to $p|4$, a contradiction. similarly, if $s_{2p} = 1 + 3r$, then we deduce that $p|8$, a contradiction. Therefore, $s_{2p} = 0$. Observe that p divides both $f(2p) = s_1 + s_2 + s_p$ and $f(p) = s_1 + s_p$. Thus $p|s_2$, so $p|(1 + 2r)$. Since $p|(2 + r)$, $p = 3$. Now, we will prove that there is no element of order p^2 in G. If $s_{p^2} = 1 + r$, then $f(p^2) = s_1 + s_p + s_{p^2} = 3 + 2r$ and since $p|f(p^2)$, we have $p|(3 + 2r)$. It follows that $p = 1$ (because p divides $2 + r$), a contradiction. If $s_{p^2} = 1 + 3r$, then $f(p^2) = 3 + 4r$ is divisible by p. Since $p|(2 + r)$, we have $p = 5$, which contradicts the fact that $p = 3$. Therefore, $s_{p^2} = 0$. Hence, $\pi_e(G) \subseteq \{1, p, 2, 4, \ldots, 2^i\}$ where $i \geqslant 2$ (because $|\tau_e(G)| = 4$). By using Lemma 2.3, we have

$$\begin{cases} p|s_2 + s_4 + s_8 + \cdots + s_{2^i} \\ p|s_4 + s_8 + \cdots + s_{2^i} \\ \vdots \\ p|s_{2^i}. \end{cases}$$

Thus, $s_2, s_4, \ldots, s_{2^i}$ are divisible by p. Since $\tau_e(G) = \{1, 1+r, 1+2r, 1+3r\}$, $s_2 = 1+2r$ and $s_p = 1+r$, at least one of s_4, s_8,…,s_{2^i} equals $1 + 3r$. Therefore, $p|(1 + 3r)$. In particular, since $s_p = 1 + r$, we have $p|(2 + r)$. Therefore, $p = 5$, which contradicts the fact that $p = 3$.

We next assume that $s_p = 1 + 3r$. Then $f(p) = 2 + 3r$ and, by Lemma 2.1, we have $p|(2 + 3r)$. Observe that $f(p^2) = 2 + 3r + s_{p^2}$ and $p|f(p^2)$. If $s_{p^2} = 1 + r$ then $p|(3 + 4r)$. It follows that $p = 1$, a contradiction. Suppose that $s_{p^2} = 1 + 3r$. Then $f(p^2) = 3 + 6r$ and $f(p) = 2 + 3r$ are both divisible by p. Thus $p = 1$, a contradiction. Therefore, $s_{p^2} = 0$. Now we consider the number of elements of order 4 and order $4p$ in G. Since

$$f(2p) = 1 + s_2 + s_p + s_{2p},$$
$$f(4p) = 1 + s_2 + s_4 + s_p + s_{2p} + s_{4p},$$

and p divides both $f(4p)$ and $f(2p)$, we have $p|(s_4 + s_{4p})$. Moreover, since $s_4, s_{4p} \in \{0, 1 + r, 1 + 3r\}$, we have

$$s_4 + s_{4p} \in \{0, 1 + r, 1 + 3r, 2 + 2r, 2 + 4r, 2 + 6r\}.$$

Thus p divides one of the numbers of the following set

$$\{0, 1 + r, 1 + 3r, 2 + 2r, 2 + 4r, 2 + 6r\}.$$

Recall that $p|(2 + 3r)$, it is easy to check that $s_4 + s_{4p} = 0$ based on the facts that $p|(2 + 3r)$ and $p|s_4 + s_{4p}$. The previous equality leads to $s_4 = s_{4p} = 0$. This implies that $s_{2^i} = s_{2^i p} = 0$ for $i \geqslant 2$. Therefore,

$$\tau_e(G) = \{1, s_2, s_p, s_{2p}\}.$$

It follows that $|G| = 1 + s_2 + s_p + s_{2p} = 4 + 6r$. Since r is odd, $|G|$ is not divisible by 4. So we can write $|G| = 2p^t$ for $t \geqslant 1$. Let N be a Sylow p-subgroup of G. Clearly, $N \trianglelefteq G$ since $|G : N| = 2$. Thus $G \cong N \rtimes Z_2$. Let z be an involution of G. Note that $\langle z\rangle$ is a Sylow 2-subgroup of G. The Sylow Theorem implies that all involutions of G are conjugate, thus

$$s_2 = \frac{|G|}{|C_G(z)|} = \frac{|N \rtimes Z_2|}{|C_N(z) \rtimes Z_2|} = \frac{|N|}{|C_N(z)|},$$

and since $s_2 = 1 + 2r$, we have $\frac{|N|}{|C_N(z)|} = 1 + 2r$. Assume $|C_N(z)| \neq p^t$. Then $\frac{|N|}{|C_N(z)|}$ is a power of p, hence $p|(1 + 2r)$. Recall that $p|(2 + 3r)$, so $p = 1$, a contradiction. Thus $|C_N(z)| = p^t$, and then $1 + 2r = 1$, implying that $r = 0$ and $|\tau_e(G)| = 1$, contradicting the fact that $|\tau_e(G)| = 4$. □

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Rulin Shen

Department of Mathematics, Hubei Minzu University, Enshi, Hubei, China

Email: shenrulin@hotmail.com

Yutao Jin

Department of Mathematics, Hubei Minzu University, Enshi, Hubei, China

Email: 282955964@qq.com

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