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## THE CLIQUE NUMBER OF THE INTERSECTION GRAPH OF SOME CYCLIC GROUPS

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**ABSTRACT.** For a nontrivial finite group  $G$ , the intersection graph  $\Gamma(G)$  of  $G$  is a simple undirected graph whose vertices are the nontrivial proper subgroups of  $G$  and two vertices are joined by an edge if and only if they have a nontrivial intersection. In this paper, we obtain the clique number of the intersection graph of cyclic groups whose orders have four prime divisors. Moreover we find the clique number of the intersection graph of cyclic groups of order  $n$  such that all powers of prime divisors of  $n$  are equal. As a special case, we find the clique number of this graph for the cyclic groups of the square-free orders.

### 1. Introduction

Let  $G$  be a group. There are several ways to associate a graph to  $G$  (see [7] and the references therein). In this paper, we define the intersection graph of  $G$ , denoted by  $\Gamma(G)$ , is a graph having all the non-trivial proper subgroups of  $G$  as its vertices and two distinct vertices in  $\Gamma(G)$  are adjacent if and only if their intersection is non-trivial. Motivated by [6], Csákány and Pollák [8] defined the intersection graph of a finite group  $G$  in 1969. Zelinka in [19] considered the intersection graphs of finite abelian groups. Taeri and Ahmadi [1] characterized all finite groups with planar intersection graphs (see also [12]).

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Kayacan in [11] considered the connectivity of the intersection graphs of finite groups. In 2010, Shen in [17] determined all finite groups with disconnected intersection graphs. In 2021, Freedman [9] proved that  $\Gamma(G)$  is connected with diameter at most 5 when  $G$  is non-abelian simple. In [14, 15], Shahsavari and Khosravi characterized some families of finite simple groups using their intersection graphs. The reader can see other results about  $\Gamma(G)$  in [1, 5, 16, 18]. Until now, there are only a few results on the clique number of the intersection graph of finite groups.

Let  $\Gamma$  be a simple graph. The set of vertices of every complete subgraph of  $\Gamma$  is called a clique of  $\Gamma$ . A maximal clique is a clique that cannot be extended by including one more adjacent vertex. A maximum clique of  $\Gamma$ , is a clique, such that there is no clique with more vertices. Moreover, the clique number of  $\Gamma$ , denoted by  $\omega(\Gamma)$ , is the number of vertices in a maximum clique in  $\Gamma$ .

In [2], we showed that if  $G$  is a finite group such that  $\omega(\Gamma(G)) < 13$ , then  $G$  is solvable and  $\omega(\Gamma(A_5)) = 13$  where  $A_5$  is the alternative group on 5 letters. Also, we determined all groups  $G$  with  $\omega(\Gamma(G)) \leq 4$ . In [4], we showed that  $\omega(\Gamma(\text{PSL}(2, 7))) = 25$  and classify all semisimple groups  $G$  with  $\omega(\Gamma(G)) \leq 25$ . We find  $\omega(\Gamma(C_n))$  where  $C_n$  is a cyclic group of order  $n$  having at most three prime divisors [3]. It seems not easy to find  $\omega(\Gamma(C_n))$  in general. So we obtain it for some special cases in this paper.

In the sequel, assume that  $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$  is a positive integer where  $r \geq 1$ ,  $p_1, p_2, \dots, p_r$  are distinct primes and  $k_1 \geq k_2 \geq \cdots \geq k_r \geq 1$ . We use  $\pi(n)$  and  $d(n)$  to denote the set of all prime divisors and the number of the divisors of  $n$ , respectively. Therefore  $d(n) = (k_1 + 1)(k_2 + 1) \cdots (k_r + 1)$ . It is easy to see that  $\omega(\Gamma(C_{p_1^{k_1}})) = d(p_1^{k_1}) - 2 = k_1 - 1$  (see [2, Lemma 4.1 (1)]). In [3], we have determined  $\omega(\Gamma(C_n))$  when  $r \leq 3$  (see Propositions 2.1 and 2.2). In the present paper we first find  $\omega(\Gamma(C_n))$  when  $r = 4$  (see Theorem 2.3) and then we obtain  $\omega(\Gamma(C_n))$  when  $r$  is an arbitrary positive integer and  $k_1 = k_2 = \cdots = k_r$  (see Theorem 3.2 and 3.3).

