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NOTES ON INFLUENCE OF CERTAIN PERMUTABLE SUBGROUPS ON FINITE SMOOTH GROUPS

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Abstract. A maximal chain of a finite group G is called smooth if any two intervals have the same length are isomorphic. A group G is called totally smooth if all maximal chains of G are smooth, and called generalized smooth if all chains from each subgroup of prime order to G are smooth. In the paper entitled "Influence of certain permutable subgroups on finite smooth groups" (A. M. Elkholy and A. A. Heliel in *Acta Math. Sin. (Engl. Ser.)*, **27** no. 8 (2011) 1547-1556), the authors investigated the structure of finite groups which have a permutable subgroup of prime order and whose maximal subgroups are totally (or generalized) smooth groups. The results obtained by the authors require further precision. In the proof of some theorems, they overlooked some cases which may represent counterexamples to these theorems. Additionally, in certain theorems, we can omit certain hypotheses and get more accurate results. In this paper, we present counterexamples to some of these results and reintroduce these theorems after modification, using simpler and more direct proofs. Furthermore, we generalize these results by replacing certain hypotheses with weaker ones.

1. Introduction and Preliminaries

All groups considered in this paper will be finite. A chain $1 = G_0 \lessdot G_1 \lessdot \cdots \lessdot G_n = G$ is called a maximal chain if G_i is a maximal subgroup of G_{i+1} for $i = 0, 1, 2, \ldots, n-1$. An interval $[G_{i+j}/G_j] = \{X \leq G : G_j \leq X \leq G_{i+j}\}$ represents the set of all subgroups of G contained within G_{i+j} and containing G_j. A maximal chain is called smooth if $[G_{i+j}/G_j] \cong [G_i/G_0]$ for all $i, j \in \mathbb{N}$

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such that $i + j \leq n$. A group G is called totally smooth if all maximal chains of G are smooth (see [4]), and called generalized smooth if all chains from each subgroup of prime order to G are smooth (see [5]).

A group G is called a P-group if either G is an elementary abelian group or $G = G_pG_q$ is a semidirect product of an elementary abelian normal Sylow p-subgroup G_p and a Sylow q-subgroup G_q of order q which induces a non-trivial power automorphism on G_p, with $q \mid p-1$ (see [12, p. 49]). The following two lemmas introduce the structure of totally smooth and generalized smooth groups.

Lemma 1.1. [4, Theorem 1] *A finite group G is totally smooth if and only if one of the following holds:*

- **(i):** *G is cyclic of prime power order.*
- **(ii):** *G is a P-group.*
- **(iii):** *G is cyclic of square free order.*

Lemma 1.2. [5, Main Theorem] *A finite group G is generalized smooth if and only if one of the following holds:*

- **(i):** *$|G| = p_1p_2p_3$, where p_1, p_2, and p_3 are not necessarily distinct primes.*
- **(ii):** *G is a totally smooth group.*
- **(iii):** *$G = G_{p_1}G_{p_2}$, where G_{p_1} is a minimal normal subgroup of order p_1^2 and G_{p_2} is cyclic of order p_2^2 such that G_{p_2} and $\Phi(G_{p_2})$ operate irreducibly on G_{p_1} where p_1, p_2 are distinct primes.*
- **(iv):** *$G = G_pA$, where $|G_p| = p_1$, $G_p \lhd G$ and A is cyclic of order $p_2p_3 \cdots p_m$ and operates faithfully on G_p where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in 1, 2, \ldots, m$.*
- **(v):** *$G \cong A_5$.*

A subgroup of a group G is called permutable (or S-permutable) in G, if it permutes with every subgroup (or Sylow subgroup) of G. The paper [3] claims the following theorems:

Theorem A. *Assume that G is a group with maximal length $n \geq 3$. Suppose that p and q are distinct primes in $\pi(G)$ such that $|G| = p^\alpha q^\beta$; $p > q$, where α and β are non-zero positive integers. If the maximal subgroups of the group G are totally smooth groups and G has a permutable subgroup of prime order, then G is a nonabelian P-group or one of the following holds:*

