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p-GROUPS WITH A SMALL NUMBER OF CHARACTER DEGREES AND THEIR NORMAL SUBGROUPS

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ABSTRACT. If G be a finite p-group and χ is a non-linear irreducible character of G, then $\chi(1) \leq |G/Z(G)|^{\frac{1}{2}}$. In [2], Fernández-Alcober and Moretó obtained the relation between the character degree set of a finite p-group G and its normal subgroups depending on whether $|G/Z(G)|$ is a square or not. In this paper we investigate the finite p-group G where for any normal subgroup N of G with $G' \nleq N$ either $N \leq Z(G)$ or $|NZ(G)/Z(G)| \leq p$ and obtain some alternate characterizations of such groups. We find that if G is a finite p-group with $|G/Z(G)| = p^{2n+1}$ and G satisfies the condition that for any normal subgroup N of G either $G' \nleq N$ or $N \leq Z(G)$, then $cd(G) = \{1, p^n\}$. We also find that if G is a finite p-group with nilpotency class not equal to 3 and $|G/Z(G)| = p^{2n}$ and G satisfies the condition that for any normal subgroup N of G either $G' \nleq N$ or $|NZ(G)/Z(G)| \leq p$, then $cd(G) \subseteq \{1, p^{n-1}, p^n\}$.

1. Introduction

In this paper, all groups are finite. By $\mathrm{Irr}(G)$ and $\mathrm{nl}(G)$ we denote the set of complex irreducible characters and the set of complex non-linear irreducible characters of the group G respectively. By $c(G)$ we denote nilpotency class of G. The characterizations of a finite group G from the set $cd(G)$ of the degrees of its complex irreducible characters have been done in many papers. For example groups with two character degrees have been characterized in [7, Chapter 12] and [6, Chapter 27].

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It can be found in [6, Lemma 7.4] that for any $\chi \in \mathrm{Irr}(G)$ we must get that $\chi(1)^2 \leq |G/Z(G)|$. A group for which this bound is attained is called a group of central type. It has also been shown in [7] that for any $\chi \in \mathrm{Irr}(G)$, $\chi(1)^2 \leq |G/Z(\chi)|$ and if $G/Z(\chi)$ is Abelian then $\chi(1)^2 = |G/Z(\chi)|$. The relation between the set of the degrees of its complex irreducible characters of a group and its normal subgroups have been studied in [2]. We begin with the following definitions.

Definition 1.1. [2] *A p-group is said to satisfy the strong condition on normal subgroups if for any $N \trianglelefteq G$ either $G' \leq N$ or $N \leq Z(G)$.*

Definition 1.2. [2] *A p-group is said to satisfy the weak condition on normal subgroups if for any $N \trianglelefteq G$ either $G' \leq N$ or $|NZ(G)/Z(G)| \leq p$.*

In [2] the authors proved that for any p-group G, if G satisfies the strong condition on normal subgroups then $c(G) \leq 3$ and if G satisfies the weak condition on normal subgroups then $c(G) \leq 4$. Moreover in [2], the character degree sets have been obtained for the p-groups satisfying the strong condition and weak condition based on the value of $|G/Z(G)|$. In this paper we obtain some alternative characterizations of a p-group G if G satisfies the strong condition on normal subgroups and the weak condition on normal subgroups. We prove the following theorems.

Theorem 1.3. *Let G be a p-group of nilpotency class 2 with two character degrees and $|G/Z(G)| = p^{2n}$. Then the following conditions are equivalent:*

(i) exp $G' = p$ and $Z(\chi)/Z(G)$ is cyclic for all $\chi \in nl(G)$.
(ii) For every normal subgroup N of G, either $G' \leq N$ or $N \leq Z(G)$.
(iii) $(G, Z(G))$ is a generalized Camina pair.