In this paper all groups will be finite and we use the usual notation. Also  $\mathcal{S}(G)$  is the set of all subgroups of  $G$ . The rest of the notation is standard and can be found mainly in [13].

## 2. cyclic groups of order with at most four prime divisors

It is known that  $C_n$  has a unique subgroup of each order dividing  $n$ . It follows that  $|\mathcal{S}(C_n)| = d(n)$  and so the number of vertices of  $\Gamma(C_n)$  is  $d(n) - 2$ . For given subgroups  $H$  and  $K$  of  $C_n$ , we have  $H \cap K \neq 1$  if and only if  $\gcd(|H|, |K|) \neq 1$ . This is useful in the sequel.

**Proposition 2.1.** *Let  $n = p_1^{k_1} p_2^{k_2}$ . Then  $\omega(\Gamma(C_n)) = d(\frac{n}{p_1}) - 1$ .*

*Proof.* See [3, Theorem 3.2]. Note that  $d(\frac{n}{p_1}) - 1 = k_1 k_2 + k_1 - 1$  is the number of proper subgroups  $H$  of  $C_n$  such that  $p_1$  divides  $|H|$ . □

**Proposition 2.2.** *Let  $n = p_1^{k_1} p_2^{k_2} p_3^{k_3}$ .*

- (1) *If  $k_1 \geq k_2 k_3$ , then  $\omega(\Gamma(C_n)) = d(\frac{n}{p_1}) - 1$ .*
- (2) *If  $k_1 < k_2 k_3$ , then  $\omega(\Gamma(C_n)) = d(\frac{n}{p_2 p_3}) - 1 + k_1(k_2 + k_3)$ .*

*Proof.* See [3, Theorem 3.3]. Note that if  $H, K < C_n$  such that  $|\pi(|H|)| \geq 2$  and  $|\pi(|K|)| \geq 2$ , then  $H \cap K \neq 1$ . □

In the following, we state and prove the first main result:

**Theorem 2.3.** *Let  $n = p_1^{k_1} p_2^{k_2} p_3^{k_3} p_4^{k_4}$ .*

- (1) *If  $k_1 \geq k_2 k_3 k_4$ , then  $\omega(\Gamma(C_n)) = d(\frac{n}{p_1}) - 1$ .*
- (2) *If  $k_1 < k_2 k_3 k_4$ , then*

$$\omega(\Gamma(C_n)) = d_3(n) + k_1 k_2 + k_1 k_3 + \max\{k_1 k_4, k_2 k_3\} - 1$$

where  $d_3(n)$  is the number of divisors of  $n$  with at least three distinct prime divisors.

*Proof.* Let  $G = C_n$ . We introduce some subsets of the vertex set of  $\Gamma(G)$  as follows:

- $V_{p_i}$  is the set of all subgroups  $H$  of  $G$  such that  $\pi(|H|) = \{p_i\}$  for  $1 \leq i \leq 4$ . Clearly  $|V_{p_i}| = k_i$ .
- $V_{p_i p_j}$  is the set of all subgroups  $H$  of  $G$  such that  $\pi(|H|) = \{p_i, p_j\}$  for  $1 \leq i \neq j \leq 4$ . Clearly  $|V_{p_i p_j}| = k_i k_j$  and  $V_{p_i p_j} = V_{p_j p_i}$ .
- $V_{p_i p_j p_l}$  is the set of all subgroups  $H$  of  $G$  such that  $\pi(|H|) = \{p_i, p_j, p_l\}$  for  $1 \leq i < j < l \leq 4$ . Clearly  $|V_{p_i p_j p_l}| = k_i k_j k_l$ .
- $V_{p_1 p_2 p_3 p_4}$  is the set of all proper subgroups  $H$  of  $G$  such that  $\pi(|H|) = \{p_1, p_2, p_3, p_4\}$ . Clearly  $|V_{p_1 p_2 p_3 p_4}| = k_1 k_2 k_3 k_4 - 1$ .
- $X := (\cup_{1 \leq i < j < l \leq 4} V_{p_i p_j p_l}) \cup V_{p_1 p_2 p_3 p_4}$ .

