https://ijgt.ui.ac.ir

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 15 No. 3 (2026), pp. 145–160.

https://www.ui.ac.ir

COMPUTATIONAL ASPECTS OF SUBINDICES AND SUBFACTORS WITH CHARACTERIZATION OF FINITE INDEX STABLE GROUPS

MOHAMMAD HADI HOOSHMAND* AND MOEIN YOUSEFIAN ARANI

Abstract. Recently, subindices and subfactors of groups with connections to number theory, additive combinatorics, and factorization of groups have been introduced and studied. Since all group subsets are considered in the theory and there are many basic open problems and questions, their computational aspects are of particular importance. In this paper, by introducing some computational methods and using theoretical approaches together, we not only solve several problems but also pave the way for studying the topic. As the most important result of the study, we completely characterize finite index stable groups.

1. Introduction

While studying periodic type sets and factors of basic algebraic structures (i.e., magmas, semigroups, groups, etc.), the first author was led to new concepts such as subfactors of groups, subindices, and index stability of group subsets. Indeed, he first encountered a challenging problem in 2014 (see: [3]) about factorization of (arbitrary) finite groups (also see: Kourovka Notebook [8], Vol. 20: Question 20.37& 19.35, and [4]). The conjecture says: for every factorization $|G| = ab$ of a finite group G, there exist subsets A, B such that $|A| = a$, $|B| = b$, and $G = AB$. It has been partially

MSC(2010): Primary: 20F99, 20E99, 20-04, 20-08.

Keywords: Factor subset, subfactors, sub-index, index stable subset, index stable group, factorization of finite groups.

Article Type: Research Paper.

Communicated by Attila Maroti.

* Corresponding author.

Received: 14 August 2024, Accepted: 25 August 2025, Published Online: 21 September 2025.

Cite this article: M. H. Hooshmand and M. Yousefian Arani, Computational aspects of subindices and subfactors with characterization of finite index stable groups, *Int. J. Group Theory*, **15** no. 3 (2026) 145–160. https://dx.doi.org/10.22108/ijgt.2025.142465.1919 .

answered in [1, 2, 4]. After that, subfactors of groups are introduced and it is shown that the concept of index of subgroups can be extended to factors and even arbitrary subsets! ([5]). In [6] more studies for subindices and subfactors of finite groups have been done. Characterization of index stable groups and subsets is a challenging problem in the theory together with many questions, conjectures, and research projects. Answers to all of these seem unlikely without the use of computational methods since the theory considers all subsets of a group. Here, the previous results together with some new computational methods enable us to prove a main theorem that completely characterizes the finite index stable groups. Also, we solve many problems, answer several questions, and prove some related conjectures by using the theoretical backgrounds from [5, 6] together with the computational methods.

2. subfactors and subindices of group subsets with computational aspects

Let A, B be subsets of a group G. We call the product AB to be direct, and denote it by $A \cdot B$ if the representation of every element of AB by $x = ab$ with $a \in A$, $b \in B$ is unique. Hence, $G = A \cdot B$ if and only if $G = AB$ and the product AB is direct (the additive notation is $G = A \dot{+} B$). We have $AB = A \cdot B$ if and only if $A^{-1}A \cap BB^{-1} = \{1\}$ where $A^{-1} = \{a^{-1} \ : \ a \in A\}$. If $G = A \cdot B$, then A (resp. B) is a left (resp. right) factor of G related to B (resp. A). We call A a left factor of G if and only if $G = A \cdot B$ for some $B \subseteq G$ (equivalently, there exists a right factor B of G related to A). For example, every subgroup is a left (resp. right) factor related to its right (resp. left) transversal, hence it is a two-sided factor of G. In reference [5], a generalization of factors is introduced that not only avoids the deficiency of factors but also leads to the important concepts of subindices for all subsets of groups.

2.1. **Subfactors of groups.** Let A be a fixed subset of G. We call B a right subfactor of G related to A if B is an inclusion-maximal subset of G with respect to the property $AB = A \cdot B$. Note that such subfactors always exist, by [5, Theorem 3.2]. Also, we say B is a right subfactor of G if it is a right subfactor related to some subset of G (left subfactors are defined analogously).