Theorem 1.4. *Let G be a finite p-group with $|G/Z(G)| = p^{2n+1}$. If G satisfies the strong condition on normal subgroups then $cd(G) = \{1, p^n\}$. Moreover, if the nilpotency class of G is 2, then all the non-linear irreducible characters are non-faithful.*

Theorem 1.5. *Let G be a p-group of nilpotency class 2 satisfying the weak condition on normal subgroups. Suppose one of the non-linear irreducible characters of G is faithful. Then*

(i) $G' = C_{p^2}$.
(ii) $K \cap Z(G)$ is the unique minimal normal G for the kernel K of any non-faithful non-linear irreducible character of G.
(iii) G has at least two distinct non-linear non-faithful irreducible character kernels each of order p^2.
(iv) $K_1 \cap K_2$ is the unique minimal normal subgroup of G for any two distinct kernels K_1 and K_2 of non-faithful non-linear irreducible characters of G.
(v) If $p = 2$ then G has exactly three distinct non-linear irreducible character kernels.

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Theorem 1.6. *Let G be a p-group of nilpotency class 2 with two character degrees and $|G/Z(G)| = p^{2n+1}$. Then the following conditions are equivalent:*

(i) exp $G' = p$ and $Z(\chi)/Z(G)$ is cyclic for all non-linear irreducible characters χ of G.

(ii) For every normal subgroup N of G, either $G' \leq N$ or $|NZ(G)/Z(G)| \leq p$.

(iii) $|Z(\chi)/Z(G)| = p$ for all non-linear irreducible characters χ of G.

Theorem 1.7. *Let G be a finite p-group of nilpotency class not equal to 3 and $|G/Z(G)| = p^{2n}$. If G satisfies the weak condition on normal subgroups then $cd(G) \subseteq \{1, p^{n-1}, p^n\}$.*

2. Preliminaries

In this section, we prove some preliminary results which will aid us in proving our main results. First we state some results of a p-group G if $|G'| = p$. These results are stated in [7] and [1].

Lemma 2.1. *Let G be a p-group such that $|G'| = p$. Then*

(i) $cd(G) = \{1, |G/Z(G)|^{\frac{1}{2}}\}$.

(ii) G satisfies the strong condition on normal subgroups.

From [7, Theorem 2.32] we get that if a group G has a faithful irreducible character, then $Z(G)$ is cyclic. The necessary and sufficient conditions when all the non-linear irreducible characters are faithful have been obtained by Doostie and Saeidi [1]. We state the result here.

Lemma 2.2. [1] *All non-linear irreducible characters of a finite group G are faithful if and only if $|G'| = p$ and $Z(G)$ is cyclic.*

Combining the above two lemmas we get the following result.

Proposition 2.3. *Let G be a p-group where all the non-linear irreducible characters are faithful. Then $cd(G) = \{1, |G/Z(G)|^{\frac{1}{2}}\}$.*

If G be a group of nilpotency class 2 and if G satisfies either the strong condition or the weak condition on normal subgroups then the exponents of the subgroup G' and the factor group $G/Z(G)$ have been obtained in [2].

Theorem 2.4. [2] *Let G be p-group of nilpotency class 2.*

(i) If G satisfies the strong condition on normal subgroups then exp $G/Z(G) =$ exp $G' = p$.

(ii) G satisfies the weak condition on normal subgroups then exp $G/Z(G) =$ exp $G' = p$ or p^2. In the later case $G/Z(G) \cong C_{p^2} \times C_{p^2}$ and $G' \cong C_{p^2}$.

Qian and Wang[5] examined the conditions under which the kernels of the non-linear characters of a non Abelian p-group form a chain with respect to inclusion and proved the following result. By Kern(G) we denote the set of kernels of non-linear irreducible characters of G.

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Theorem 2.5. *Let G be a finite non-Abelian p-group. Then the following are statements are equivalent:*

(i) $Kern(G)$ is a chain with respect to inclusion

(ii) Whenever $N < G'$ is a normal subgroup of G, N is a member of $Kern(G)$.

(iii) G is one of the following groups:

(a) G' is a unique minimal normal subgroup of G.

(b) G is of maximal class.

Next we prove that a p-group of nilpotency class 2 satisfying the strong condition on normal subgroups must have that all the non-linear irreducible characters are either faithful or non-faithful.