Clearly,

$$\{V_{p_i}, V_{p_j p_l}, X : 1 \leq i \leq 4, 1 \leq j < l \leq 4\}$$

forms a partition for  $\mathcal{S}(G) \setminus \{1, G\}$ .

Now set

$$(*) \quad B_{p_i} := V_{p_i p_j} \cup V_{p_i p_l} \cup V_{p_i p_t}$$

and

$$(**) \quad A_{p_i p_j p_l} := V_{p_i p_j} \cup V_{p_i p_l} \cup V_{p_j p_l}$$

where  $i, j, l, t$  are distinct. Clearly  $B_{p_i}$  and  $A_{p_i p_j p_l}$  are cliques in  $\Gamma(G)$  for every  $i, j, l$ . If  $T < G$  such that  $|\pi(|T|)| = 2$  and  $T \notin B_{p_i} \cup A_{p_j p_l p_t}$  for some  $i, j, l, t$ , then  $B_{p_i} \cup \{T\}$  and  $A_{p_j p_l p_t} \cup \{T\}$  are not cliques. So  $B_{p_i}$  and  $A_{p_j p_l p_t}$  are all maximal cliques whose members are of orders with two distinct prime divisors. Assume first that

$$P_i = \{H < G : p_i \text{ divides } |H|\},$$

$$D_i = B_{p_i} \cup X \quad 1 \leq i \leq 4$$

and

$$E_{ijl} := A_{p_i p_j p_l} \cup X \quad 1 \leq i < j < l \leq 4.$$

It is clear that  $P_i, D_i$  and  $E_{ijl}$  are cliques in  $\Gamma(G)$  for every distinct  $i, j, l$ . We claim that all maximal cliques in  $\Gamma(G)$  are  $P_i, D_i$  and  $E_{ijl}$  for all distinct  $i, j, l$ .

Suppose that  $W$  is a maximal clique in  $\Gamma(G)$  and  $H \in W$  such that  $|\pi(|H|)|$  has the minimum value among all elements in  $W$ . It is easy to see that  $|\pi(|H|)| \leq 2$ . Now we consider two cases:

**Case 1.** If  $|\pi(|H|)| = 1$ , then  $\pi(|H|) = \{p_i\}$  for some  $1 \leq i \leq 4$ . Since  $W$  is a clique and  $H \in W$ ,  $p_i$  divides the order of every element of  $W$  and so  $W \subseteq P_i$ . By the maximality of  $W$ , we have  $W = P_i$ , as claimed.

**Case 2.** Suppose that  $|\pi(|H|)| = 2$ . Note that if  $K, L < G$  such that  $|\pi(|K|)| \geq 2$  and  $|\pi(|L|)| \geq 3$ , then  $\gcd(|K|, |L|) \neq 1$  and so  $K \cap L \neq 1$ . Since  $W$  is a maximal clique in  $\Gamma(G)$ , we have  $X \subset W$ . By the properties of  $B_{p_i}, A_{p_j p_i p_l}$  and the maximality of  $W$ , we get either  $W = B_{p_i} \cup X$  for some  $i$  or  $W = A_{p_i p_j p_l} \cup X$  for some distinct  $i, j, l$ . Therefore either  $W = D_i$  for some  $i$  or  $W = E_{ijl}$  for some distinct  $i, j, l$ . This completes the proof of the claim. So

$$\omega(\Gamma(G)) = \max\{|P_i|, |D_i|, |E_{ijl}| : 1 \leq i < j < l \leq 4\}.$$

Clearly,  $|P_i| = d(\frac{n}{p_i}) - 1, |D_i| = |B_{p_i}| + |X|$  for each  $i$  and  $|E_{ijl}| = |A_{p_i p_j p_l}| + |X|$ . By (\*) and (\*\*), we see that  $|B_{p_i}| = k_i k_j + k_i k_l + k_i k_t$  and  $|A_{p_i p_j p_l}| = k_i k_j + k_i k_l + k_j k_l$ . Since  $k_1 \geq k_2 \geq k_3 \geq k_4$ , we conclude that  $|P_1| \geq |P_i|, |D_1| \geq |D_i|$  for each  $i > 1$  and  $|E_{123}| \geq |E_{ijl}|$  for each  $(i, j, l) \neq (1, 2, 3)$ . Therefore  $\omega(\Gamma(G)) = \max\{|P_1|, |D_1|, |E_{123}|\}$ .