It is proved that every subset of a group has related right and left subfactors. Also, B is a right subfactor (of G) related to A if and only if

$$A^{-1}A \cap BB^{-1} = \{1\} \ , \ A^{-1}AB = G. \tag{2.1}$$

Putting

$$\mathrm{Fac}_r(A) = \mathrm{Fac}_r(G : A) = \{B \subseteq G : B \text{ is a right factor of } G \text{ related to } A\},$$

$$\mathrm{SubF}_r(A) = \mathrm{SubF}_r(G : A) = \{B \subseteq G : B \text{ is a right subfactor of } G \text{ related to } A\},$$

we have $\mathrm{Fac}_r(A) \subseteq \mathrm{SubF}_r(A) \neq \emptyset$. But $\mathrm{Fac}_r(A) \neq \emptyset$ if and only if A is a left factor of G.

Computational aspects of subfactor. Since there are many right subfactors B related to a subset

https://dx.doi.org/10.22108/ijgt.2025.142465.1919

A, in practice, we need to know the limitations on such B, and applicable methods and algorithms for computing subfactors (of finite groups). The following are some useful facts for the computational aspect.

(a) It is enough to consider subsets B containing the identity 1. Because putting

$$\mathrm{SubF}^1_r(A) = \mathrm{SubF}^1_r(G:A) = \{B_1 \in \mathrm{SubF}_r(A) : 1 \in B_1\},$$

(this agrees with the notation $X^1 = X \cup \{1\}$ for every $X \subseteq G$), we have $B \in \mathrm{SubF}_r(A)$ if and only if $B = B_1\beta$, for some $B_1 \in \mathrm{SubF}^1_r(A)$ and $\beta \in B$ (note that $B \neq \emptyset$, and consider $B_1 = Bb_0^{-1}$ for a $b_0 \in B$). Hence

$$\{B_1 g : B_1 \in \mathrm{SubF}^1_r(A), g \in G\} = \mathrm{SubF}_r(A).$$

(b) We have that $B_1 \subseteq (A^{-1}A)^c \cup \{1\}$ for every $B_1 \in \mathrm{SubF}^1_r(A)$ (because $A^{-1}A \cap B_1B_1^{-1} = \{1\}$ and $B_1 \subseteq B_1{B_1}^{-1}$). Hence, for finite groups G we have

$$\mathrm{SubF}^1_r(A) \subseteq \left\{\mathbf{B} \subseteq (A^{-1}A)^c \cup \{1\} : 1 \in \mathbf{B}, \left\lceil \frac{|G|}{|A^{-1}A|} \right\rceil \leq |\mathbf{B}| \leq \left\lfloor \frac{|G|}{|A|} \right\rfloor \right\}. \tag{2.2}$$

Therefore it is enough to check only elements of the right hand of (2.2) for finding the right subfactors of G related to A, that in this case, the calculations will be much less. Hence, we can write a GAP code for computing $\mathrm{SubF}_r(A)$ as follows (link to code).

Example 2.1. Consider the additive group $G = \mathbb{Z}_6$ and $A = \{0, 1\} \subseteq G$. By using the above code, we obtain $\mathrm{SubF}(A) = \{\{0, 2, 4\}, \{0, 3\}, \{1, 3, 5\}, \{1, 4\}, \{2, 5\}\}$ and $\mathrm{SubF}^1(A) = \{\{0, 2, 4\}, \{0, 3\}\}$. Also, see (link to more examples).

(c) Another way for computing subfactors is applying an algorithm in [6]. The next theorem states an improved version of the algorithm (with a different expression) that prove by [6, conjecture V]

Theorem 2.2. *Every right subfactor of G related to A can be obtained from the following algorithm (i.e., the set of all outputs B of the algorithm is equal to $\mathrm{SubF}_r(A)$.) Let $g_0 \in G$ be chosen arbitrary. Let $g_0, \ldots, g_n$, $n \geq 0$ be already constructed put $M_n = A^{-1}Ag_0 \cup \cdots \cup A^{-1}Ag_n$. If $M_n = G$ then the algorithm is ended otherwise, continue by choosing an element $g_{n+1} \notin M_n$ Then, $\{g_0, \ldots, g_N\}$ is a right subfactor of G related to A where N is the least integer such that $M_N = G$.*

