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LIFTING AUTOMORPHISMS OF SUBGROUPS OF DIRECT PRODUCTS OF CYCLIC p-GROUPS

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Abstract. Let Γ be a finite group. A subgroup H of Γ is called "fully liftable" in Γ if every automorphism of H is the restriction of an automorphism of Γ. Let $G = C_{p^{k_1}} \times C_{p^{k_2}}$, where $1 \leq k_1 \leq k_2$ and p is prime. Using information about the subgroup structure of G and knowledge of $\mathrm{Aut}(G)$, we characterize all fully liftable subgroups of G. It turns out that all cyclic subgroups of G are fully liftable, and non-cyclic subgroups are fully liftable if and only if they are automorphic to certain subproducts of G, where two subgroups H and K are automorphic in G if there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(H) = K$. Further, we compare the fully liftable subgroups of G with the characteristic subgroups of G, which are similarly characterized by certain subproducts. Finally, we exhibit some interesting lattice features of both fully liftable subgroups of G and characteristic subgroups of G.

1. Introduction

Finite abelian p-groups are foundational in the study of group theory. While they seem to be among the easiest groups to understand, their structure is rich and surprisingly complicated. Counting and describing the subgroups of abelian groups goes back to at least 1904 in a paper by Miller [13]. The same problem was revisited many times in the last 120 years (see [12], [4], [15]) including a paper published in 2020 [5]. Similarly, the study of characteristic subgroups of abelian groups goes back at least to 1905 (once again in a paper by Miller [14]) and was revisited in 1935 by Baer [2], in 2011 by Kerby and Rode [11], and most recently by Humam and Astuti in 2022 [10]. In recent years, there has

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been interest in lattice-theoretic properties of groups, including finite abelian groups (see [10], [1], and at least 16 papers from Tărnăuceanu including the book [17]).

In this paper we continue the tradition of diving into the structural depths of finite abelian p-groups, use many of the results in the papers mentioned above, and contribute new results on the interplay between automorphisms of a subgroup of a group and the automorphisms of the parent group.

Let $B \leq A$ be groups. "Looking down" from A toward B, we have long been interested in automorphisms of A that are invariant on B and, hence, in characteristic subgroups of A. Here we will "look up" from B toward A and determine when automorphisms of B "lift" to automorphisms of A, or, equivalently, are induced by automorphisms of A. When every member of $\mathrm{Aut}(B)$ lifts to $\mathrm{Aut}(A)$, B is "fully liftable." Note that a group is of "injective type" every subgroup is fully liftable (see [3]).

There is a symmetry in the ideas of looking down and looking up to identify characteristic and fully liftable subgroups. While some subgroups can be both characteristic and fully liftable, we will see that corresponding lattice diagrams are fairly complementary.

The structure of this paper is as follows. After a bit of notation in section 2, we define fully liftable subgroups in section 3 and state our main theorems. In section 4 we state some needed information on automorphism groups; similarly, in section 5 we establish notation and information concerning subgroups of rank 2 abelian p-groups. Section 6 focuses on fully liftable subgroups and proving the main theorems, while in section 7 we investigate the lattices of characteristic and fully liftable subgroups.

2. Notation

- Γ will denote a generic finite group, while G will specifically refer to the rank 2 abelian p-group
$$G = C_{p^{k_1}} \times C_{p^{k_2}},$$
where $1 \leq k_1 \leq k_2$, and p is prime.
- $|X|$ is the cardinality of the set X, while $o(g)$ is the order of $g \in \Gamma$.
- We will consider G multiplicatively, with $G = \langle s_1, s_2 \rangle$ where $o(s_i) = p^{k_i}$.
- We say that G has type (p^{k_1}, p^{k_2}).
- If $t \in \mathbb{Z}$ then t_p is the highest power of the prime p that divides t

3. Fully liftable subgroups

We begin with some definitions, examples, and analogies between fully liftable subgroups and characteristic subgroups.

Definition 3.1. *Let H be a subgroup of the group Γ. The element $\theta \in \mathrm{Aut}(H)$* **lifts** *to $\mathrm{Aut}(\Gamma)$ if there exists $\hat{\theta} \in \mathrm{Aut}(\Gamma)$ such that $\hat{\theta}|_H = \theta$. If every member of $\mathrm{Aut}(H)$ lifts to $\mathrm{Aut}(\Gamma)$, then we say H is* **fully liftable** *(FL) in Γ.*

Example 3.2. *Let $G = C_p \times C_{p^2} = \langle s_1, s_2 \rangle$. The subgroup $H = \langle s_1, s_2^p \rangle$ is not FL in G because the automorphism of H that swaps the generators does not lift to $\mathrm{Aut}(G)$.*

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Definition 3.3. *Let* $\mathrm{Aut}(\Gamma)$ *act naturally on the set of subgroups of* Γ. *Let* $H \le \Gamma$.

(1) *Denote the orbit of* H *by* $\mathcal{O}(H)$. *If* $K \in \mathcal{O}(H)$ *we will say that* H *and* K *are* **automorphic**, *and write* $H \sim K$. *We will call* $\mathcal{O}(H)$ *the* **automorphic class** *of* H.

(2) *The stabilizer of* H *under the action of* $\mathrm{Aut}(\Gamma)$ *is the set of* **characterizers** *of* H *in* G. *That is,*

$$\mathrm{Char}_{\mathrm{Aut}(\Gamma)}(H) = \{\phi \in \mathrm{Aut}(\Gamma) \,|\, \phi(H) = H\}.$$

Clearly H is characteristic in Γ if and only if $\mathrm{Char}_{\mathrm{Aut}(\Gamma)}(H) = \mathrm{Aut}(\Gamma)$.

Proposition 3.4. *Let* H *be a subgroup of the group* Γ. *If* H *is FL in* Γ *then we must have*

$$|\mathrm{Aut}(H)| \mid |\mathrm{Char}_{\mathrm{Aut}(\Gamma)}(H)| \mid |\mathrm{Aut}(\Gamma)|.$$

Proof. Whether H is FL in Γ or not, we must have $|\mathrm{Char}_{\mathrm{Aut}(\Gamma)}(H)| \mid |\mathrm{Aut}(\Gamma)|$ because the set of characterizers of H is a stabilizer subgroup of $\mathrm{Aut}(\Gamma)$.

Let $\rho : \mathrm{Char}_{\mathrm{Aut}(\Gamma)}(H) \to \mathrm{Aut}(H)$ be the restriction homomorphism so that $\rho(\phi) = \phi|_H$. Then H is FL in Γ if and only if ρ is surjective, in which case we must have $|\mathrm{Aut}(H)| \mid |\mathrm{Char}_{\mathrm{Aut}(\Gamma)}(H)|$. □

Just as the property of being a characteristic subgroup satisfies a transitive relation, we have the following transitivity result for FL subgroups. The proof is clear, but we state the result in order to reference it later.

Proposition 3.5. *Let* H, K, *and* Γ *be groups with* $H \le K \le \Gamma$. *If* H *is FL in* K *and* K *is FL in* Γ, *then* H *is FL in* Γ.