- **(i):** *$G = PQ$, where P is a Sylow p-subgroup of G of order p and Q is an elementary abelian Sylow q-subgroup of G of order q^e with $e \geq 2$.*
- **(ii):** *$G = PQ$, where $|P| = p^2$ and $|Q| = q$.*

Theorem B. *Assume that G is a group with maximal length $n \geq 3$ and $|\pi(G)| \geq 3$. If the maximal subgroups of the group G are totally smooth groups and G has a permutable subgroup of prime order, then G is cyclic of square free order or one of the following holds:*

(i): *$n = 3$, $G/P \cong K$, where $|P| = p$, and $|K| = qr$; p, q and r are primes.*

(ii): *$n \geq 4$, $G = PK$, where $|P| = p$, P is normal in G and K is cyclic of square free order operating faithfully on P.*

Theorem C. *Assume that G is a group with maximal length $n \geq 4$. suppose that p and q are distinct primes in $\pi(G)$ such that $|G| = p^{\alpha}q^{\beta}$, where α and β are non-zero positive integers. If the maximal subgroups of the group G are generalized smooth groups and G has a permutable subgroup of prime order, then G is a nonabelian P-group or one of the following holds:*

(i): *$G = PQ$, where P is a Sylow p-subgroup of G of order p^2 and Q is a Sylow q-subgroup of order q^2.*

(ii): *$G = PQ$, where Q is a non-normal subgroup of G of order q and P is an elementary abelian normal subgroup of G such that $|P| = p^e$, $e \geq 3$.*

Theorem D. *Assume that G is a group with maximal length $n \geq 4$ and $|\pi(G)| \geq 3$. If the maximal subgroups of the group G are generalized smooth groups and G has a permutable subgroup of prime order, then one of the following holds:*

(i): *G is cyclic of square free order.*

(ii): *$G = PK$, where $|P| = p_1$, $P \lhd G$ and K is cyclic of order $p_2p_3 \cdots p_n$ and operates faithfully on P where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in \{1, 2, \ldots, n\}$.*

(iii): *$|G| = p_1p_2p_3p_4$, where p_1, p_2, p_3 and p_4 are prime numbers, not necessary different in pairs.*

Notes on the previous theorems:

(1) Any cyclic group of order pq^2 with $p > q$ is a counterexample to Theorem A, for example a cyclic group of order 12.

(2) Any abelian group of order p^3q is a counterexample to Theorem C, for example abelian group of order 24.

(3) The statement of these theorems needs improvement. For instance, in Theorem B, it is impossible for G to satisfy structure (ii). Therefore, it must be omitted from the theorem's results. Additionally, we can omit the hypothesis that G has a permutable subgroup of prime order and still obtain the same results.

In this paper, in addition to modifying the previous notes, we will introduce new proofs for the modified theorems which will be simpler and more direct. We will also generalize these theorems by replacing certain hypotheses with weaker ones, as demonstrated in the next section.

2. Main results

Lemma 2.1. *Assume that G is a group with $|\pi(G)| \geq 3$. Then all maximal subgroups of G are totally smooth if and only if G is cyclic of square free order or $|G| = p_1p_2p_3$ where p_1, p_2 and p_3 are distinct primes.*

Proof. Assume that all maximal subgroups of G are totally smooth. Since a totally smooth group is supersolvable, G is solvable by [9, Satz 9.6, p. 718]. Suppose for a contradiction that $|G|$ is divisible by p^2 for some prime p. Since G possesses a Sylow basis, G_pG_q is a subgroup of G where $q \in \pi(G)$ with $q \neq p$. Our hypothesis and Lemma 1.1 imply that G_pG_q is a nonabelian P-group. It follows that every subgroup of G_p is normal in G_pG_q and consequently normal in G as q is an arbitrary prime in $\pi(G)$. Hence G has a maximal subgroup of index p which is not totally smooth, a contradiction. Thus G is of square free order. If $|\pi(G)| = 3$, then $|G| = p_1p_2p_3$ where p_1, p_2 and p_3 are distinct primes and we are done. So let $|\pi(G)| > 3$. By Lemma 1.1, all maximal subgroups of G are cyclic and hence G is cyclic of square free order. The converse is evident as a subgroup of cyclic group is cyclic and a group of maximal length 2 is totally smooth. □