Proof. Let $X \in \mathrm{SubF}_r(A)$ and represent its members by $X = \{x_0, \ldots, x_{m+1}\}$ where $m = |X| - 2$ (thus $m \geq -1$). Now in the algorithm choose $g_0 = x_0$ (since g_0 is arbitrary in it). If $A^{-1}A = G$, then $|X| = 1$, $N = 0$ (in the algorithm) and $X = \{x_0\} = \{g_0\}$ thus we are done. Otherwise, suppose that $g_0, \ldots, g_n$ take the values $x_0, \ldots, x_n$, respectively, for some $n < m+1$. Since the product $A(\{x_0, \ldots, x_n\} \cup \{x_{n+1}\})$ is direct,

$$x_{n+1} \in \bigcap_{i=0}^{n} (A^{-1}A)^c x_i = \bigcap_{i=0}^{n} (A^{-1}A)^c g_i = (\bigcup_{i=0}^{n} A^{-1}Ag_i)^c.$$

So g_{n+1} can take the value x_{n+1} in the algorithm process. Therefore $X \subseteq \{g_0, \ldots, g_N\}$ and so $X = \{g_0, \ldots, g_N\}$ (and $N = m$) since $X, \{g_0, \ldots, g_N\} \in \mathrm{SubF}_r(A)$. □

By using Theorem 2.2, we are now able to write another GAP code for computing the whole $\mathrm{SubF}_r(A)$ as (link to code). This also gives us a constructive method to compute an arbitrary subfactor of G related to A which is much more efficient in larger groups, see (link to code). Note that by choosing $g_0 = 1$ at the beginning of the algorithm, all members of $\mathrm{SubF}^1_r(G : A)$ are obtained.

Example 2.3. Consider the dihedral group

$$G = D_8 \cong \langle a, b | a^4 = b^2 = 1, ba = a^3 b\rangle = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\}.$$

If $A = \{a, a^2, b\}$, then we obtain

$$\mathrm{SubF}_r(A) = \Big\{\{1, b\}, \{1, a^2\}, \{1, ba^3\}, \{b, ba^2\}, \{a, b\},$$

$$\{a, ba\}, \{ba, ba^3\}, \{a, a^3\}, \{a^2, ba^2\}, \{a^3, ba^2\}, \{a^3, ba^3\}\Big\},$$

by using the second code. Also, see (link to more examples).

2.2. **Subindices of group subsets.** For each subset A of a group G we assign subindices of A as follows:

$$|G : A|^+ = \sup\{|B| : B \in \mathrm{SubF}_r(A)\} \ : \text{ right upper index of } A \text{ (in } G);$$

$$|G : A|^- = \inf\{|B| : B \in \mathrm{SubF}_r(A)\} \ : \text{ right lower index of } A \text{ (in } G).$$

The left notations $|G : A|_\pm$ are defined analogously. Now, we call A:

(a) right (resp. left) index stable in G if $|G : A|^+ = |G : A|^-$ (resp. $|G : A|_+ = |G : A|_-$), and we use the notation $|G : A|_r$ (resp. $|G : A|_\ell$) for the common value and call it the right (resp. left) index of A in G.

(b) index stable (in G) if all of its four subindices are equal (equivalently $|G : A|_r = |G : A|_\ell$), and the common value is denoted by $|G : A|$ and is called the index of A in G (a unique cardinal number corresponding to A).

Also, a group is called index stable (resp. right index stable) if all its subsets are index stable (resp. right index stable).

It is worth noting that if G is a group and H a subgroup, then H (as a subset) is always index stable in G, but as an independent group, H may be not index stable (i.e., it contains a subset that is not index stable in H).

There are some basic properties of subindices in arbitrary and finite groups (see [5, 6]). If G is finite, $A \subseteq H \leq G$, and A is right index stable in G, then it is so in H and we have

$$|G : A|_r = |G : H||H : A|_r.$$

Therefore, if G is index stable, then every $H \leq G$ is so, and $|G : A| = |G : H||H : A|$, for all $A \subseteq H$. Hence, every finite group containing a non-index stable subgroup is non-index stable.