We see in the next proposition that FL subgroups fall into automorphic classes.

Proposition 3.6. *Let* H *and* K *be automorphic subgroups of* Γ. *If* H *is FL in* Γ, *then so is* K.

Proof. Let $\phi \in \mathrm{Aut}(\Gamma)$ such that $\phi(H) = K$. Let $\theta \in \mathrm{Aut}(K)$, then $\alpha = ((\phi|_H)^{-1} \circ \theta \circ \phi|_H) \in \mathrm{Aut}(H)$. Since H is FL in Γ, there is a lift $\hat{\alpha}$ of α in $\mathrm{Aut}(\Gamma)$. It is routine to check that $\hat{\theta} = \phi \circ \hat{\alpha} \circ \phi^{-1}$ is a lift of θ. □

Proposition 3.7. *Let* H, K, *and* Γ *be groups with* $H \le K \le \Gamma$. *If* H *is not FL in* K *and* K *is characteristic in* Γ, *then* H *is not FL in* Γ.

Proof. Suppose by way of contradiction that H is FL in Γ, then every $\theta \in \mathrm{Aut}(H)$ lifts to $\hat{\theta} \in \mathrm{Aut}(\Gamma)$. Now K characteristic in Γ means that $\hat{\theta}|_K \in \mathrm{Aut}(K)$. But $\hat{\theta}|_K$ is clearly a lift of θ in $\mathrm{Aut}(K)$, contradicting the fact that H is not FL in K. □

We end this section by stating the main result characterizing fully liftable subgroups of

$$G = C_{p^{k_1}} \times C_{p^{k_2}} = \langle s_1, s_2 \rangle.$$

Its proof requires results in each of the next three sections.

Main Theorem. *Let $H \leq G$ and assume $0 \leq a_i \leq k_i$ for $i = 1, 2$.*

(1) *When $k_1 < k_2$, H is FL in G if and only if H is cyclic or is automorphic to $\langle s_1^{p^{a_1}}, s_2^{p^{a_2}} \rangle$ with $a_1 \geq a_2$.*

(2) *When $k_1 = k_2$, H is FL in G if and only if H is cyclic or is automorphic to $\langle s_1^{p^{a_1}}, s_2^{p^{a_2}} \rangle$ for any a_1, a_2.*

4. Automorphisms of G

Recall that $G = C_{p^{k_1}} \times C_{p^{k_2}} = \langle s_1, s_2 \rangle$, with $1 \leq k_1 \leq k_2$. From [9] and [7], every automorphism ϕ of G can be associated with a matrix

$$M_\phi = \begin{bmatrix} i & j \\ qp^m & r \end{bmatrix}$$

where $m = k_2 - k_1$, $0 \leq i, j < p^{k_1}$, $0 \leq qp^m, r < p^{k_2}$, and $\det(M_\phi) \not\equiv 0 \bmod p$. The notation means that $\phi(s_1) = s_1^i s_2^{qp^m}$ and $\phi(s_2) = s_1^j s_2^r$. For simplicity, we will often write $\phi = M_\phi$. Note that if $k_1 = k_2 = k$, then $\mathrm{Aut}(G) = \mathrm{GL}_2(\mathbb{Z}_{p^k})$.

From [9, Theorem 4.1],

$$|\mathrm{Aut}(G)| = \begin{cases} p^{4k-3}(p-1)^2(p+1), & \text{if } k_1 = k_2 = k \\ p^{3k_1+k_2-2}(p-1)^2, & \text{if } k_1 < k_2. \end{cases}$$

5. Subgroups of G

5.1. Subgroup structure.

From [8, Theorem 1] there is a one-to-one correspondence between the subgroups of G and triples of the form (p^{a_1}, p^{a_2}, t) where $0 \leq a_i \leq k_i$, $0 \leq t \leq p^d - 1$, and $d = \min(a_1, k_2 - a_2)$. Every triple is associated with

$$H(a_1, a_2, t) = \langle s_1^{p^{a_1}}, s_1^\sigma s_2^{p^{a_2}} \rangle$$

where $\sigma = tp^{a_1 - d}$.

Notes about $H(a_1, a_2, t)$:

- The order of $H(a_1, a_2, t)$ is easily seen to be $p^{k_1 - a_1 + k_2 - a_2}$.
- Fixing a_1 and a_2 clearly fixes the order of $H(a_1, a_2, t)$ but not its type. The [8, Theorem 2] shows that

  $$H(a_1, a_2, t) \cong C_{p^\alpha} \times C_{p^\beta}$$

  where $p^\alpha = \gcd(p^{k_1 - a_1}, p^{k_2 - a_2}, tp^{k_2 - a_2 - d})$ and $\beta = k_1 - a_1 + k_2 - a_2 - \alpha$. Thus, the type of $H(a_1, a_2, t)$ depends on t_p once the a_i are fixed.
- Different values of t_p might lead to different types of subgroups of the form $H(a_1, a_2, t)$ which will then not be automorphic; on the other hand, different values of t that have equal t_p will always lead to the same type subgroup. We will see in Proposition 5.1 that having the same value of t_p always results in automorphic subgroups.

- $H = H(a_1, a_2, t)$ is a subproduct of G (meaning that $H = H_1 \times H_2$ where $H_i \leq C_{p^{a_i}}$) if and only if $t = 0$. Further, $H(a_1, a_2, 0)$ is cyclic if and only if $a_1 = k_1$ or $a_2 = k_2$.
- When $t = 0$, the subgroups $H(a_1, a_2, 0)$ are exactly the same as the subgroups $R(k_1 - a_1, k_2 - a_2)$ in [11]. We will use this fact later.
- When $t = 0$ we will often write $H(a_1, a_2)$ or $H(\mathbf{a})$, where $\mathbf{a} = (a_1, a_2)$.

5.2. **Maximal subgroups.** There are $p+1$ maximal subgroups of G. Here we associate them with the notation introduced above.

In order for $H(a_1, a_2, t)$ to be a maximal subgroup of G, we must have $a_1 + a_2 = 1$. If $a_1 = 0$ and $a_2 = 1$, then $d = 0$ and we get the subgroup

$$H(0,1,0) = \langle s_1, s_2^p \rangle \cong C_{p^{k_1}} \times C_{p^{k_2-1}}.$$

If $a_1 = 1$ and $a_2 = 0$, then $d = 1$ and we get p subgroups of the form

$$H(1,0,t) = \langle s_1^p, s_1^t s_2 \rangle \cong C_{p^{k_1-1}} \times C_{p^{k_2}}, 0 \leq t \leq p-1.$$

Unless $k_1 = k_2$, the maximal subgroups fall into two different types and different types are not automorphic.

5.3. **Automorphic subgroups.** Next we will partition subgroups of G of the same order into subsets, all of whose members are automorphic, but this partition is not necessarily a decomposition into automorphic classes.

