https://ijgt.ui.ac.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 15 No. 2 (2026), pp. 55–56.
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ON p-LOCAL FACTORIZATION IN A FACTORIZED FINITE GROUP

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Abstract. Factorizations play an important role in the analysis of the structure of finite groups. In this short note we extend a basic p-local Factorization Theorem in the textbook *The Theory of Finite Groups* by H. Kurzweil and B. Stellmacher that is applied in their Signalizer Functor Chapter. Our result is somewhat surprising and immediately implies their Factorization Theorem and should have other important applications. The proof of this basic result is tricky and elementary and could have been proved several years ago. The proof requires only well known facts about finite group factorizations and Sylow p-subgruops of finite groups. It resulted from the goal of simplifying the proof of the p-Local Factorization Theorem in the excellent H. Kurzweil and B. Stellmacher textbook *Theory of Finite Groups*. Hopefully this elementary result will inspire and lead to new important results on the structure of finite proofs, as is illustrated by the application to the Signalizer Functor Chapter in the textbook.

In the analysis of the structure of a finite group, factorizations play a vital role. In their textbook *The Theory of Finite Groups*, Professors Kurzweil and Stellmacher present a p-Local Factorization Theorem (Theorem 3.2.9) which plays an essential role in their Signalizer Functor Chapter. This very short note presents a very basic, somewhat surprising, result in p-Local Factorization Theory which has their Theorem as an immediate consequence and should have other important applications.

Let p be a prime integer. Let G be a finite group, let N be a normal subgroup of G and let H be a subgroup of G such that $G = NH = HN$.

MSC(2010): Primary: 20E25; Secondary: 20E34.

Keywords: Factorized Finite Groups, p-Local Factorization, covering p-group normalizer.

Article Type: Research paper.

Communicated by Victor Bovdi.

Received: 28 March 2024, Accepted: 05 May 2025, Published Online: 11 May 2025.

Cite this article: M. E. Harris, On p-local factorization in a factorized finite group, *Int. J. Group Theory*, **15** no. 2 (2026) 55–56.
https://dx.doi.org/10.22108/ijgt.2025.141044.1899 .

Let $N_1 = N \cap H$ so that $N_1 \trianglelefteq H$. Let Q be a p-subgroup of H such that $Q \cap N_1 \in \mathrm{Syl}_p(N_1)$. Thus $Q \in \mathrm{Syl}_p(QN_1)$ and $H \cap (QN) = QN_1$.

Our main result (Theorem 1) extends and implies the Kurzweil–Stellmacher Theorem [1, Theorem 3.2.9]:

Theorem 0.1. $N_G(Q) = N_N(Q)N_H(Q)$.

Proof. Here $N_G(Q) \leq N_G(QN) = N_G(QN) \cap (NH) = NN_H(QN) \leq NN_H(QN_1)$ since $H \cap QN = QN_1$. As $N_H(QN_1) = N_1N_H(Q)$ we have $N_G(Q) \leq NN_1N_H(Q) = NN_H(Q)$. Thus $N_G(Q) = N_N(Q)N_H(Q)$ and we are done. □

Remark 0.2. *If $|N|_p = 1$, then we obtain* [1, Theorem 3.2.9].

For our final result, let $N \trianglelefteq G$, $R \in \mathrm{Syl}_p(N)$ and set $K = N_G(R)$. Thus $G = NK$.

Corollary 0.3. *Let P be a p-subgroup of K such that $R \leq P$. Then $N_G(P) = N_K(P)$.*

Proof. Here $G = NK$ and $P \cap N = R \in \mathrm{Syl}_p(N \cap K)$ and $N_N(P) \leq N_N(R) \leq K$. Theorem 0.1 completes our proof. □

References

[1] H. Kurzweil and B. Stellmacher, *The theory of finite groups: an introduction*, translated from the 1998 German original, Springer-Verlag, New York, 2004.

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https://dx.doi.org/10.22108/ijgt.2025.141044.1899