There is an important property for subindices of finite group subsets that if $|A| > \frac{|G|}{2}$, then $|G : A| = 1$. The converse is not true (e.g., if $G = \mathbb{Z}_6, A = \{0, 1, 3\}$, then $|G : A| = 1$). But as a weak converse, if $|G : A| = 1$, then the inequality $|A^{-1}A| \leq |A|^2 - |A| + 1$ implies that $\frac{|G|}{|A|^2 - |A| + 1} \leq 1$, and so $|A| \geq \frac{1}{2} + \sqrt{|G| - \frac{3}{4}} > \sqrt{|G|}$ (if $G \neq 1$). Due to recent facts, for computational aspects of subindices, one may consider the partition $\{\mathcal{A}_r\}_{r=1}^{|G|}$ for $2^G \setminus \{\emptyset\}$ where

$$\mathcal{A}_r = \mathcal{A}_r(G) = \left\{ A \subseteq G : \frac{|G|}{r+1} < |A| \leq \frac{|G|}{r} \right\}.$$

All subindices of every element of $\mathcal{A}_r$ are $\leq r$ (because $\frac{|G|}{r+1} < |A| \leq \frac{|G|}{r}$ if and only if $\left\lfloor \frac{|G|}{|A|} \right\rfloor = r$).
As a weak converse, let r be an integer such that $1 \leq r \leq |G|$. If one of the subindices of A is $\leq r$, then

$$|A| \geq \frac{1}{2} + \sqrt{\frac{|G|}{r} - \frac{3}{4}} > \sqrt{\frac{|G|}{r}}.$$

Hence, $\sqrt{\frac{|G|}{r}} < |A| \leq \frac{|G|}{r}$ if $r \leq \frac{|G|}{|A|}$ (but we cannot conclude that $A \in \mathcal{A}_r(G)$). Since for $r > \frac{|G|}{2}$ and $r = 1$ all elements of $\mathcal{A}_r(G)$ are index stable, it is enough to study $\mathcal{A}_r(G)$ for $2 \leq r \leq \frac{|G|}{2}$ (i.e., $\{\mathcal{A}_r\}_{r=2}^{\left\lfloor \frac{|G|}{2} \right\rfloor}$). It is worth noting that all elements of $\mathcal{A}_2(G)$ are left and right index stable with subindices 1 or 2, and they are index stable if G is abelian (see [6, Corollary 3.4]).

Computational aspects of subindices. To calculate $|G : A|^+$, the straightforward way is computing $\mathrm{SubF}_r^1(A)$, and then the maximum sizes of its elements. But with a closer look, it can be seen that there is another algorithm, since according to (2.2) it is enough to do the next steps:
(1) Start with subsets $B \ni 1$ of $(A^{-1}A)^c \cup \{1\}$ of sizes $\left\lfloor \frac{|G|}{|A|} \right\rfloor$, and then all subsets of sizes $\left\lfloor \frac{|G|}{|A|} \right\rfloor - 1$, and so on.
(2) Find the first such B for which $B \in \mathrm{SubF}_r^1(A)$ and denote it by B_0 .
(3) $|G : A|^+ = |B_0|$.
Analogous algorithm exists for computing $|G : A|^-$ (and other subindices).
We are now able to write an appropriate GAP code for computing the right subindices and checking the right index stability of subsets as (link to code).

Example 2.4. If $G = C_4 \times C_2 \times C_2$, $A = \{000, 001, 010, 100\}$, then we have $|G : A|^- = 2$ and $|G : A|^+ = 4$. (link to more examples).

2.3. **A table of subindices for k-index stability of groups of small orders.** In the theory of subindices, we observe that the cardinality of subsets plays an important role in index stability. Hence we recall a definition from [5, 6].

Definition 2.5. *Let G be a finite group and $1 \leq k \leq |G|$ a given integer number. We call G k-index stable if all its subsets of size k are index stable (analogously for left and right k-index stabilities).*

Note that a group is right k-index (resp. index) stable if and only if it is left k-index (resp. index) stable, since