Fix a_1 and a_2 where $0 \leq a_i < k_i$ (we will not consider cyclic subgroups here). Let X_{a_1,a_2} be the set of subgroups of G of index $p^{a_1+a_2}$. The members of X_{a_1,a_2} are subgroups of G of the form $H(a_1, a_2, t)$, where $0 \leq t \leq p^d - 1$ and $d = \min(a_1, k_2 - a_2)$, hence $|X_{a_1,a_2}| = p^d$. If $d > 0$, partition X_{a_1,a_2} as follows: let

$$\begin{aligned} A_0 &= \{H(a_1, a_2, 0)\} \\ A_1 &= \{H(a_1, a_2, t) \,|\, t \neq 0, t_p = 1\} \\ A_p &= \{H(a_1, a_2, t) \,|\, t \neq 0, t_p = p\} \\ \vdots &= \qquad \vdots \\ A_{p^{d-1}} &= \{H(a_1, a_2, t) \,|\, t \neq 0, t_p = p^{d-1}\}, \end{aligned}$$

then $A_i \cap A_j = \emptyset$ when $i \neq j$ and

$$X_{a_1,a_2} = A_0 \cup A_1 \cup A_p \cup \cdots \cup A_{p^{d-1}}.$$

We see that $|A_{p^i}| = p^{d-i-1}(p-1)$ for $i = 0, 1, \ldots, d-1$. If $d = 0$, then $X_{a_1,a_2} = A_0 = \{H(a_1, a_2, 0)\}$.

From the notes in section 5, we see that every subgroup in the set A_{p^i} has the same type. Further, we will show that all of the subgroups in A_{p^i} are automorphic to each other.

Proposition 5.1. *Assume $d = \min(a_1, k_2 - a_2) \neq 0$. Let A_{p^i}, $i = 0, 1, \ldots, d-1$, be as described above. Let $H(a_1, a_2, t) \in A_{p_i}$, then $H(a_1, a_2, t)$ is automorphic to $H(p^{a_1}, p^{a_2}, t_p)$.*

Proof. Since $H(a_1, a_2, t) \in A_{p^i}$ we know that $t \neq 0$ and $t_p = p^i$. Let $t = qp^i$ where $p \nmid q$. As $H(a_1, a_2, t_p)$ is automorphic to itself, we will assume $q \neq 1$ and show $H(a_1, a_2, qp^i) \sim H(a_1, a_2, p^i)$.

Recall that $H(a_1, a_2, t) = \langle s_1^{p^{a_1}}, s_1^{\sigma} s_2^{p^{a_2}} \rangle$, where $\sigma = tp^{a_1 - d}$.

Consider $\phi \in \mathrm{Aut}(G)$ given by $\phi(s_1) = s_1^q$ and $\phi(s_2) = s_2$. Then we have

$$\phi(s_1^{p^{a_1}}) = s_1^{qp^{a_1}}$$
$$\phi(s_1^{p^{a_1-d+i}} s_2^{p^{a_2}}) = s_1^{qp^{a_1-d+i}} s_2^{p^{a_2}}.$$

Hence,

$$\phi(H(a_1, a_2, p^i)) = \langle s_1^{qp^{a_1}}, s_1^{qp^{a_1-d+i}} s_2^{p^{a_2}} \rangle = \langle s_1^{p^{a_1}}, s_1^{qp^{a_1-d+i}} s_2^{p^{a_2}} \rangle = H(a_1, a_2, qp^i).$$

□

By transitivity (Proposition 3.5), every member of the set A_{p^i} is automorphic to every other member of the set. However, an entire automorphic class may be split up among several A_{p^i}'s.

5.4. **Characteristic subgroups.** We end this section with a description of the characteristic subgroups of G, when $p > 2$.

Theorem 5.2. [11, Thm 2.2] *Let $p > 2$. The subgroup $H(a_1, a_2, t)$ is characteristic in G if and only if the following three conditions hold:*

(1) *$t = 0$ (so that $H(a_1, a_2, 0) = R(k_1 - a_1, k_2 - a_2)$ in the [11] notation),*
(2) *$k_1 - a_1 \leq k_2 - a_2$, and*
(3) *$a_1 \leq a_2$.*

6. Fully liftable subgroups of G

6.1. Cyclic subgroups.

Proposition 6.1. *Every cyclic subgroup of G is FL in G.*

Proof. Let $H = \langle x \rangle$ where $x = s_1^{b_1} s_2^{b_2}$ with $0 \leq b_i < p^{k_i}$. As $H = \langle e \rangle$ is clearly FL in G, we will assume that b_1 and b_2 are not both 0.

Let $o(x) = p^b$ for some $1 \leq b \leq k_2$. An element of $\mathrm{Aut}(H)$ is θ defined by $\theta(x) = x^c$, where $1 \leq c < p^b$ and $\gcd(c, p) = 1$. Set $c_1 \equiv c \bmod p^{k_1}$, where $0 \leq c_1 < p^{k_1}$. We will show that $\hat{\theta}$, defined below, is a lift of θ.

$$\hat{\theta} = \begin{bmatrix} c_1 & 0 \\ 0 & c \end{bmatrix}$$

First note that $\hat{\theta}$ is, indeed, a member of $\mathrm{Aut}(G)$ because $c_1 c \not\equiv 0 \bmod p$.

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Next, we have

$$\begin{aligned}\hat{\theta}(x) &= \hat{\theta}(s_1^{b_1} s_2^{b_2}) \\ &= s_1^{c_1 b_1} s_2^{c b_2} \\ &= (s_1^{b_1} s_2^{b_2})^c \\ &= x^c \\ &= \theta(x).\end{aligned}$$

□

For the rest of this section, we will restrict to non-cyclic subgroups of G.

6.2. **Reduction to subproducts.** In this subsection, assume $0 \leq a_i < k_i$ so that $H(a_1, a_2, t)$ is non-cyclic.

From Proposition 5.1, we see that the automorphic class of $H(a_1, a_2, t)$ must include entire sets of the form A_{p^i}. Certainly the subset A_{t_p} is in $\mathcal{O}(H(a_1, a_2, t))$, but other members of the partition of X_{a_1,a_2} from section 5.3 might also be included $\mathcal{O}(H(a_1, a_2, t))$. Regardless, we will see that $H(a_1, a_2, t)$ is FL in G if and only if $H(a_1, a_2, t)$ is automorphic to the subproduct $H(a_1, a_2, 0)$.

Proposition 6.2. *If the non-cyclic subgroup $H(a_1, a_2, t)$ is FL in G, then $H(a_1, a_2, t)$ is automorphic to $H(a_1, a_2, 0)$.*

Proof. Suppose $H(a_1, a_2, t)$ is not automorphic to $H(a_1, a_2, 0)$, then the automorphism orbit of $H(a_1, a_2, t)$ consists of subsets of the form A_{p^i} and specifically does not include A_0. Thus, there exist $i_1, i_2, \dots, i_n$ with $0 \leq i_j \leq d$ (where p^d is the number of subgroups of G having the same order as $H(a_1, a_2, t)$) such that $i_1 < i_2 < \cdots < i_n$ and

$$\begin{aligned}|\mathcal{O}(H(a_1, a_2, t))| &= |A_{p^{i_1}}| + |A_{p^{i_2}}| + \cdots + |A_{p^{i_n}}| \\ &= (p^{d-i_1-1} + p^{d-i_2-1} + \cdots + p^{d-i_n-1})(p-1) \\ &= p^{d-i_n-1}(p^{i_n-i_1} + p^{i_n-i_2} + \cdots + 1)(p-1) \\ &= rp^{d-i_n-1}(p-1),\end{aligned}$$

where $r = p^{i_n-i_1} + p^{i_n-i_2} + \cdots + 1$.