Lemma 2.6. *Let G a p-group of nilpotency class 2 satisfying the strong condition on normal subgroups. Then the non-linear irreducible characters are either all faithful or all non-faithful.*

Proof. Suppose G is p-group of nilpotency class 2 satisfying the strong condition. Suppose at least one of the non-linear irreducible characters of G is faithful. Then $Z(G)$ is cyclic and since the nilpotency class of G is 2, we get that $G' \subseteq Z(G)$. Since $Z(G)$ is a cyclic p-group, its subgroups form a chain with respect to inclusion. Since the group satisfies the strong condition, we get that the kernels of the irreducible characters are subsets of $Z(G)$ and hence it follows that all the non-linear character kernels form a chain with respect to inclusion. Thus by Theorem 2.5 either G is of maximal class or G' is a unique minimal normal subgroup. If G is of maximal class , then $|G| = p^3$ and hence all the characters are faithful. Suppose G' is a unique minimal normal subgroup. Since $Z(G)$ is cyclic and $G' \subseteq Z(G)$ we get that $|G'| = p$. Then by Lemma 2.2 all the non-linear characters are faithful. □

It has been proved in [2] that if the group G satisfies the strong condition on normal subgroups then $c(G) \leq 3$ and if G satisfies the weak condition on normal subgroups then $c(G) \leq 4$.

Lemma 2.7. [2] *Suppose G be a non Abelian p-group which satisfies the strong condition on normal subgroups. Then $c(G) \leq 3$.*

Lemma 2.8. [2] *Suppose G be a non Abelian p-group which satisfies the weak condition on normal subgroups. Then $c(G) \leq 4$.*

A pair (G, N) is said to be a generalized Camina pair (abbreviated GCP) if N is normal in G and all the non-linear irreducible characters of G vanish outside N. The notion of GCP was introduced by Mark L. Lewis [3]. Equivalently a pair (G, N) is a GCP if and only if for $g \in G \setminus N$, the conjugacy class of g in G is gG'. In [4] the authors obtained the following results.

Theorem 2.9. [4] *Let $(G, Z(G))$ be a GCP. Then we have the following.*

(i) $cd(G) = \{1, |G/Z(G)|^{\frac{1}{2}}\}$.

(ii) The number of non-linear irreducible characters of G is $|Z(G)| - |Z(G)/G'|$.

3. Main Results

In [2] the authors characterized the p-groups G satisfying the strong condition on normal subgroups by its character degree set if $G/Z(G)$ is an even power of p . In the following theorem we provide some alternate characterizations.

Theorem 3.1. *Let G be a p-group of nilpotency class 2 with two character degrees and $|Z/Z(G)| = p^{2n}$. Then the following conditions are equivalent:*

(i) $exp\ G' = p$ and $Z(\chi)/Z(G)$ is cyclic for all $\chi \in nl(G)$.

(ii) For every normal subgroup N of G, either $G' \leq N$ or $N \leq Z(G)$.

(iii) $(G, Z(G))$ is a generalized Camina pair.

Proof. $(i) \Rightarrow (iii)$ Since the nilpotency class of G is 2 and $exp\ G' = p$, it follows from [8, Lemma 4.4] that $exp\ (G/Z(G)) = p$ and hence $G/Z(G)$ is elementary Abelian. If $|G'| = p$, it follows from Lemma 2.1 that $cd(G) = \{1, |G/Z(G)|^{\frac{1}{2}}\}$. So we assume that $|G'| > p$. Let us choose a normal subgroup N of G such that $[G'/N] = p$. Let us consider an non-linear irreducible character $\chi \in \mathrm{Irr}(G/N)$. Then $N \subseteq Ker\ \chi = K$ and hence $N = K \cap G'$. Hence $|(G/K)'| = [G'/K \cap G'] = p$. By lemma 2.2 we get $\mathrm{cd}(G/K) = \{1, |G/Z(\chi)|^{\frac{1}{2}}\}$. Combining the fact that $|G/Z(G)| = p^{2n}$ and $|G/Z(\chi)|$ has square order we get that $|Z(\chi)/Z(G)|$ has square order. Since $exp\ (Z(\chi)/Z(G) = p$ and $Z(\chi)/Z(G)$ is cyclic, we get that $Z(\chi) = Z(G)$. Hence $cd(G) = cd(G/K) = \{1, |G/Z(G)|^{\frac{1}{2}}\}$. Let $\chi \in \mathrm{nl}(G)$. Since the group $G/Z(\chi)$ is Abelian, it follows by [7, Theorem 2.31] that $\chi(1)^2 = |G/Z(\chi)| = |Z/Z(G)|$. Hence we get that $Z(\chi) = Z(G)$. Hence by [7, Corollary 2.30] it follows that χ vanishes off $Z(G)$. Thus $(G, Z(G))$ is a generlized Camina pair.