Lemma 2.2. *Assume that G is a group with $|\pi(G)| = 3$. Then all maximal subgroups of G are generalized smooth if and only if $|G| = p_1^e p_2 p_3$ $(e = 1, 2)$ where p_1, p_2 and p_3 are distinct primes.*

Proof. Since a group with a maximal length at most 3 is generalized smooth, the sufficient condition holds. Now, suppose that all maximal subgroups of G are generalized smooth. Let G be a simple group. By [8, Theorem 1], G is isomorphic to one of the groups: $PSL(2,5)$, $PSL(2,8)$, $PSL(2,7)$, $PSL(2,9)$, $PSL(2,17)$, $PSL(3,3)$, $U_3(3)$ or $U_4(2)$. According to Atlas of finite groups [2], only $PSL(2,5)$ among these groups has generalized smooth maximal subgroups. Therefore $|G| = 2^2.3.5$ and we are done. So assume that G is a non-simple group. We aim to show that G is solvable. Suppose L is a minimal normal subgroup of G. By Lemma 1.2, if L is not solvable, then $L \cong A_5$. Since no generalized smooth group can contain A_5 as a proper subgroup, A_5 is a maximal subgroup of G of prime index. By [13, Theorem 6.2.2, p. 136], if $N_G(G_2) \leq A_5$, where G_2 is a Sylow 2-subgroup of A_5, then $N_G(A_5) = A_5$, a contradiction. Therefore $|N_G(G_2)| = 2^2.3.p$ where $p = [G : A_5]$. According to Lemma 1.2, if $p = 2$ or 5, then $N_G(G_2)$ is not generalized smooth which contradicts our hypothesis. Thus $p = 3$ and consequently, $|G| = 180$. By [14], there exists a unique non-solvable group of order 180 which is $GL(2,4) \cong A_5 \times Z_3$. However, in this instance, $N_G(G_2) = A_4Z_3$ is not generalized smooth, a contradiction. This contradiction confirms that L is solvable. Consequently, L is elementary abelian.

Suppose, for a contradiction, that G is not solvable. By the renowned theorem of Feit and Thompson [6], a group of odd order is solvable. So assume that G is of even order and G_2 is a Sylow 2-subgroup of G. According to Lemma 1.2, G_2 is cyclic, elementary abelian or of order 2^3. Additionally, a group whose order is divisible by 2^3q, where q is odd prime, cannot be generalized smooth which implies that if $|G_2| \geq 2^3$, then G_2 must be a maximal subgroup of G. Therefore by [10, Exercise 10.5.7, p.

309] and [13, Theorem 6.2.11, p. 138], we can conclude that, G_2 is either elementary abelian of order 2^2 or non-abelian maximal subgroup of order 2^3. Firstly, assume that G_2 is elementary abelian of order 2^2. Clearly, if $L \leq G_2$, then G/L is solvable and by [13, Theorem 2.6.3, p. 39], G is solvable, a contradiction. So assume that L is of odd order. By [7, Lemma 2.4, p. 422], if G_2 is self-centralizing, then $N_G(G_2) \cong L_2(3)$ and hence $L \nleq N_G(G_2)$. According to Lemma 1.2, $LN_G(G_2)$ cannot be generalized smooth, then $G = LN_G(G_2)$. Hence $G/L \cong N_G(G_2)$ is solvable which follows that G is solvable, a contradiction. Thus G_2 is a proper subgroup of its centralizer. If $C_G(G_2) \lneqq N_G(G_2)$, then $N_G(G_2)$ is not generalized smooth, a contradiction. Thus $C_G(G_2) = N_G(G_2)$ and by Burnside's theorem [13, Theorem 6.2.9, p. 137], G is 2-nilpotent and hence solvable which again leads to a contradiction. By this final contradiction, we can assume that G_2 is a non-abelian maximal subgroup of order 2^3. Then $N_G(G_2) = G_2$ is an extra-special group. According to [7, Theorem 4.2, p. 252], $G_2 \cap G^\backprime = G_2^\backprime$. Furthermore, by [7, Lemma 3.1, p. 246], G contains a normal subgroup K of G such that $G/K \cong G_2/G_2^\backprime$. Consequently, both G/K and K are solvable groups and hence G is solvable, a contradiction. This final contradiction shows that G is solvable.