$$|G : A^{-1}|^{+} = |G : A|_{+} \ , \ |G : A^{-1}|^{-} = |G : A|_{-} \ , \ |A^{-1}| = |A|,$$

for all $A \subseteq G$ (see [5, Theorem 3.12(c)]). For finite groups, we prove in the next section that right, left, and two-sided index stability of finite groups are equivalent but this is not true for k-index stability (for the first counterexample, A_4, D_{12} are right and left 6-index stable but not two-sided 6-index stable). Now, using the GAP code (link to code) which is obtained according to the stated facts and algorithms, we present a complete table for right and two-sided k-index (and index) stability of finite groups of orders ≤ 27. It is worth noting that many cases of the table have also theoretical evidence in [5, 6]. Note that in the following table, there are columns that indicate the state of the right and two-sided index stability of subsets of the mentioned size with an ordered pair of 0's and 1's. The first component of the ordered couple corresponds to the k-right index stability and the next one to the k-index stability, where the number zero means that it is not established and one indicates that the related property is satisfied. Hence, the second component is less than or equal to the first one. For example, the column 6 for A_4 indicates that A_4 is right 6-index stable but not 6-index stable. Note that since every group of order≤ 5 is index stable and all finite groups G are k-index stable for $k > \left\lfloor \frac{|G|}{2} \right\rfloor$, we do not mention these cases in the table. Also, notice that every $A \in \mathcal{A}_2(G)$ is right index stable (but not necessarily index stable), thus the first component in the kth column is 1 for all $\left\lceil \frac{|G|}{3} \right\rceil + 1 \leq k \leq \left\lfloor \frac{|G|}{2} \right\rfloor$.

Table 1: k-index stability of groups; $6 \leq |G| \leq 16$

Group	**k=2**	**k=3**	**k=4**	**k=5**	**k=6**	**k=7**	**k=8**	**(right) index stability**
S_3	1-1	1-1						Index Stable
C_6	0-0	1-1						None (right) index stable
C_7	1-1	1-1						Index Stable
C_8	0-0	1-1	1-1					None (right) index stable
$C_4 \times C_2$	1-1	1-1	1-1					Index Stable
D_8	1-1	1-1	1-1					Index Stable
Q_8	1-1	1-1	1-1					Index Stable
$C_2 \times C_2 \times C_2$	1-1	1-1	1-1					Index Stable
C_9	0-0	0-0	1-1					None (right) index stable
$C_3 \times C_3$	1-1	1-1	1-1					Index Stable

Table 1: k-index stability of groups; $6 \leq |G| \leq 16$

Group	**k=2**	**k=3**	**k=4**	**k=5**	**k=6**	**k=7**	**k=8**	**(right) index stability**
D_{10}	1-1	0-0	1-1	1-1				None (right) index stable
C_{10}	0-0	0-0	1-1	1-1				None (right) index stable
C_{11}	0-0	1-1	1-1	1-1				None (right) index stable
$C_3 : C_4$	0-0	0-0	0-0	1-1	1-1			None (right) index stable
C_{12}	0-0	0-0	0-0	1-1	1-1			None (right) index stable
A_4	1-1	0-0	0-0	1-1	1-0			None (right) index stable
D_{12}	0-0	0-0	0-0	1-1	1-0			None (right) index stable
$C_6 \times C_2$	0-0	0-0	0-0	1-1	1-1			None (right) index stable
C_{13}	0-0	0-0	0-0	1-1	1-1			None (right) index stable
D_{14}	1-1	0-0	0-0	1-1	1-0	1-0		None (right) index stable
C_{14}	0-0	0-0	0-0	1-1	1-1	1-1		None (right) index stable
C_{15}	0-0	0-0	0-0	0-0	1-1	1-1		None (right) index stable
C_{16}	0-0	0-0	0-0	0-0	1-1	1-1	1-1	None (right) index stable
$C_4 \times C_4$	1-1	0-0	0-0	1-1	1-1	1-1	1-1	None (right) index stable
$(C_4 \times C_2) : C_2$	1-1	1-1	0-0	1-1	1-0	1-0	1-0	None (right) index stable
$C_4 : C_4$	1-1	1-1	0-0	1-1	1-1	1-1	1-1	None (right) index stable
$C_8 \times C_2$	0-0	0-0	0-0	1-1	1-1	1-1	1-1	None (right) index stable
$C_8 : C_2$	0-0	0-0	0-0	1-1	1-0	1-0	1-0	None (right) index stable
D_{16}	0-0	0-0	0-0	0-0	1-1	1-0	1-0	None (right) index stable
QD_{16}	0-0	1-1	0-0	1-1	1-0	1-0	1-0	None (right) index stable
Q_{16}	0-0	0-0	0-0	1-1	1-1	1-1	1-1	None (right) index stable
$C_4 \times C_2 \times C_2$	1-1	1-1	0-0	1-1	1-1	1-1	1-1	None (right) index stable
$C_2 \times D_8$	1-1	1-1	0-0	1-1	1-1	1-0	1-0	None (right) index stable
$C_2 \times Q_8$	1-1	1-1	0-0	1-1	1-1	1-1	1-1	None (right) index stable
$(C_4 \times C_2) : C_2$	1-1	1-1	0-0	1-1	1-1	1-0	1-0	None (right) index stable
$C_2 \times C_2 \times C_2 \times C_2$	1-1	1-1	1-1	1-1	1-1	1-1	1-1	Index Stable