$(iii) \Rightarrow (ii)$ It follows from Theorem 2.9 that $cd(G) = \{1, p^n\}$ and hence by [2, theorem B] we get that for any normal subgroup N of G, either $G' \leq N$ or $N \leq Z(G)$.

$(ii) \Rightarrow (i)$ By Theorem 2.4 we get that $exp\ G' = p$. Let $\chi \in \mathrm{nl}(G)$ and K be the kernel of χ. Then $Z(\chi)/K$ is a cyclic group. Since $K \leq Z(G)$ we get that $Z(G)/K$ is also a cyclic group. Hence it follows that $Z(\chi)/Z(G)$ is a cyclic group. □

In the following theorem we obtain the character degree set of a p group G under the condition that $|G/Z(G)| = p^{2n+1}$ if G satisfies the strong condition on normal subgroups.

Theorem 3.2. *Let G be a finite p-group with $|G/Z(G)| = p^{2n+1}$. If G satisfies the strong condition on normal subgroups then $cd(G) = \{1, p^n\}$. Moreover, if the nilpotency class of G is 2, then all the non-linear irreducible characters are non-faithful.*

Proof. By Lemma 2.7 we get that $c(G) \leq 3$.

Case I:: nilpotency class of G is 2.

Let χ be any non-linear irreducible character of G and let ...$K = \ker\chi$. Since $K \leq Z(G)$, the group $Z(G)/K$ is cyclic. Hence $Z(\chi)/Z(G)$ is a cyclic group. Since $|G/Z(G)| = p^{2n+1}$ and

$\chi(1)^2 = |G/Z(\chi)|$, we get that $|Z(\chi)/Z(G)|$ is an odd power of p. By [2, Theorem 4.3] we get that $exp\ Z(\chi)/Z(G) = p$. Hence $|Z(\chi)/Z(G)| = p$ for all non-linear irreducible characters of G. This gives that all the non-linear irreducible characters are non-faithful because if χ is a faithful non-linear irreducible character of G then by [7, Lemma 2.27] we get that $Z(\chi) = Z(G)$.

Since $\chi(1)^2 = |G/Z(\chi)|$, it follows that $cd(G) = \{1, p^n\}$.

Case II:: nilpotency class of G is 3.

By [2, Theorem F] we get that $|G/Z(G)| = p^3$ and hence $cd(G) = \{1, p\}$.

□

The structures of Abelian groups have been discussed in [9] and [10]. In particular we state the following results.

Proposition 3.3. [9] *Let G be a finite Abelian group and U be a cyclic subgroup of maximal order in G. Then there exists a complement V of U in G.*

Lemma 3.4. [10] *Let G be a finite Abelian group and $H \leq G$. Then G contains a subgroup isopmorphic to G/H.*

Theorem 3.5. *Let G be a p-group of nilpotency class 2 satisfying the weak condition on normal subgroups. Suppose one of the non-linear irreducible characters of G is faithful. Then*

(i) $G' = C_{p^2}$.

(ii) *$K \cap Z(G)$ is the unique minimal normal subgroup of G for the kernel K of any non-faithful non-linear irreducible character of G.*

(iii) *G has at least two distinct non-linear non-faithful irreducible character kernels each of order p^2.*

(iv) *$K_1 \cap K_2$ is the unique minimal normal subgroup of G for any two distinct kernels K_1 and K_2 of non-faithful non-linear irreducible characters of G.*

(v) *If $p = 2$ then G has exactly three distinct non-linear irreducible character kernels.*

Proof. Since one of the non-linear irreducible characters is faithful, we get that $Z(G)$ is cyclic.