From the orbit-stabilizer theorem we know that $|\mathcal{O}(H(a_1, a_2, t))|$ must divide $|\mathrm{Aut}(G)|$. From the formula for $|\mathrm{Aut}(G)|$ in section 4, we conclude that r must divide $p^{3k_1+k_2-2-\alpha}(p-1)$. However, $\gcd(r, p) = 1$ and $r > p - 1$, so we have a contradiction. □

Thus, we may concentrate on determining which subproducts of G are FL in G.

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6.3. **Fully liftable subproducts of G.** In this section we will switch to the more compact notation of $H(a_1, a_2, 0) = H(\mathbf{a})$, when referring to non-cyclic subproducts of G. Given a_i with $0 \leq a_i < k_i$, let $b_i = k_i - a_i$. Then

$$H(\mathbf{a}) = \langle s_1^{p^{a_1}}, s_2^{p^{a_2}} \rangle \cong C_{p^{b_1}} \times C_{p^{b_2}}.$$

Definition 6.3. *The* **factor distance** *of $H(\mathbf{a}) \cong C_{p^{b_1}} \times C_{p^{b_2}}$ is*

$$d_{\mathbf{a}} = (k_2 - a_2) - (k_1 - a_1) = b_2 - b_1.$$

Note that we do not require $b_1 \leq b_2$ in the definition above, so it is possible for $d_{\mathbf{a}}$ to be negative. On the other hand, since we insist that $k_1 \leq k_2$, the factor distance of $G = H(0,0)$ is $d_G = k_2 - k_1 \geq 0$. (Note that $d_G = m$, from the description of $\mathrm{Aut}(G)$ in section 4.)

Except when $k_1 = k_2$, we will ultimately see that $H(\mathbf{a})$ is FL in G if and only if the factor distance of G is a lower bound for the factor distance of $H(\mathbf{a})$.

Proposition 6.4. *Let $G = C_{p^{k_1}} \times C_{p^{k_2}}$ with $1 \leq k_1 < k_2$. Let $H(\mathbf{a})$ be a non-cyclic subproduct of G. If $d_{\mathbf{a}} < d_G$, then $H(\mathbf{a})$ is not FL in G.*

Proof. Note that since $H(\mathbf{a})$ is non-cyclic, neither b_1 nor b_2 is 0.

Case 1: Assume $b_2 \geq b_1$.

Let $\alpha_i = s_i^{p^{a_i}}$ so that $H(\mathbf{a}) = \langle \alpha_1, \alpha_2 \rangle$. Consider $\theta \in \mathrm{Aut}(H(\mathbf{a}))$ given below

$$\theta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We claim that θ does not have a lift in $\mathrm{Aut}(G)$.

Suppose a lift of θ exists and has the form

$$\hat{\theta} = \begin{bmatrix} i & j \\ qp^m & r \end{bmatrix}.$$

Then $\hat{\theta}(\alpha_2) = \theta(\alpha_2)$ implies

$$s_1^{jp^{a_2}} s_2^{rp^{a_2}} = \alpha_1 \alpha_2 = s_1^{p^{a_1}} s_2^{p^{a_2}}.$$

In particular, we must be able to solve $jp^{a_2} \equiv p^{a_1} \bmod p^{k_1}$. By hypothesis, $b_2 - b_1 < k_2 - k_1$, hence $a_1 < a_2$, and it is impossible to solve $jp^{a_2 - a_1} \equiv 1 \bmod p^{b_1}$.

Case 2: Assume $b_2 < b_1$.

In this case, we essentially swap the generators of $H(\mathbf{a})$ so that automorphisms are of the form given in section 4, but

$$\phi = \begin{bmatrix} i & j \\ qp^{b_1 - b_2} & r \end{bmatrix}$$

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means $\phi(\alpha_2) = \alpha_2^i \alpha_1^{qp^{b_1-b_2}}$ and $\phi(\alpha_1) = \alpha_2^j \alpha_1^r$. We will show that

$$\theta = \begin{bmatrix} 1 & 0 \\ p^{b_1-b_2} & 1 \end{bmatrix}$$

does not have a lift in $\mathrm{Aut}(G)$.

Suppose a lift of θ exists and has the form

$$\hat{\theta} = \begin{bmatrix} i & j \\ qp^m & r \end{bmatrix}.$$

Then $\hat{\theta}(\alpha_2) = \theta(\alpha_2)$ implies

$$s_1^{jp^{a_2}} s_2^{rp^{a_2}} = \alpha_2 \alpha_1^{p^{b_1-b_2}} = s_2^{p^{a_2}} s_1^{p^{a_1+b_1-b_2}}.$$

In particular, we must be able to solve

$$jp^{a_2} \equiv p^{a_1+b_1-b_2} \bmod p^{k_1}. \tag{6.1}$$

Now

$$a_1 + b_1 - b_2 = a_1 + (k_1 - a_1) - (k_2 - a_2) = k_1 - k_2 + a_2 < a_2$$

and

$$a_1 + b_1 - b_2 = k_1 - b_1 + b_1 - b_2 = k_1 - b_2 < k_1.$$

Thus, solving equation 6.1 is equivalent to solving

$$jp^{a_2-(a_1+b_1-b_2)} \equiv 1 \bmod p^{k_1-(a_1+b_1-b_2)},$$

which is impossible. □

By the proposition above, when $k_1 < k_2$ we need only look among the groups $H(\mathbf{a})$ with factor distance at least $d_G = k_2 - k_1$ to find non-cyclic FL subgroups. It turns out that all such subgroups are FL in G. To show this, we will follow a chain of subproducts, each FL in the next, up to the end of the chain, whose factor distance equals d_G and is FL in G.

We begin by looking at the maximal subgroups of G. From section 5.2 we see that the maximal subgroups of G that are subproducts are $H(0,1)$ and $H(1,0)$. These subproducts are non-cyclic as long as $k_1, k_2 > 1$.

Theorem 6.5. *Assume $2 \le k_1 \le k_2$. The maximal subgroup $H(1,0)$ is FL in G and the maximal subgroup $H(0,1)$ is FL in G only when $k_1 = k_2$.*

Proof. First consider the subgroup $H(1,0) = \langle s_1^p, s_2 \rangle$. Let

$$\theta = \begin{bmatrix} i & j \\ qp^{k_2-k_1+1} & r \end{bmatrix} \in \mathrm{Aut}(H(1,0)).$$

We will show that a lift of θ is

$$\hat{\theta} = \begin{bmatrix} i & pj \\ qp^{k_2-k_1} & r \end{bmatrix}.$$

First note that $\hat{\theta}$ is, indeed, a member of $\mathrm{Aut}(G)$. Next, we have

$$\begin{aligned} \hat{\theta}(s_1^p) &= (s_1^i s_2^{qp^{k_2-k_1}})^p \\ &= s_1^{ip} s_2^{qp^{k_2-k_1+1}} \\ &= \theta(s_1^p) \end{aligned}$$

and

$$\begin{aligned} \hat{\theta}(s_2) &= s_1^{pj} s_2^r \\ &= \theta(s_2). \end{aligned}$$

Thus, we see that $H(1,0)$ is FL in G.