On the other hand we have  $P_1 = V_{p_1} \cup B_{p_1} \cup (X \setminus \{V_{p_2 p_3 p_4}\})$  and so  $|P_1| = k_1 + |B_{p_1}| + (|X| - k_2 k_3 k_4)$ . Moreover  $|D_1| = |B_{p_1}| + |X|$  and  $|E_{123}| = |A_{p_1 p_2 p_3}| + |X|$ .

First, assume that  $k_1 \geq k_2 k_3 k_4$ . Then  $|P_1| \geq |D_1| \geq |E_{123}|$  and so  $\omega(\Gamma(G)) = |P_1| = d(\frac{n}{p_1}) - 1$ , as desired.

Now assume that  $k_1 < k_2 k_3 k_4$ . Then  $|D_1| > |P_1|$  and so  $\omega(\Gamma(G)) = \max\{|D_1|, |E_{123}|\}$ . Note that  $|B_{p_1}| = k_1 k_2 + k_1 k_3 + k_1 k_4$  and  $|A_{p_1 p_2 p_3}| = k_1 k_2 + k_1 k_3 + k_2 k_3$ . If  $k_1 k_4 \geq k_2 k_3$ , then  $\omega(\Gamma(G)) = |D_1|$ , as wanted. If  $k_1 k_4 \leq k_2 k_3$ , then  $\omega(\Gamma(G)) = |E_{123}|$ . This completes the proof.  $\square$

### 3. the powers of all prime divisors of $n$ are equal

In this section we find  $\omega(\Gamma(C_n))$  where  $n = p_1^k p_2^k \cdots p_r^k$  such that  $k \geq 1$  and  $r > 2$ .

For positive integers  $i \leq r$ , we denote  $C(r, i) = \frac{r!}{i!(r-i)!}$ . Also we define  $C(r, 0) = C(r, r) = 1$ . Clearly,  $C(r, i) = C(r, r - i)$  for each  $i \leq r$ . It is well-known that if  $\Omega$  is a set with  $r$  elements, then the number of subsets of  $\Omega$  with  $i$  elements is  $C(r, i)$ . So we have

$$(1) \quad C(r, 0) + C(r, 1) + \cdots + C(r, r - 1) + C(r, r) = 2^r.$$

If  $r > 1$  is odd, then

$$(2) \quad \sum_{i=1}^{\frac{r-1}{2}} C(r, i) = \sum_{i=\frac{r+1}{2}}^{r-1} C(r, i) = 2^{r-1} - 1.$$

If  $r$  is even, then

$$(3) \quad \sum_{i=1}^{\frac{r}{2}-1} C(r, i) + \frac{1}{2}C(r, \frac{r}{2}) = \sum_{i=\frac{r}{2}+1}^{r-1} C(r, i) + \frac{1}{2}C(r, \frac{r}{2}) = 2^{r-1} - 1.$$

The following result is useful in the sequel.

**Lemma 3.1.** (see [13, 1.3.11 ]) *Let  $G$  be a group and  $H$  and  $K$  be subgroups of  $G$ . Then  $|G : H \cap K| \leq |G : H||G : K|$ , with equality if the indices  $|G : H|$  and  $|G : K|$  are coprime.*

First, we state and prove the main result for the cyclic groups of orders having odd prime divisors with equal powers.

**Theorem 3.2.** *Let  $n = p_1^k p_2^k \cdots p_r^k$  where  $r > 1$  is an odd integer. Then*

$$\omega(\Gamma(C_n)) = \sum_{i=\frac{r+1}{2}}^{r-1} C(r, i)k^i + k^r - 1.$$

*Proof.* Assume that  $G = C_n$  and  $\omega = \sum_{i=\frac{r+1}{2}}^{r-1} C(r, i)k^i + k^r - 1$ . Also  $\mathfrak{S}_i$  denotes the set of all subgroups of  $G$  of orders having  $i$  prime divisors for  $1 \leq i \leq r$ . It is easy to see that  $|\mathfrak{S}_i| = C(r, i)k^i$ .