Consider $G = G_{p_1}G_{p_2}G_{p_3}$ where G_{p_i} is a Sylow p_i-subgroup of G $(i = 1, 2, 3)$. Suppose, for a contradiction, that $|G_{p_1}| \geq p_1^3$. Solvability of G implies that, $G_{p_1}G_{p_j} < G$ for $j = 2, 3$. By our hypothesis, $G_{p_1}G_{p_j}$ is a generalized smooth group. Applying Lemma 1.2, $G_{p_1}G_{p_j}$ is a nonabelian P-group with $p_1 > p_j$ $(j = 2, 3)$. It follows that every subgroup of G_{p_1} is normal in $G_{p_1}G_{p_j}$ $(j = 2, 3)$ and consequently, normal in G. Therefore G has a maximal subgroup of index p_1 which is not generalized smooth, a contradiction. Thus $|G_{p_1}| \leq p_1^2$ and so as $|G_{p_j}| \leq p_j^2$ for $j = 2, 3$. If G is of square free order, then $|G| = p_1p_2p_3$ and we are done. Without loss of generality, assume that $|G_{p_1}| = p_1^2$. We argue that, $|G_{p_j}| = p_j$ for $j = 2, 3$. Suppose, for a contradiction, that $|G_{p_2}| = p_2^2$. By hypothesis, $G_{p_1}G_{p_2}$ is a generalized smooth group of order $p_1^2p_2^2$. According to Lemma 1.2, G_{p_1} is an elementary abelian minimal normal subgroup of $G_{p_1}G_{p_2}$, G_{p_2} is cyclic and each subgroup of G_{p_2} operates irreducibly on G_{p_1}. If $L \nleq G_{p_1}G_{p_2}$, where L is a minimal normal subgroup of G, then G has a maximal subgroup of index p_2 which is not generalized smooth, a contradiction. Thus $L < G_{p_1}G_{p_2}$ and consequently, $L = G_{p_1}$. If G_{p_3} is not cyclic, then LG_{p_3} is not generalized smooth which contradicts our hypothesis. Therefore G_{p_3} is cyclic and by [13, Theorem 13.3.1, p. 383], $G_{p_2}G_{p_3}$ is supersolvable. It follows that $G_{p_2}G_{p_3}$ has a subgroup K, say, of index p_2 and hence LK is a proper subgroup of G but is not generalized smooth, a contradiction. This final contradiction establishes that $|G_{p_j}| = p_j$ $(j = 2, 3)$. Hence $|G| = p_1^2p_2p_3$, completing our proof. □

Proposition 2.3. *Assume that G is a group with $|\pi(G)| \geq 2$ and all maximal subgroups of G are generalized smooth. Then the following conditions are equivalent:*

- **(i):** *G has a normal subgroup of prime order.*
- **(ii):** *G has a permutable subgroup of prime order.*

(iii): *G has an S-permutable subgroup of prime order.*

Proof. Clearly, $(i) \Rightarrow (ii) \Rightarrow (iii)$. So assume that G has an S-permutable subgroup H of order p. By [11, Lemma A, p. 287], $O^p(G) \leq N_G(H)$. If G_p is abelian, then $N_G(H) = G_pO^p(G) = G$ and hence $H \lhd G$. Now, assume that G_p is not abelian. According to Lemma 1.2, $|G_p| = p^3$. Assume that H is not normal in G. Since G_p has a maximal subgroup G_p^*, such that $H \lhd G_p^*$, we get $N_G(H) = G_p^*O^p(G)$ is a maximal subgroup of G. Applying Lemma 1.2, $|N_G(H)| = p^2q$. It is obvious that both G_p and $N_G(H)$ are supersolvable groups with $[G : G_p] = q$ and $[G : N_G(H)] = p$. By [1, Theorem 5, p. 5], G is supersolvable and hence G has a normal subgroup of prime order. □