Table 2: k-index stability of groups; $17 \leq |G| \leq 27$

Group	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	
C_{17}	0-0	0-0	0-0	0-0	1-1	1-1	1-1						None (right) index stable
D_{18}	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0					None (right) index stable
C_{18}	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1					None (right) index stable
$C_3 \times S_3$	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0					None (right) index stable
$(C_3 \times C_3) : C_2$	1-1	1-1	0-0	0-0	0-0	1-1	1-0	1-0					None (right) index stable
$C_6 \times C_3$	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1					None (right) index stable
C_{19}	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1					None (right) index stable
$C_5 : C_4$ (Dic20)	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1				None (right) index stable
C_{20}	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1				None (right) index stable
$C_5 : C_4$ ($GA(1,5)$)	1-1	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0				None (right) index stable
D_{20}	0-0	0-0	0-0	0-0	0-0	1-1	1-0	1-0	1-0				None (right) index stable
$C_{10} \times C_2$	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1				None (right) index stable
$C_7 : C_3$	1-1	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-1				None (right) index stable
C_{21}	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1				None (right) index stable
D_{22}	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0			None (right) index stable
C_{22}	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1			None (right) index stable

https://dx.doi.org/10.22108/ijgt.2025.142465.1919

Table 2: k-index stability of groups; $17 \leq |G| \leq 27$

Group	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	
C_{23}	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1			None (right) index stable
$C_3 : C_8$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-1		None (right) index stable
C_{24}	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1		None (right) index stable
$SL(2,3)$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
$C_3 : Q_8$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
$C_4 \times S_3$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
D_{24}	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
$C_2 \times (C_3 : C_4)$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
$(C_6 \times C_2) : C_2$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
$C_{12} \times C_2$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1		None (right) index stable
$C_3 \times D_8$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
$C_3 \times Q_8$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-1	1-1		None (right) index stable
S_4	1-1	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
$C_2 \times A_4$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
$C_2 \times C_2 \times S_3$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0		None (right) index stable
$C_6 \times C_2 \times C_2$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1		None (right) index stable

https://dx.doi.org/10.22108/ijgt.2025.142465.1919

Table 2: k-index stability of groups; $17 \leq |G| \leq 27$

Group	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	
C_{25}	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1		None (right) index stable
$C_5 \times C_5$	1-1	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1	1-1	1-1		None (right) index stable
D_{26}	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0	1-0	1-0	None (right) index stable
C_{26}	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1	1-1	None (right) index stable
C_{27}	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1	1-1	None (right) index stable
$C_9 \times C_3$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-1	1-1	1-1	1-1	None (right) index stable
$(C_3 \times C_3) : C_3$	1-1	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-1	1-1	1-1	1-1	None (right) index stable
$C_9 : C_3$	0-0	0-0	0-0	0-0	0-0	0-0	0-0	0-0	1-0	1-0	1-0	1-0	None (right) index stable
$C_3 \times C_3 \times C_3$	1-1	1-1	0-0	1-1	1-1	1-1	1-1	1-1	1-1	1-1	1-1	1-1	None (right) index stable

3. Complete determination of all finite index stable groups: main theorem

Now, we are ready to completely characterize finite index stable groups. The next main theorem provides answers to two questions: [6]IIIa and [5]IIIa (also the finite case of [5]I). Hence, it solves one of the main open problems of the theory. We recall the important fact from [5, Corollary 3.23], which states that every finite group containing a non-index stable subgroup is non-index stable.