If  $H \in \mathfrak{S}_m$  and  $T \in \mathfrak{S}_l$  such that  $m \geq l \geq \frac{r+1}{2}$ , then  $|G : H|$  and  $|G : T|$  have  $r - m$  and  $r - l$  prime divisors respectively. Therefore  $|G : H \cap T|$  has at most  $2r - (m + l)$  prime divisors by Lemma 3.1. Since  $m + l \geq r + 1$ , we have  $|G : H \cap T|$  has at most  $r - 1$  prime divisors. But  $|G|$  has  $r$  prime divisors which implies that  $H \cap T \neq 1$ . Hence  $\mathfrak{S}_{\frac{r+1}{2}} \cup \mathfrak{S}_{\frac{r+1}{2}+1} \cup \cdots \cup \mathfrak{S}_{r-1}$  forms a clique in  $\Gamma(G)$ . Moreover, if  $H \in \mathfrak{S}_r \setminus \{G\}$  and  $K \in \mathcal{S}(G) \setminus \{1, G\}$ , then  $\gcd(|H|, |K|) \neq 1$  and so  $H \cap K \neq 1$ . Consequently  $(\bigcup_{i=\frac{r+1}{2}}^{r-1} \mathfrak{S}_i) \cup (\mathfrak{S}_r \setminus \{G\})$  forms a clique in  $\Gamma(G)$ . By the definition of clique number of a graph, we have  $\omega(\Gamma(G)) \geq |(\bigcup_{i=\frac{r+1}{2}}^{r-1} \mathfrak{S}_i) \cup (\mathfrak{S}_r \setminus \{G\})|$ . Since  $|\mathfrak{S}_i| = C(r, i)k^i$  and  $\mathfrak{S}_i \cap \mathfrak{S}_j = \emptyset$  for each  $i \neq j$ , we deduced that

$$(*) \quad \omega(\Gamma(G)) \geq \sum_{i=\frac{r+1}{2}}^{r-1} C(r, i)k^i + k^r - 1 = \omega.$$

Now suppose that  $W$  is an arbitrary clique in  $\Gamma(G)$ . It is enough to show that  $|W| \leq \omega$ .

Set  $W_i = W \cap \mathfrak{S}_i$  for each  $1 \leq i \leq r$ . If  $1 \leq i \leq \frac{r-1}{2}$ , then define  $f_i : W_i \cup W_{r-i} \rightarrow \mathfrak{S}_{r-i}$  by

$$f_i(H) = \begin{cases} H & H \in W_{r-i} \\ H' & H \in W_i \end{cases}$$

where  $H'$  is defined as follows.

Note that if  $H \in W_i$ , then  $|H| = p_{j_1}^{\alpha_{j_1}} \cdots p_{j_i}^{\alpha_{j_i}}$  such that  $1 \leq j_1 < j_2 < \cdots < j_i \leq r$  and  $1 \leq \alpha_{j_i} \leq k$  for every  $t$ . Suppose that

$$\{p_1, p_2, \dots, p_r\} \setminus \{p_{j_1}, \dots, p_{j_i}\} = \{p_{l_1}, \dots, p_{l_{r-i}}\}$$

where  $l_1 < l_2 < \dots < l_{r-i}$ . Now we consider  $H' < G$  such that  $|H'| = p_{l_1}^{\alpha_{j_1}} \dots p_{l_i}^{\alpha_{j_i}} p_{l_{i+1}} \dots p_{l_{r-i}}$ . Therefore  $H \cap H' = 1$  and  $H' \in \mathfrak{S}_{r-i}$ . Since  $H \in W_i$ , we have  $H' \in \mathfrak{S}_{r-i} \setminus W_{r-i}$ . We claim that  $f_i$  is a one to one map for each  $1 \leq i \leq \frac{r-1}{2}$ .