(i) By [2, Theorem D] we get that either $exp\ G' = p$ or $G' = C_{p^2}$. Since the nilpotency class of G is 2 and G has a faithful non-linear irreducible character, we get that $Z(G)$ is cyclic. If $exp\ G' = p$ then $G' = C_p$.

Let K be the kernel of a non-faithful non-linear irreducible character χ. Now if $G' \subseteq K \cap Z(G)$ we get that χ is a faithful character . Hence we must get that $K \cap Z(G) < G'$. If $G' = C_p$, this will give that $K \cap Z(G) = 1$, a contradiction. Thus it follows that $G' = C_{p^2}$.

(ii) As observed in (i) above we get that $K \cap Z(G) < G'$ for the kernel K of any non-faithful non-linear irreducible character of G and hence it follows that $|K \cap Z(G)| = p$. Thus $K \cap Z(G)$ is the unique minimal normal subgroup of G.

(iii) Let $N = K \cap Z(G)$. Then we have that $|(G/N)'| = p$. Hence by Lemma 2.1 we get that G/N is a group of nilpotency class 2 satisfying the strong condition on normal subgroups. Hence all the non-linear irreducible characters of G/N are either faithful or non-faithful. If all the non-linear irreducible characters of G/N are faithful we get that $N = K$ for the kernel K of each of the non-faithful non-linear irreducible characters of G. Thus G is a p-group of nilpotency class 2 where the number of distinct non-linear irreducible character kernels is 2. Hence by [1, Theorem 1.1] we get that G is a group of order p^4 and nilpotency class 3, a contradiction. Thus all the non-linear irreducible characters of G/N are non-faithful and hence $N = K \cap Z(G) \subset K$ for the kernel K of any non-faithful non-linear irreducible character of G. This gives that $|K| = p^2$. If G has only one distinct kernel K of non-faithful non-linear irreducible character, then by [1, Theorem 1.1] all the non-linear irreducible characters of G/N are faithful. This contradiction proves that G has at least two distinct non-linear non-faithful irreducible character kernels.

(iv) Since $N \subseteq K_1 \cap K_2$ and $|N| = |K_1 \cap K_2| = p$ we get that $K_1 \cap K_2$ is the unique minimal normal subgroup of G for any two distinct kernels K_1 and K_2 of non-faithful non-linear irreducible characters of G.

(v) The result follows from [1, Theorem 1.1].

□

In the following theorem we obtain some alternative characterizations of p groups of nilpotency class 2 which satisfies the weak condition.

Theorem 3.6. *Let G be a p-group of nilpotency class 2 with two character degrees and $|G/Z(G)| = p^{2n+1}$. Then the following conditions are equivalent:*

(i) $exp\ G' = p$ and $Z(\chi)/Z(G)$ is cyclic for all non-linear irreducible characters χ of G.

(ii) For every normal subgroup N of G, either $G' \leq N$ or $|NZ(G)/Z(G)| \leq p$.

(iii) $|Z(\chi)/Z(G)| = p$ for all non-linear irreducible characters χ of G.

Proof. (i) ⇒ (iii)