Lemma 2.4. *Let G be a group with $|\pi(G)| > 3$. Then all maximal subgroups of G are generalized smooth and G has an S-permutable subgroup of prime order if and only if one of the following holds:*

- **(i):** *$|G| = p_1p_2p_3p_4$, where p_1, p_2, p_3 and p_4 are distinct primes.*
- **(ii):** *G is cyclic of square free order.*
- **(iii):** *$G = G_pK$, where $|G_p| = p_1$, $G_p \lhd G$ and K is cyclic of order $p_2p_3 \cdots p_m$ and operates faithfully on G_p where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in 1, 2, \ldots, m$.*

Proof. Assume that all maximal subgroups of G are generalized smooth groups and G has an S-permutable subgroup of prime order. By Proposition 2.3, G has a normal subgroup H of prime order p. We argue that G is of square free order. Let G_q be a Sylow q-subgroup of G for some prime $q \in \pi(G)$ with $q \neq p$ and $|G_q| > q$. By hypothesis, HG_q is a generalized smooth group with $H \lhd HG_q$. Applying Lemma 1.2, $|G_q| = q^2$ and HG_q must be a maximal subgroup of G as no generalized smooth group can contain it. Hence G is a non-solvable group as $[G : HG_q]$ is divisible at least by two different primes. If G_q is self normalizing, then by Burnside's theorem [9, Hauptsatz 2.6], G has a normal q-complement U, say. By Lemma 1.2, U is of square free order as U is a generalized smooth group with $|\pi(U)| \geq 3$ and $H < U$. Therefore U and G/U are solvable groups and hence G is solvable, a contradiction. Similarly, if $G_q \lhd G$, then we get the same contradiction as $G/G_q \cong U$ is solvable and hence G is solvable. Thus $G_q \lneqq N_G(G_q) \lneqq G$. Since HG_q is a maximal subgroup of G and H is a permutable subgroup in G, we get $N_G(G_q) = H \times G_q$ is abelian. By [10, Exercise 10.5.7, p. 309], G is solvable and once again we get a contradiction. Thus $|G_q| = q$ for all $q \in \pi(G)$ with $q \neq p$. Let $|G_p| > p$. Suppose that G has a maximal subgroup M which is not supersolvable. Since $|G_q| = q$ for all $q \in \pi(G)$ with $q \neq p$, we get, by Lemma 1.2, either $|M| = p^2q$ or $M \cong A_5$ and consequently, $H \nleq M$. This implies that $G = HM$ and hence $|\pi(M)| = |\pi(G)| > 3$, a contradiction. Therefore all maximal subgroups of G are supersolvable and hence G is solvable. Since $|\pi(G)| > 3$, then all maximal subgroups of G contains G_p is not generalized smooth, a contradiction. Thus $|G_p| = p$ and hence G is of square free order.

If $n = 4$, then $|G| = p_1p_2p_3p_4$, where p_1, p_2, p_3 and p_4 are distinct primes. Additionally, if all maximal subgroups of G are cyclic, then G is cyclic of square free order. Now assume that G has a maximal subgroup $M = M_{p_1}A$ where M_{p_1} is a normal Sylow p_1-subgroup of M, A is cyclic and operates faithfully on M_{p_1}. Suppose G has a maximal subgroup K with $[G : K] = p_1$. If K is not cyclic, then K satisfies (iv) in Lemma 1.2. Consider $K = K_{p_2}B$ and B operates faithfully on K_{p_2}. However, each maximal subgroup of G contains both M_{p_1} and K_{p_2} is not generalized smooth, a contradiction. Therefore K is cyclic and (iii) holds. For the converse, clearly, if G satisfies (i), (ii) or (iii), then all maximal subgroups of G are generalized smooth and G has a normal subgroup of prime order, concluding our proof. □