Next consider the subgroup $H(0,1) = \langle s_1, s_2^p \rangle$, with $k_1 < k_2$. Let $\theta \in \mathrm{Aut}(H(0,1))$ be defined by $\theta(s_1) = s_1$ and $\theta(s_2^p) = s_1 s_2^p$. That is

$$\theta = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Suppose θ lifts to $\hat{\theta} \in \mathrm{Aut}(G)$, where

$$\hat{\theta} = \begin{bmatrix} i & j \\ p^m q & r \end{bmatrix}.$$

Then $\hat{\theta}(s_2^p) = \theta(s_2^p)$ implies $s_1^{jp} s_2^{rp} = s_1 s_2^p$. In particular, we must be able to solve $jp \equiv 1 \bmod p^{k_1}$, which is impossible.

Finally, we consider the subgroup $H(0,1) = \langle s_1, s_2^p \rangle$, with $k_1 = k_2$. Using the automorphism of G that swaps generators, $H(0,1)$ and $H(1,0)$ are automorphic. Hence, by Proposition 3.6, $H(0,1)$ is FL in G. □

Proposition 6.6. *Let $H(\mathbf{a})$ be a non-cyclic subgroup of G with $d_{\mathbf{a}} = d_G$. Then $H(\mathbf{a})$ is FL in G.*

Proof. Since $d_{\mathbf{a}} = d_G$ we know that $a_1 = a_2 = a$ for some $0 \leq a \leq k_1$. We can assume $a \neq 0$ so that $H(\mathbf{a}) \neq G$, and $a \neq k_1$ since $H(\mathbf{a})$ is not cyclic.

Since $b_2 - b_1 = k_2 - k_1 = d_G = m$, a typical automorphism of $H(\mathbf{a})$ is

$$\theta = \begin{bmatrix} i & j \\ qp^m & r \end{bmatrix},$$

where $0 \leq i, j < p^{b_1}$, $0 \leq qp^m, r < p^{b_2}$, and $\det(\theta) \not\equiv 0 \bmod p$.

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Consider θ as a member of $\mathrm{Aut}(G)$ to get a lift. That is, define $\hat{\theta} \in \mathrm{Aut}(G)$ by

$$\hat{\theta} = \begin{bmatrix} i & j \\ qp^m & r \end{bmatrix}.$$

We see that $\hat{\theta}$ is, indeed, in $\mathrm{Aut}(G)$, and it clearly induces θ when restricted to $H(\mathbf{a})$. □

Theorem 6.7. *Let $G = C_{p^{k_1}} \times C_{p^{k_2}}$ with $1 \leq k_1 \leq k_2$ and $d_G = k_2 - k_1$. Let $H(\mathbf{a})$ be a non-cyclic subgroup of G where $\mathbf{a} = (a_1, a_2)$, $b_i = k_i - a_i$, and $d_{\mathbf{a}} = b_2 - b_1$.*

1. *When $k_1 < k_2$, $H(\mathbf{a})$ is FL in G if and only if $d_{\mathbf{a}} \geq d_G$.*
2. *When $k_1 = k_2$, $H(\mathbf{a})$ is always FL in G.*

Proof. We begin by showing that $H(\mathbf{a})$ is FL in G whenever $d_{\mathbf{a}} \geq d_G$ for $k_1 \leq k_2$. If $d_{\mathbf{a}} = d_G$, then $H(\mathbf{a})$ is FL in G by Proposition 6.6. When $d_{\mathbf{a}} > d_G$ we let $n = d_{\mathbf{a}} - d_G$. Then $n = a_1 - a_2 > 0$. Applying Theorem 6.5 to the group $H(a_1 - i, a_2)$, for $i = 1, 2, \ldots, n$, we see that $H(a_1 - i + 1, a_2)$ is maximal and FL in $H(a_1 - i, a_2)$. Thus, we get the following sequence of subgroups, each maximal and FL in the next:

$$H(a_1, a_2) \leq H(a_1 - 1, a_2) \leq H(a_1 - 2, a_2) \leq \cdots \leq H(a_1 - n, a_2).$$

On the righthand end of the sequence, $a_1 - n = a_2$, so $d_{(a_2,a_2)} = (k_2 - a_2) - (k_1 - a_2) = d_G$ and we see that $H(a_1 - n, a_2)$ is FL in G by Proposition 6.6. By transitivity (Proposition 3.5), $H(\mathbf{a})$ is FL in G.

To complete part (1) of the theorem, we assume $k_1 < k_2$ and note that if $d_{\mathbf{a}} < d_G$, then $H(\mathbf{a})$ is not FL in G by Proposition 6.4.

To complete part (2) of the theorem, we assume $k_1 = k_2$ and must show that $H(\mathbf{a})$ is FL in G when $d_{\mathbf{a}} < d_G = 0$. In this case, we will see that we essentially swap the roles of the generators of G and use part (1).

Let $G' = \langle s_2, s_1 \rangle$ and set $H'(a_2, a_1) = \langle s_2^{p^{a_2}}, s_1^{p^{a_1}} \rangle$. Then $G' = G$ and $H'(a_2, a_1) = H(a_1, a_2)$. Now $d_{(a_2,a_1)} = -d_{\mathbf{a}} \geq 1$, so $H'(a_2, a_1)$ is FL in $H'(1, 0)$ by part (1) of the Theorem. We know $H'(1, 0)$ is FL in G' by Proposition 6.5, thus, by transitivity, $H'(a_2, a_1)$ is FL in G'. Finally, by swapping generators again, we see that $H(a_1, a_2)$ is FL in G. □

Combining the theorem above with the description of characteristic subgroups in Theorem 5.2, we get the following description of subproducts that are both FL and characteristic in G when $p > 2$.

Corollary 6.8. *Let $G = C_{p^{k_1}} \times C_{p^{k_2}}$ with $1 \leq k_1 \leq k_2$ and $p > 2$. Let $H(\mathbf{a})$ be a (possibly cyclic) subproduct of G where $0 \leq b_i \leq k_i$.*

1. *When $k_1 < k_2$, $H(\mathbf{a})$ is both FL and characteristic in G if and only if (i) $H(\mathbf{a})$ is non-cyclic and $a_1 = a_2$, or (ii) $H(\mathbf{a}) = H(k_1, a_2)$ is cyclic with $a_2 \geq k_1$.*
2. *When $k_1 = k_2$, $H(\mathbf{a})$ is both FL and characteristic in G if and only if $a_1 = a_2$.*

Proof. For the first part of the theorem, assume $k_1 < k_2$. When $H(\mathbf{a})$ is non-cyclic, we need $a_1 \geq a_2$ in order for $H(\mathbf{a})$ to be FL in G by Theorem 6.7 and we need $a_1 \leq a_2$ in order for $H(\mathbf{a})$ to be characteristic in G by Theorem 5.2, hence $a_1 = a_2$. If $H(\mathbf{a})$ is cyclic it is always FL in G by Proposition 6.1, but characteristic in G only when $d_{\mathbf{a}} \leq d_G$ by Theorem 5.2.