Theorem 3.1. *There are only* 14 *finite* (*right*) *index stable groups as follows:*

$$C_1,\ C_2,\ C_2 \times C_2,\ C_2 \times C_2 \times C_2,\ C_2 \times C_2 \times C_2 \times C_2,$$

$$C_3,\ C_3 \times C_3,\ C_4,\ C_4 \times C_2,\ C_5,\ C_7,\ S_3,\ D_8,\ Q_8.$$

(*i.e., all groups of orders* ≤ 9 *except* C_6, C_8, C_9 *together with* $C_2 \times C_2 \times C_2 \times C_2$).

https://dx.doi.org/10.22108/ijgt.2025.142465.1919

Proof. First note that all finite (left, right) index stable groups are among groups G with order $|G| = 2^q \cdot 3^r \cdot 5^s \cdot 7^t$ for non-negative integers q, r, s, t, by [6, Theorem 3.12]. We prove this theorem by six steps.

Step 1. The groups $C_2 \times C_2 \times C_2 \times C_2 \times C_2$, $C_3 \times C_3 \times C_3$, $C_5 \times C_5$ and $C_7 \times C_7$ are the first powers of C_2, C_3, C_5 and C_7 that are not index stable. From Tables 1 and 2, it is evident that groups of this form with smaller orders are index stable. Additionally, Table 1,2 shows that $C_3 \times C_3 \times C_3$ and $C_5 \times C_5$ are not index stable.

Moreover, $C_2 \times C_2 \times C_2 \times C_2 \times C_2$ is not index stable. For if

$$A = \{00000, 10000, 01000, 00100, 00010, 00001\} \subseteq C_2 \times C_2 \times C_2 \times C_2 \times C_2,$$

i.e., the set of elements with at most one coordinate equal to 1. Then, for any $b \in C_2 \times C_2 \times C_2 \times C_2 \times C_2$, the elements of $A + b$ will be those which differ from b in at most one coordinate; hence for elements b, b', the sets $A + b$ and $A + b'$ will be disjoint if and only if b and b' differ in at least 3 coordinates. So let $B_1 = \{00000, 11100, 00111, 11011\}$, and $B_2 = \{00000, 11111\}$. It is easy to check that every element of $C_2 \times C_2 \times C_2 \times C_2 \times C_2$ agrees in at least 3 coordinates with an element of B_2, hence disagrees with such an element in at most 2 coordinates; so no additional elements can be added to B_2 and keep its product with A direct; so B_2 is a subfactor related to A. Similarly, B_1 is also such a subfactor, and so A is not index stable.

Also, applying the GAP code (link to code) for $A = \{00, 01, 10\} \subseteq C_7 \times C_7$ shows that

$$B_1 = \{00, 02, 22, 24, 26, 30, 41, 44, 52, 65\}$$

of size 10, and

$$B_2 = \{00, 20, 12, 04, 32, 14, 52, 25, 44, 64, 66\}$$

of size 11 are two subfactors of $C_7 \times C_7$ related to A. (link to recorded output).

Therefore, $C_2 \times C_2 \times C_2 \times C_2 \times C_2$, $C_3 \times C_3 \times C_3$, $C_5 \times C_5$ and $C_7 \times C_7$ are the first powers of C_2, C_3, C_5 and C_7 that are not index stable. Because in subsequent powers, we will have a subgroup that is isomorphic to these groups, therefore, they are not index stable.

Step 2. All finite right index stable groups G are of the order

$$|G| \in \{2^q \cdot 3^r \cdot 5^s \cdot 7^t : q = 0, 1, 2, 3, 4,\ r = 0, 1, 2,\ s = 0, 1,\ t = 0, 1\}$$
$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 16, 18, \dots, 1680, 2520, 5040\}$$

because groups of larger order have a subgroup of order 32, 27, 25 or 49. It is clear from Table 1,2 that there is no index stable group of orders 25 and 27. For the order of 49, we have two groups, $C_7 \times C_7$ and C_{49}, both of which are not index stable. For the order of 32, after considering the proper subgroups of these groups and examining Tables 1 and 2, only the group $C_2 \times C_2 \times C_2 \times C_2 \times C_2$ has the property that all of its proper subgroups are index stable. However, as we have seen, that group itself is not index stable.

Step 3. Right index stability of groups G of orders $1 < |G| \leq 2^4$: from Table 1,2, it can be deduced that only groups of orders ≤ 9, except C_6, C_8, C_9, along with $C_2 \times C_2 \times C_2 \times C_2$, are index stable.