Since the nilpotency class of G is 2 and $exp\ G' = p$, it follows that $exp\ (G/Z(G)) = p$ and hence $G/Z(G)$ is elementary Abelian. If $|G'| = p$, it follows from Lemma 2.1 that $\chi(1)^2 = |G/Z(G)|$. This contradicts that $|G/Z(G)| = p^{2n+1}$. So we get that $|G'| > p$. Let us choose a normal subgroup N of G' such that $[G'/N] = p$. Then G/N is a non Abelian group and $cd(G/N) = cd(G)$. Let us consider an non-linear irreducible character $\chi \in \mathrm{Irr}(G/N)$. Then $N \subseteq Ker\ \chi = K$ and hence $N = K \cap G'$. Hence $|(G/K)'| = [G'/K \cap G'] = p$. By Lemma 2.1 we get $\mathrm{cd}(G/K) = \{1, |G/Z(\chi)|^{\frac{1}{2}}\}$. Combining the fact that $|G/Z(G)| = p^{2n+1}$ and $|G/Z(\chi)|$ has square order we get that $|Z(\chi)/Z(G)|$ is an odd power of p. Since $exp\ (Z(\chi)/Z(G) = p$ and $Z(\chi)/Z(G)$ is cyclic, we get that $|Z(\chi)/Z(G)| = p$. Hence $cd(G) = cd(G/K) = \{1, |G/Z(\chi)|^{\frac{1}{2}}\} = \{1, p^n\}$. Hence by [2, Theorem C] we get that for every normal subgroup N of G, either $G' \leq N$ or $|NZ(G)/Z(G)| \leq p$.

$(ii) \Rightarrow (iii)$ Let K be the kernel of a non-linear irreducible character χ of G. First assume that $|KZ(G)/Z(G)| = p$ for the kernel K of a non-linear irreducible character of G. Since $KZ(G)/Z(G) \leq Z(\chi)/Z(G)$, by Lemma 3.4 we get that the Abelian group $Z(\chi)/Z(G)$ contains a subgroup isomorphic to $Z(\chi)/KZ(G)$, which is cyclic. Since $|KZ(G)/Z(G)| = p$, either $|Z(\chi)/KZ(G)| = 1$ or $Z(\chi)/KZ(G)$ is of maximal order. If $|Z(\chi)/KZ(G)| = 1$, then we get that $Z(\chi) = KZ(G)$ and hence $Z(\chi)/Z(G)| = |KZ(G)/Z(G)| = p$. If $Z(\chi)/KZ(G)$ is of maximal order, then by Proposition 3.3 we get that $Z(\chi)/Z(G) \cong C_{p^k} \times C_p$, where $p^k = |Z(\chi)/KZ(G)|$. Since $|G/Z(G)| = p^{2n+1}$ and G satisfies the weak condition on normal subgroups, by Theorem 2.4 we get that $exp\ G' = exp\ G/Z(G) = p$. This gives that $exp\ Z(\chi)/Z(G) = p$ and consequently $k = 0$ or $k = 1$. If $k = 1$, then $|Z(\chi)/Z(G)| = p^2$ and hence $|G/Z(\chi)| = p^{2n-1}$. This contradicts that $|G/Z(\chi)| = \chi(1)^2$. Thus $k = 0$ and hence $Z(\chi)/Z(G) \cong C_p$. Since $|KZ(G)/Z(G)| = p$ and $KZ(G)/Z(G) \leq Z(\chi)/Z(G)$, we get that $KZ(G)/Z(G) = Z(\chi)/Z(G)$ and this in turn gives that $|Z(\chi)/Z(G)| = p$. Next assume that $|KZ(G)/Z(G)| = 1$ so that $K \leq Z(G)$. Then $Z(\chi)/Z(G)$ is cyclic and $exp\ Z(\chi)/Z(G) = p$. This gives that either $Z(\chi) = Z(G)$ or $|Z(\chi)/Z(G)| = p$. If $Z(\chi) = Z(G)$ we get that $|G/Z(\chi)| = |G/Z(G)| = p^{2n+1}$, which contradicts that $\chi(1)^2 = |G/Z(\chi)|$. Hence $|Z(\chi)/Z(G)| = p$.

$(iii) \Rightarrow (ii)$ Obvious.

$(iii) \Rightarrow (i)$ By (ii) We get that the group G satisfies the weak condition on normal subgroups of G and hence by Theorem 2.4 we get that $exp\ G' = p$ or p^2. If $exp\ G' = p^2$ it follows from Theorem 2.4 that $|G/Z(G)| = p^4$. This contradiction gives that $exp\ G' = p$. Since $|Z(\chi)/Z(G)| = p$, we get that $Z(\chi)/Z(G)$ is cyclic. □

In [2, Theorem C] the authors found that for any p-group G satisfying the weak condition on normal subgroups, $cd(G) = \{1, p^n\}$ if $|G/Z(G)| = p^{2n+1}$. In the following theorem we obtain information on character degree sets of such groups under the condition that $|G/Z(G)| = p^{2n}$.