For the second part of the theorem, assume $k_1 = k_2$. We know that $H(\mathbf{a})$ is always FL in G, but characteristic in G if and only if $a_1 = a_2$. □

We know from [11] that a subgroup of G is characteristic in G if and only if it equals $H(a_1, a_2, 0)$ for some $a_1 \leq a_2$ with $k_1 - a_1 \leq k_2 - a_2$. From Proposition 6.2 we know that if a non-cyclic subgroup of G is FL in G then it is automorphic to some $H(a_1, a_2, 0)$. In the next proposition we will see when that happens.

Proposition 6.9. *Let $H = H(a_1, a_2, t)$ be a subgroup of G. When $t > 0$, let $t_p = p^a$ with $0 \leq a \leq d-1$ where $d = \min(a_1, k_2 - a_2)$. Then H is FL in G if and only if H is cyclic or one of the following conditions holds when H is non-cyclic:*

(1) *$t = 0$, and if $k_1 < k_2$ then we must also have $a_1 \geq a_2$; or*
(2) *$t \neq 0$, $k_1 - a_1 \leq k_2 - a_2 - d - a$, $a_1 > a_2$, and $a_1 - d + a \geq a_2$.*

Proof. To prove the sufficiency of the conditions, we begin with the fact that if H is cyclic then it is FL in G by Proposition 6.1.

Assume H is non-cyclic, so neither $a_1 = k_1$ nor $a_2 = k_2$. If $t = 0$ then $H = H(a_1, a_2)$. By Theorem 6.7 we know that $H(\mathbf{a})$ is always FL in G when $k_1 = k_2$, and when $k_1 < k_2$ we must have $a_1 \geq a_2$ in order for $H(\mathbf{a})$ to be FL in G.

Suppose $t \neq 0$, $k_1 - a_1 \leq k_2 - a_2 - d - a$, $a_1 > a_2$, and $a_1 - d + a > a_2$. From Proposition 5.1 we know that $H(a_1, a_2, t)$ is automorphic to $H(a_1, a_2, t_p)$, so it suffices to show that $H(a_1, a_2, p^a)$ is automorphic to a FL subgroup of G.

From Section 5 we know that

$$H = H(a_1, a_2, p^a) \cong C_{p^\alpha} \times C_{p^\beta}$$

where $p^\alpha = \gcd(p^{k_1 - a_1}, p^{k_2 - a_2}, p^{k_2 - a_2 - d + a})$ and $\beta = k_1 - a_1 + k_2 - a_2 - \alpha$. Now $a \leq d - 1$ implies

$$k_2 - a_2 - d + a \leq k_2 - a_2 - 1 < k_2 - a_2.$$

Hence $\alpha = \min(k_1 - a_1, k_2 - a_2 - d + a)$.

By assumption, $k_1 - a_1 \leq k_2 - a_2 - d - a$, thus $\alpha = k_1 - a_1$ and $\beta = k_2 - a_2$. Since $a_1 - d + a - a_2 < a_1 < k_1$, the following function is in $\mathrm{Aut}(G)$

$$\phi = \begin{bmatrix} 1 & p^{k_1} - p^{a_1 - d + a - a_2} \\ 0 & 1 \end{bmatrix}. \tag{6.2}$$

Recall that

$$H(a_1, a_2, p^a) = \langle s_1^{p^{a_1}}, s_1^{\sigma} s_2^{p^{a_2}} \rangle \text{ and } H(a_1, a_2, 0) = \langle s_1^{p^{a_1}}, s_2^{p^{a_2}} \rangle,$$

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where $\sigma = p^{a_1-d+a}$. Now

$$\phi(s_1^{p^{a_1}}) = s_1^{p^{a_1}}$$

and

$$\phi(s_1^{p^{a_1-d+a}} s_2^{p^{a_2}}) = s_1^{p^{a_1-d+a}} (s_1^{p^{k_1}-p^{a_1-d+a-a_2}} s_2)^{p^{a_2}} = s_2^{p^{a_2}}.$$

Thus $\phi(H) = H(a_1, a_2, 0) = H(\mathbf{a})$. By assumption, $a_1 > a_2$, thus $d_{\mathbf{a}} > d_G$ and we know that $H(\mathbf{a})$ is FL in G. Hence H is FL in G by Proposition 3.6.

To prove the necessity of the conditions in the theorem, assume that H is FL in G and that it is not cyclic.

If $t = 0$ then $H = H(a_1, a_2)$ and we must have $a_1 \geq a_2$ when $k_1 < k_2$.

Next, assume $t > 0$. By Proposition 6.2, $H = H(a_1, a_2, t)$ must be automorphic to $H(a_1, a_2, 0)$, which must also be FL in G. In particular, H must have the same type as $H(\mathbf{a}) = H(a_1, a_2)$. The exponent of $H(\mathbf{a})$ is $\max(k_1 - a_1, k_1 - a_2)$ while the exponent of H is β.

To begin with, assume that $k_2 - a_2 - d + a < k_1 - a_1$ so that $\alpha = k_2 - a_2 - d + a$. Then

$$\beta = k_1 - a_1 + k_2 - a_2 - \alpha = k_1 - a_1 + d - a.$$

Now $\beta \neq k_1 - a_1$ since $d \neq a$, and $\beta \neq k_2 - a_2$ because that would imply $\alpha = k_1 - a_1$. Since H does not have the same type as $H(\mathbf{a})$, it cannot be automorphic to $H(\mathbf{a})$, so is not FL in G when $k_2 - a_2 - d + a < k_1 - a_1$.

Next, assume $k_1 - a_1 \leq k_2 - a_2 - d + a$ so that $\alpha = k_1 - a_1$ and $\beta = k_2 - a_2$. We know $k_1 - a_1 \leq k_2 - a_2$ from [8, Theorem 2], thus if $a_1 \leq a_2$ then $H(\mathbf{a})$ is characteristic in G by Theorem 5.2. Characteristic subgroups are the only ones in their automorphic class, so $H \sim H(\mathbf{a})$ if and only if $t = 0$. Since $t \neq 0$, we must have $a_1 > a_2$.

If $a_1 - d + a < a_2$ we will show that $H \not\sim H(\mathbf{a})$. Suppose we have $\phi(H) = H(\mathbf{a})$ where

$$\phi = \begin{bmatrix} i & j \\ qp^m & r \end{bmatrix} \in \mathrm{Aut}(G).$$

Then

$$\phi(s_1^{p^{a_1-d+a}} s_2^{p^{a_2}}) = s_1^{ip^{a_1-d+a}+jp^{a_2}} s_2^{qp^{m+a_1-d+a}+rp^{a_2}}.$$

If $\phi(s_1^{p^{a_1-d+a}} s_2^{p^{a_2}}) \in H(\mathbf{a})$, then we must be able to solve

$$ip^{a_1-d+a} + jp^{a_2} \equiv 0 \bmod p^{a_1}.$$

This is equivalent to solving

$$i + jp^{a_2-a_1+d-a} \equiv 0 \bmod p^{d-a},$$

which is impossible since $i \not\equiv 0 \bmod p$. □

Finally we note that Proposition 6.1, Proposition 6.2, Proposition 3.6, and Theorem 6.7 prove the Main Theorem from section 3.