Assume that  $H_1, H_2 \in W_s \cup W_{r-s}$  for some  $1 \leq s \leq \frac{r-1}{2}$  such that  $f_s(H_1) = f_s(H_2)$ . If  $H_1 \in W_{r-s}$  and  $H_2 \in W_s$ , then  $H_1 = H_2'$ . By the definition of  $H_2'$ , we have  $H_2 \cap H_2' = 1$  and so  $H_2 \cap H_1 = 1$ , a contradiction since  $H_1, H_2 \in W$ . Therefore either  $H_1, H_2 \in W_s$  or  $H_1, H_2 \in W_{r-s}$ . The first case gives  $H_1' = H_2'$  since  $f_s(H_1) = f_s(H_2)$ . It follows that

$$|H_1'| = |H_2'| = p_{i_1}^{\alpha_{i_1}} p_{i_2}^{\alpha_{i_2}} \dots p_{i_s}^{\alpha_{i_s}} p_{i_{s+1}} \dots p_{i_{r-s}}.$$

By the definition of  $f_s$ , we get  $|H_1| = |H_2| = p_{j_1}^{\alpha_{i_1}} \dots p_{j_s}^{\alpha_{i_s}}$  where

$$\{p_{j_1}, p_{j_2}, \dots, p_{j_s}\} = \{p_1, \dots, p_r\} \setminus \{p_{i_1}, p_{i_2}, \dots, p_{i_{r-s}}\}.$$

Since  $G$  is cyclic, we have  $H_1 = H_2$ , as wanted. The second case gives  $H_1 = H_2$  by the definition of  $f_s$ . This completes the proof of the claim. Hence  $|W_i \cup W_{r-i}| \leq |\mathfrak{S}_{r-i}|$  for each  $1 \leq i \leq \frac{r-1}{2}$ . Since  $W_r \subseteq \mathfrak{S}_r \setminus \{G\}$ , we deduce that

$$\begin{aligned} |W| &= \left| \bigcup_{i=1}^r W_i \right| = \left| \bigcup_{i=1}^{\frac{r-1}{2}} (W_i \cup W_{r-i}) \right| + |W_r| \\ (3.1) \quad &\leq \left( \sum_{i=1}^{\frac{r-1}{2}} |\mathfrak{S}_{r-i}| \right) + k^r - 1 = \omega \end{aligned}$$

Hence (\*) and (3.1) give the result. □

In the following, we prove the third main result. More precisely, we find the clique number of the intersection graph of cyclic groups of orders having even prime divisors with equal powers.

**Theorem 3.3.** *Let  $n = p_1^k p_2^k \dots p_r^k$  be a positive integer such that  $r \geq 4$  is even. Then*

$$\omega(\Gamma(C_n)) = \sum_{i=\frac{r}{2}+1}^{r-1} C(r, i) k^i + \frac{1}{2} C(r, \frac{r}{2}) k^{\frac{r}{2}} + k^r - 1.$$

*Proof.* Assume that  $G = C_n$  and  $\omega = \sum_{i=\frac{r}{2}+1}^{r-1} C(r, i) k^i + \frac{1}{2} C(r, \frac{r}{2}) k^{\frac{r}{2}} + k^r - 1$ . Let  $\mathfrak{S}_i$  denote the set of all subgroups of  $G$  of orders having  $i$  prime divisors where  $1 \leq i \leq r$ . It is easy to see that  $|\mathfrak{S}_i| = C(r, i) k^i$  for every  $i$ .

By a similar argument to that of the proof of Theorem 3.2, we see that  $\mathfrak{S}_{\frac{r}{2}+1} \cup \mathfrak{S}_{\frac{r}{2}+2} \cup \dots \cup \mathfrak{S}_{r-1} \cup (\mathfrak{S}_r \setminus \{G\})$  is a clique in  $\Gamma(G)$ . Set

$$\mathfrak{S}_{\frac{r}{2}}^1 := \{H \in \mathfrak{S}_{\frac{r}{2}} : p_1 \text{ divides } |H|\}.$$