Lemma 2.5. *Assume that G is a group with $|\pi(G)| = 2$ and $n > 3$. Then all maximal subgroups of G are totally smooth and G has an S-permutable subgroup of prime order if and only if G is a nonabelian P-group.*

Proof. Let $G = G_pG_q$ where p and q are distinct primes. Firstly, assume that all maximal subgroups of G are totally smooth and G has an S-permutable subgroup of prime order. By Proposition 2.3, we can assume that G possesses a subgroup H with $H \lhd G$ and $|H| = p$. As $n > 3$, if $H = G_p$, then $|G_q| \geq q^3$, and hence G has a maximal subgroup of index q which is not totally smooth, a contradiction. Thus H is a proper subgroup of G_p. By hypothesis, HG_q is a totally smooth group with $H \lhd HG_q$. According to Lemma 1.1, $|G_q| = q$. Since G_p is totally smooth, G_p is either cyclic or elementary abelian. If G_p is cyclic, then by [13, Theorem 13.3.1, p. 383], G is supersolvable. It follows that G has a maximal subgroup of index p which is not totally smooth, a contradiction. Thus G_p is elementary abelian. By [13, Theorem 9.3.7, p. 225], H has a complement U, say, in G. By hypothesis, U is a totally smooth group with $|\pi(U)| \geq 2$. Applying Lemma 1.1, U is either cyclic of square free order or a nonabelian P-group. If U is of square free order, then $n = 3$ which contradicts our hypothesis that $n > 3$. Thus U is a nonabelian P-group with $p > q$. Therefore G_q normalizes every subgroup of G_p and hence G is a nonabelian P-group. Now assume that $G = G_pG_q$ is a nonabelian P-group with $p > q$. Then all subgroups of G are totally smooth and each subgroup of G_p is normal in G. This completes the proof. □

Lemma 2.6. *Suppose that G is a group with $|\pi(G)| = 2$ and $n > 4$. Suppose further that G has an S-permutable subgroup of prime order. Then all maximal subgroups of G are generalized smooth if and only if all maximal subgroups of G are totally smooth.*

Proof. According to Lemma 1.2, each totally smooth group is generalized smooth, thus establishing the sufficiency condition. So assume that all maximal subgroups of G are generalized smooth and G has an S-permutable subgroup H of prime order p. Consider $G = G_pG_q$, where G_p, G_q are Sylow subgroups of G and p, q are distinct primes. We argue that $|G_q| = q$, so assume, for a contradiction, $|G_q| > q$. If $|G_q| \geq q^3$, then HG_q is a proper subgroup of G but is not generalized smooth, a contradiction. Thus $|G_q| = q^2$. Since $n > 4$, $|G_p| \geq p^3$. Let $G_q \lhd G$ and choose G_p^* be a maximal subgroup of G_p containing

H. Then $G_qG_p^*$ is a proper subgroup of G but is not generalized smooth, a contradiction. Additionally, if G_q is self normalizing, then by Burnside's theorem [9, Hauptsatz 2.6], $G_p \lhd G$ and hence G_q has a subgroup G_q^*, say, such that $G_pG_q^*$ is a maximal subgroup of G. According to Lemma 1.2, $G_pG_q^*$ is a nonabelian P-group with $p > q$. Then G_p is elementary abelian and G_q^* normalizes every subgroup of G_p. By [13, Theorem 9.3.7, p. 225], H has a complement in G which is not generalized smooth, a contradiction. Thus $G_q \lneqq N_G(G_q) \lneqq G$. If $H \nleq N_G(G_q)$, then $G = HN_G(G_q)$ and once again, G has a maximal subgroup of index p containing H and is not generalized smooth, a contradiction. Thus $H < N_G(G_q)$. According to Lemma 1.2, since $N_G(G_q)$ is generalized smooth, we conclude that $N_G(G_q) = G_q \times H$ is abelian group. Consequently, $N_G(G_q) = C_G(G_q)$ and by [13, Theorem 6.2.9, p. 137], $G_p \lhd G$ which leads to the same previous contradiction. Thus $|G_q| = q$ and hence $|G_p| \geq p^4$ as $n > 4$.