7. Lattices

A number of papers concerning lattices of subgroups of a group have been published, with growing interest in the last 20 years (see [10], [1], [16]). Properties of full subgroup lattices, characteristic subgroup lattices, and centralizer lattices appear in [11], [18], and [6]. Here we mention a few interesting properties about the lattice of FL subgroups of G.

We continue using the same notation as in the earlier sections, but now we restrict to $p > 2$ and we will allow $H(\mathbf{a})$ to be cyclic. Recall that $H(\mathbf{a})$ is cyclic when one of $a_i = k_i$, and note that $H(k_1, k_2) = \{e\}$. We will consider three lattices of subgroups of G, each with a partial ordering given by inclusion.

- $\mathcal{S}(G)$ is the lattice of all subproducts of G. Hence,

$$\mathcal{S}_G = \{H(a_1, a_2) \,|\, 0 \leq a_i \leq k_i\}.$$

- $\mathcal{C}(G)$ is the lattice of all characteristic subproducts of G ([11, Theorem 2.2] shows that this is the same as the lattice of all characteristic subgroups of G).
- $\mathcal{FL}(G)$ is the lattice of all fully liftable subproducts of G.

Example 7.1. *In Figure 1 we show the Hasse diagram for all three lattices for $G = C_{p^3} \times C_{p^7}$, $p > 2$. To save space, we write just* $\mathbf{a}$ *instead of $H(\mathbf{a})$. Subproducts that are both FL and characteristic in G are in bold face, characteristic subgroups that are not FL in G are in red, and FL subgroups that are not characteristic in G are in blue. The full lattice is $\mathcal{S}(G)$. The red and bold face nodes form $\mathcal{C}(G)$ while the blue and bold face nodes form $\mathcal{FL}(G)$.*

The shape of the graph in Figure 1 is typical; we will refer to the four borders as the NE, NW, SE, and SW borders. For example, the SE border in Figure 1 consists of the subproducts with labels $(3, 7)$, $(2, 7)$, $(1, 7)$, and $(0, 7)$.

Proposition 7.2. *Let $p > 2$. The lattice of all subproducts of G that are both FL and characteristic in G, $\mathcal{C}(G) \cap \mathcal{FL}(G)$, is a chain of length k_2.*

Proof. We know what the members of $\mathcal{B} = \mathcal{C}(G) \cap \mathcal{FL}(G)$ are from Corollary 6.8. When $k_1 < k_2$, $\mathcal{B}$ is the following chain:

$$\underbrace{H(k_1, k_2) \leq H(k_1, k_2 - 1) \leq \cdots \leq H(k_1, k_1)}_{k_2 - k_1 + 1} \leq \underbrace{H(k_1 - 1, k_1 - 1) \leq \cdots \leq H(0, 0)}_{k_1}.$$

When $k_1 = k_2 = k$, $\mathcal{B}$ is the following chain:

$$\underbrace{H(k, k) \leq H(k - 1, k - 1) \leq \cdots \leq H(0, 0)}_{k+1}.$$

□

From [11, Theorem 3.3] we know that $\mathcal{C}(G)$ is a chain if and only $k_2 = k_1 + 1$. The next proposition shows $\mathcal{FL}(G)$ is never a chain.

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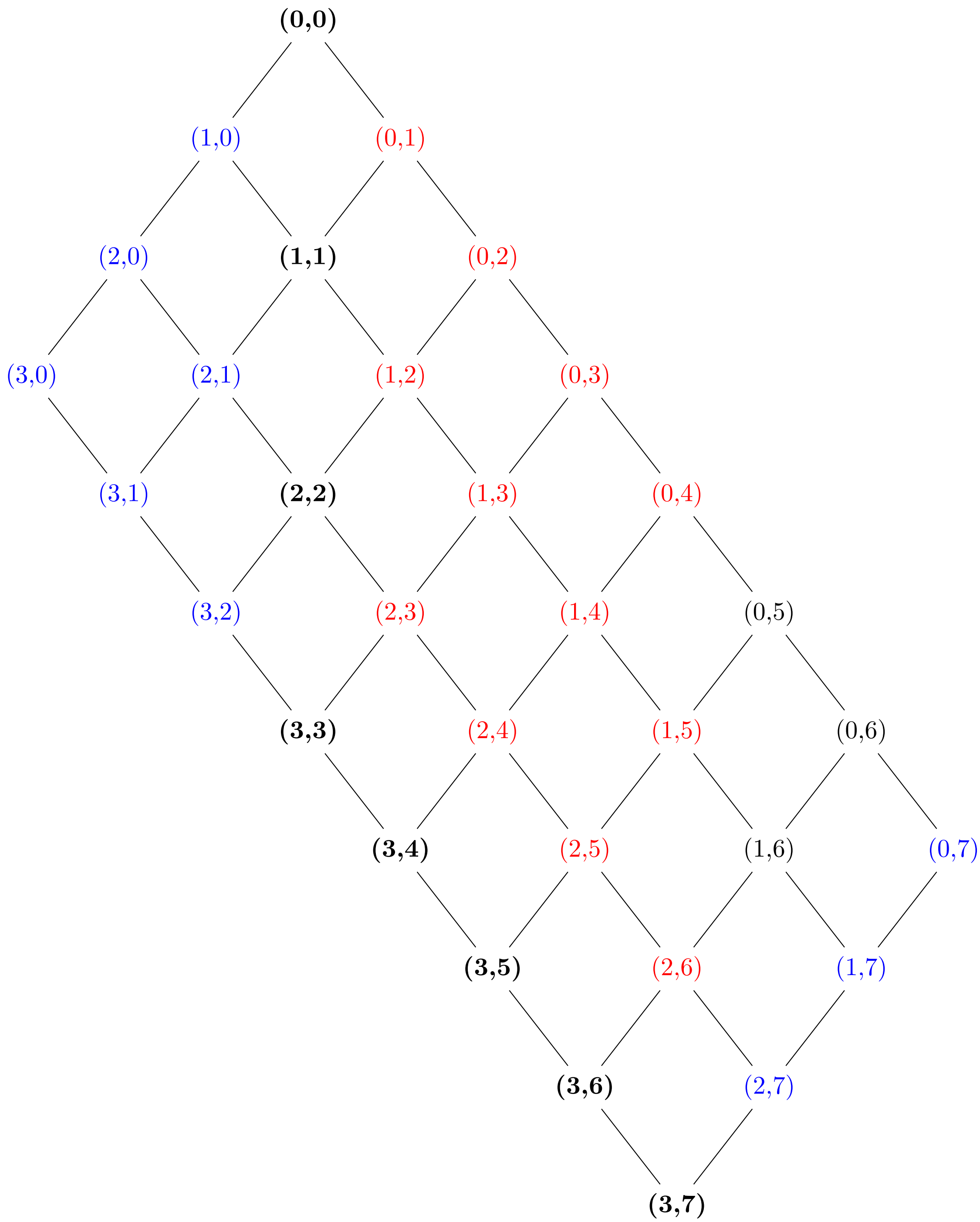

FIGURE 1. Lattices for $C_{p^3} \times C_{p^7}$

Proposition 7.3. *Let $p > 2$. The lattice $\mathcal{FL}(G)$ is never a chain.*

Proof. As cyclic subgroups, both $H(k_1, k_2 - 1) = \langle s_2^{p^{k_2-1}} \rangle$ and $H(k_1 - 1, k_2) = \langle s_1^{p^{k_1-1}} \rangle$ are FL in G, but they are not comparable so cannot be part of a chain. □

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Next we consider maximal and minimal FL subgroups.