Therefore  $\mathfrak{S}_{\frac{r}{2}}^1$  is a clique in  $\Gamma(G)$  and  $|\mathfrak{S}_{\frac{r}{2}}^1| = C(r-1, \frac{r}{2}-1) k^{\frac{r}{2}} = \frac{1}{2} C(r, \frac{r}{2}) k^{\frac{r}{2}}$ . Moreover,

$$\mathfrak{S}_{\frac{r}{2}+1} \cup \mathfrak{S}_{\frac{r}{2}+2} \cup \dots \cup \mathfrak{S}_{r-1} \cup (\mathfrak{S}_r \setminus \{G\}) \cup \mathfrak{S}_{\frac{r}{2}}^1$$

forms a clique in  $\Gamma(G)$  (similar to the proof of Theorem 3.2, applying Lemma 3.1). By the definition of the clique number, we have

$$(*) \quad \omega(\Gamma(G)) \geq \sum_{i=\frac{r}{2}+1}^{r-1} C(r, i)k^i + \frac{1}{2}C(r, \frac{r}{2})k^{\frac{r}{2}} + k^r - 1 = \omega.$$

Now suppose that  $W$  is an arbitrary clique in  $\Gamma(G)$ . We prove that  $|W| \leq \omega$ .

Set  $W_i = W \cap \mathfrak{S}_i$  for each  $1 \leq i \leq r$ . As in the proof of Theorem 3.2, we have  $|W_i \cup W_{r-i}| \leq |\mathfrak{S}_{r-i}|$  for each  $1 \leq i \leq \frac{r}{2} - 1$ . It remains to prove  $|W_{\frac{r}{2}}| \leq \frac{1}{2}|\mathfrak{S}_{\frac{r}{2}}|$ .

Define  $g : W_{\frac{r}{2}} \rightarrow \mathfrak{S}_{\frac{r}{2}} \setminus W_{\frac{r}{2}}$  by  $g(H) = H'$  where  $|H| = p_{j_1}^{\alpha_{j_1}} \cdots p_{j_{\frac{r}{2}}}^{\alpha_{j_{\frac{r}{2}}}}$ ,  $|H'| = p_{l_1}^{\alpha_{j_1}} p_{l_2}^{\alpha_{j_2}} \cdots p_{l_{\frac{r}{2}}}^{\alpha_{j_{\frac{r}{2}}}}$  and  $\{p_1, \dots, p_r\} = \{p_{j_1}, \dots, p_{j_{\frac{r}{2}}}\} \cup \{p_{l_1}, \dots, p_{l_{\frac{r}{2}}}\}$  such that  $j_1 < \dots < j_{\frac{r}{2}}$  and  $l_1 < \dots < l_{\frac{r}{2}}$ . It is easy to see that  $g$  is one to one.

Since  $W_r \subseteq \mathfrak{S}_r \setminus \{G\}$ , we conclude that

$$(3.2) \quad \begin{aligned} |W| &= \left| \bigcup_{i=1}^r W_i \right| = \left| \left( \bigcup_{i=1}^{\frac{r}{2}-1} (W_i \cup W_{r-i}) \right) \cup W_{\frac{r}{2}} \cup W_r \right| \\ &\leq \sum_{i=1}^{\frac{r}{2}-1} |\mathfrak{S}_{r-i}| + \frac{1}{2}|\mathfrak{S}_{\frac{r}{2}}| + k^r - 1 = \omega \end{aligned}$$

By (\*) and (3.2), we have the result. □

As an application, we find  $\omega(\Gamma(C_n))$  when  $n$  is square-free.

**Corollary 3.4.** *Let  $n = p_1 p_2 \cdots p_r$ . Then  $\omega(\Gamma(C_n)) = 2^{r-1} - 1$ .*

*Proof.* Apply Theorems 3.2 and 3.3 for  $k = 1$ . If  $r > 1$  is odd, then by Theorem 3.2, we have  $\omega(\Gamma(C_n)) = \sum_{i=\frac{r+1}{2}}^{r-1} C(r, i)$  and so the result follows from (2).

If  $r$  is even, then by Theorem 3.3, we have

$$\omega(\Gamma(C_n)) = \sum_{i=\frac{r}{2}+1}^{r-1} C(r, i) + \frac{1}{2}C(r, \frac{r}{2}),$$

and hence the result follows from (3). This completes the proof. □

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