Theorem 3.7. *Let G be a finite p-group of nilpotency class not equal to 3 and $|G/Z(G)| = p^{2n}$. If G satisfies the weak condition on normal subgroups then $cd(G) \subseteq \{1, p^{n-1}, p^n\}$.*

Proof. By Lemma 2.8 we get that $c(G) \leq 4$.
First we consider the case that the nilpotency class of G is 2. Let K be the kernel of a non-linear irreducible character χ of G. First we assume that $|KZ(G)/Z(G)| = p$. Since $KZ(G)/Z(G) \leq Z(\chi)/Z(G)$, by Lemma 3.4 we get that the Abelian group $Z(\chi)/Z(G)$ contains a subgroup isomorphic to $Z(\chi)/KZ(G)$, which is cyclic. Since $|KZ(G)/Z(G)| = p$, either $|Z(\chi)/KZ(G)| = 1$ or $Z(\chi)/KZ(G)$ is of maximal order. If $|Z(\chi)/KZ(G)| = 1$, then we get $Z(\chi) = Z(G)$ and hence $|Z(\chi)/Z(G)| = p$. Consequently it follows that $\chi(1)^2 = |G/Z(\chi)| = |G/Z(G)| = p^{2n}$. If

$Z(\chi)/KZ(G)$ is of maximal order, then by Proposition 3.3 we get that $Z(\chi)/Z(G) \cong C_{p^k} \times C_p$, where $p^k = |Z(\chi)/KZ(G)|$. Since G satisfies the weak condition on normal subgroups, by Theorem 2.4 we get that $exp\ G' = exp\ G/Z(G) = p$ or p^2. This gives that $exp\ Z(\chi)/Z(G) = p$ or p^2. If $exp\ Z(\chi)/Z(G) = p$ we get that $k = 0$ or $k = 1$. If $k = 1$, then $|Z(\chi)/Z(G)| = p^2$ and hence $\chi(1)^2 = |G/Z(\chi)| = p^{2n-2}$. If $k = 0$ we get that $Z(\chi)/Z(G) \cong C_p$. Since $|KZ(G)/Z(G)| = p$ and $KZ(G)/Z(G) \leq Z(\chi)/Z(G)$, we get that $KZ(G)/Z(G) = Z(\chi)/Z(G)$ and this in turn gives that $|Z(\chi)/Z(G)| = p$. Thus $|G/Z(\chi)| = p^{2n-1}$ and this contradicts that $\chi(1)^2 = |G/Z(\chi)|$. If $exp\ G' = p^2$ we get that $Z(\chi)/Z(G) \cong C_{p^2} \times C_p$. This gives that $|Z(\chi)/Z(G)| = p^3$ which in turn gives that $\chi(1)^2 = |G/Z(\chi)| = p^{2n-3}$, a contradiction.

Next assume that $|KZ(G)/Z(G)| = 1$ so that $K \leq Z(G)$. Then $Z(\chi)/Z(G)$ is cyclic. Since $exp\ Z(\chi)/Z(G) = p$ or p^2, it follows that $Z(\chi) = Z(G)$ or $Z(\chi)/Z(G) \cong C_p$ or C_{p^2}. If $Z(\chi)/Z(G) \cong C_p$ we get that $\chi(1)^2 = |G/Z(\chi)| = p^{2n-1}$, a contradiction. Hence $Z(\chi) = Z(G)$ or $Z(\chi)/Z(G) \cong C_{p^2}$. Thus $\chi(1)^2 = |G/Z(\chi)| = p^{2n}$ or p^{2n-2}.

Next we consider the case that the nilpotency class of G is 4. By [2, Theorem F] we get that $|G/Z(G)| = p^4$ and hence $cd(G) \subseteq \{1, p, p^2\}$. □

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