Proposition 7.4. *Let $p > 2$ and $1 \leq k_1 < k_2$.*

1. *$H(1,0)$ is the unique maximal subgroup in $\mathcal{FL}(G)$.*
2. *If $d_G \geq k_1$ then $H(k_1, j)$ is the unique maximal subgroup in $\mathcal{FL}(G)$ of order p^i for $j = k_1 - 1, k_1, \ldots, k_2 - k_1 - 1$.*
3. *If $d_G < k_1$, then there are no non-trivial unique maximal subgroups in $\mathcal{FL}(G)$ of any order except $p^{k_1+k_2-1}$.*

Proof. Since $k_1 < k_2$ we know from subsection 5.2 that $H(1,0)$ is the only member of $\mathcal{S}(G)$ of its type. When $k_1 > 1$, $H(1,0)$ is FL in G by Theorem 6.5. When $k_1 = 1$, $H(1,0)$ is FL in G by Proposition 6.1. Thus we have proved part (1).

From Figure 1 we see that the entire lefthand (NW and SW) border of $\mathcal{S}(G)$ consists of FL subgroups. We will show that most of them are not the only FL subgroups in their rows, so are not unique among FL subgroups of their particular order.

From order $p^{k_1+k_2+1}$ down to order p^{k_2+1}, the NW border of $\mathcal{S}(G)$ consists of the rank 2 subgroups $H(i,0)$, $i = 0, 1, \ldots, k_1 - 1$. From order p^{k_2} down to order 1, the SW border of $\mathcal{S}(G)$ consists of the cyclic subgroups $H(k_1, j)$, $j = 0, 1, \ldots, k_2$.

When $i = 0$, $H(i,0) = G$ and when $i = 1$, $H(1,0)$ is maximal by part (1), so we need not consider these cases. Consider the subgroups $H(i,0)$, $i = 2, \ldots, k_1 - 1$. The subgroup to the immediate right of $H(i,0)$ in $\mathcal{S}(G)$ is $H(i-1,1)$. The factor distance of $H(i-1,1)$ is

$$(k_2 - 1) - (k_1 - i + 1) = d_G + i - 2 \geq d_G.$$

By Theorem 6.7, $H(i-1,1)$ is FL in G so there are at least two FL subgroups of order $p^{k_1+k_2-i}$ when $i = 2, \ldots, k_1 - 1$.

Next consider the subgroups $H(k_1, j)$, $j = 0, 1, \ldots, k_2$. When $j = k_2$, $H(k_1, j) = \{e\}$, so we need not consider this case. For $j = 0, 1, \ldots, k_2 - 1$, the subgroup to the immediate right of $H(k_1, j)$ is $H(k_1 - 1, j + 1)$. The factor distance of $H(k_1 - 1, j + 1)$ is $k_2 - j - 2$ and $k_2 - j - 2 \geq d_G$ if and only if $j \leq k_1 - 2$. Thus, as long as $k_1 \geq 2$ and $j \leq k_1 - 2$, there are at least two FL subgroups of order $p^{k_2 - j}$ when $j = 0, \ldots, k_2 - 1$.

If $k_1 = 1$, then $d_G \geq k_1$ and each row of $\mathcal{S}(G)$ containing non-trivial, proper subgroups contains exactly the two subgroups $H(1,j)$ and $H(0, j+1)$, $j = 0, 1, \ldots, k_2 - 1$. The factor distance of $H(0, j+1)$ is less than d_G so the subgroup is not FL in G as long as it is not cyclic, which happens when $j = k_2 - 1$. Thus, every subgroup of the form $H(1,j)$, $j = 0, 1, \ldots, k_2 - 2$, is the unique FL subgroup of its order in $\mathcal{S}(G)$. This proves statement (2) of the theorem for $k_1 = 1$.

Now assume $k_1 \geq 2$. The SE border of $\mathcal{S}(G)$ consists of cyclic FL subgroups of the form $H(i, k_2)$ for $i = 0, 1, \ldots, k_1$. The two cyclic subgroups $H(k_1, j)$ and $H(i, k_2)$ are in the same row when $k_1 + j = k_2 + i$. In this case, $0 \leq i \leq k_1$ implies $0 \leq k_1 - k_2 + j \leq k_1$, or, equivalently, $k_2 - k_1 \leq j \leq k_2$.

When $0 \leq j \leq k_1 - 2$ we know from above that $H(k_1, j)$ is in the same row as the FL subproduct $H(k_1 - 1, j+1)$. Hence, the only hope for $H(k_1, j)$ to be the unique FL subproduct of order $p^{k_2 - j}$ is if $k_1 - 1 \leq j \leq k_2 - k_1 - 1$. Note that if $k_1 > d_G$, this will never happen.

Finally, we will show that $H(k_1, j)$ is, indeed, the unique FL subproduct in its row when $k_1 - 1 \leq j \leq k_2 - k_1 - 1$. The row in $\mathcal{S}(G)$ containing $H(k_1, j)$ consists of the subgroups $H(k_1 - a, j + a)$ for $a = 0, 1, \ldots, k_1$. The distance factor of $H(k_1 - a, j + a)$ is

$$k_2 - (j + a) - (k_1 - (k_1 - a)) = k_2 - j - 2a.$$

Since $j \geq k_1 - 1$ we know

$$k_2 - j - 2a \leq k_2 - k_1 + 1 - 2a.$$

When $1 \leq a \leq k_1$, we see that the distance factor of $H(k_1 - a, j + a)$ is less than d_G, so $H(k_1 - a, j + a)$ is not FL in G by Theorem 6.7. Finally, $H(k_1, j)$ is the unique FL subgroup of its order as long as $d_G \geq k_1$ and $j = k_1 - 1, k_1, \ldots, k_2 - k_1 - 1$, thus proving parts (2) and (3) of the theorem. □

Example 7.5. *Let $G = C_{p^3} \times C_{p^7}$, then $d_G \geq k_1$ implies that $H(k_1, j)$ is the unique maximal FL subproduct of G for $j = 2, 3$. Indeed, we can see this in Figure 1.*

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