Suppose that G has a maximal subgroup M, say, which is not totally smooth. Since $|G_q| = q$ and $|G_p| \geq p^4$, we get by Lemma 1.2 that $|M| = p^2q$. Then G is not supersolvable as $[G : M] \geq p^2$. By Lemma 1.2, G_p is either cyclic or elementary abelian. By [13, Theorem 13.3.1, p. 383], if G_p is cyclic, then G is supersolvable, a contradiction. Thus G_p is elementary abelian. By [13, Theorem 9.3.7, p. 225], H has a complement in G of order p^eq ($e \geq 3$) which is a nonabelian P-group with $q \mid p-1$. By [15, Corollary 1.10, p. 6], G is supersolvable and once again we get a contradiction. This final contradiction proves that M is totally smooth and hence all maximal subgroups of G are totally smooth and we are done. □

By Lemma 2.5 and Lemma 2.6, we get the following corollary:

Corollary 2.7. *Let G be a group with $|\pi(G)| = 2$ and $n > 4$. Then all maximal subgroups of G are generalized smooth and G has an S-permutable subgroup of prime order if and only if G is a nonabelian P-group.*

By Lemma 1.1, Lemma 1.2 and our results, we can conclude the following corollaries:

Corollary 2.8. *Assume that G is a group with $n > 3$ and $|\pi(G)| \geq 2$. Then the following conditions are equivalent:*

- **(i):** *G is a totally smooth group.*
- **(ii):** *all maximal subgroups of G are totally smooth groups and G has an S-permutable subgroup of prime order.*

Corollary 2.9. *Assume that G is a group with $n > 4$ and $|\pi(G)| \geq 2$. Then the following conditions are equivalent:*

- **(i):** *G is a generalized smooth group.*
- **(ii):** *all maximal subgroups of G are generalized smooth groups and G has an S-permutable subgroup of prime order.*

It is well known that, a group G with $n = 3$ (*or* $n = 4$) is a group of order $p_1p_2p_3$ (*or* $p_1p_2p_3p_4$), where p_1, p_2, p_3 and p_4 are not necessary distinct primes. Clearly, all maximal subgroups of G are totally (or generalized) smooth, but it is not necessary that G has an S-permutable subgroup of prime order. Therefore by the previous lemmas, we can rewrite Theorem A, Theorem B, Theorem C and Theorem D as follows:

Theorem $A^\backprime$. Assume that G is a group with $|\pi(G)| = 2$. Assume further that G has an S-permutable subgroup of prime order. Then all maximal subgroups of G are totally smooth if and only if G is a nonabelian P-group or $|G| = p_1^e p_2$ $(e = 1, 2)$ where p_1 and p_2 are distinct primes.

Theorem $B^\backprime$. Assume that G is a group with $|\pi(G)| \geq 3$. Then all maximal subgroups of G are totally smooth if and only if G is cyclic of square free order or $|G| = p_1p_2p_3$ where p_1, p_2 and p_3 are distinct primes.

Theorem $C^\backprime$. Assume that G is a group with $|\pi(G)| = 2$. Assume further that G has an S-permutable subgroup of prime order. Then all maximal subgroups of G are generalized smooth if and only if G is a nonabelian P-group or G is of order $p_1^i p_2^j$ where p_1 and p_2 are distinct primes and i, j are positive integers with $i + j \leq 4$.

Theorem $D^\backprime$. Assume that G is a group with $|\pi(G)| \geq 3$. Assume further that G has an S-permutable subgroup of prime order. Then all maximal subgroups of G are generalized smooth if and only if one of the following holds:

- **(i):** $|G| = p_1p_2p_3p_4$ or $p_1^e p_2p_3$ $(e = 1, 2)$, where p_1, p_2, p_3 and p_4 are distinct primes.
- **(ii):** G is cyclic of square free order.
- **(iii):** $G = G_pA$, where $|G_p| = p_1$, $G_p \lhd G$ and A is cyclic of order $p_2p_3 \cdots p_m$ and operates faithfully on G_p where p_i are primes with $p_i \neq p_j$ for $i \neq j$ and $i, j \in 1, 2, \ldots, m$.

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