Step 4. Index stability of abelian groups G of orders $|G| > 2^4$: such an index stable group does not exist; according to [6, Theorem 3.12], it must be a p-group with the specified order. If p is 3, 5, or 7, and $|G| > 2^4$, they do not conform to the form mentioned earlier. In the case of a 2-group, considering the abelian decomposition, it can only be a power of C_2; otherwise, it does not possess the property that all of its proper subgroups are index stable. This is evident from Table 1,2, where C_8, $C_4 \times C_4$, and $C_2 \times C_2 \times C_4$ are not index stable, and consequently, groups containing subgroups of this form are also not index stable. Furthermore, if $|G| > 2^4$ and it is a power of C_2, it must include $C_2 \times C_2 \times C_2 \times C_2 \times C_2$, which implies it is not index stable.

For continuation of the proof, using the code (link to code), we consider non-abelian groups of the specified order form within intervals of powers of 2. For each group G with order between 2^4 and 2^5, all of its proper subgroups have orders ranging from 1 to 2^4, and the index stability of these subgroups is known. In each step of this procedure, we examine the index stability of groups that possess the property of having all of their proper subgroups index stable. This process helps us determine all index stable groups up to order 2^5. In this manner, an inductive approach allows us to determine the status of groups from 2^n to 2^{n+1} once it is established up to 2^n. Any special cases with all subgroups being index stable are noted below and if such cases do not exist, we have moved on from that interval. Since the orders must adhere to the form mentioned in step two, this process is finite.

Step 5. Right index stability of non-abelian groups G of order $2^4 < |G| \leq 1680$: we have two cases :

Case 1. Right index stability of G with $2^4 < |G| \leq 2^7$: only $(C_2 \times C_2 \times C_2) : C_7$ and $(C_2 \times C_2 \times C_2 \times C_2) : C_5$ have the property that all their proper subgroups are index stable. For these two groups, we have examples of non right index stable subsets as follows. Let $G \cong (C_2 \times C_2 \times C_2) \rtimes C_7$, where $C_7 = \langle t \rangle$ acts on $N \cong C_2 \times C_2 \times C_2 = \langle x, y, z \rangle$ via $\varphi(t) \in \mathrm{Aut}(N)$ defined by $txt^{-1} = yz$, $tyt^{-1} = x$, $tzt^{-1} = y$. Then every element of G can be uniquely written as nt^i with $n \in N$ and $0 \leq i \leq 6$. If we put $A = \{1, t, x, y, z, xyz\,t^4\}$, we have subfactors $B_1 = \{1, yz\,t, yz\,t^3, t^5, xy\,t^6\}$ and $B_2 = \{t, x, t^3, yz\,t^2\}$ with respect to A. Similarly, let $G \cong (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5$, where $C_5 = \langle t \rangle$, $N \cong C_2 \times C_2 \times C_2 \times C_2 = \langle x, y, z, w \rangle$, and $\varphi(t) \in \mathrm{Aut}(N)$ is defined by $txt^{-1} = xy$, $tyt^{-1} = yz$, $tzt^{-1} = zw$, $twt^{-1} = x$. We have the subset $A = \{1, t, x, y, z, w\}$ and subfactors $B_1 = \{1, yz\,t, xzw\,t, yzw, xyzw\,t^3, xy\,t^4, z\,t^3, x\,t^2\}$, $B_2 = \{y, xt, t^3, yw\,t^2, xzw, xyw\,t^3, z\,t^2, yw\,t^4, xzw\,t^4\}$ where $|B_1| \neq |B_2|$. (link to recorded output).

Case 2. Right index stability of G with $2^7 < |G| \leq 1680$: all such groups G contain a non right index stable subgroup of order less than 2^7, whose index stability was determined in the previous steps (see link to code). Hence, there are no index stable groups in this case.

Step 6. The remains groups are those whose orders are $2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = \frac{7!}{2}$ or $5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 = 7!$ which GAP does not support them. Fortunately, we can give a theoretical proof for their non-index stability as follows (the idea belongs to Derek Holt, see his comment on the question in [7]). It is enough to show that every group G of order 2520 or 5040 has a proper subgroup H with $|H| > 16$. Let P be a Sylow 3-subgroup of G and put $N = N_G(P)$. Thus $|P| = 9$, P is abelian and 9 divides $|N|$. If $N = P$ or $N = G$, then there is a complement H of P by Burnside's Transfer Theorem and Schur–Zassenhaus Theorem. Otherwise, put $H = N$. In any case, H is a proper subgroup of G and $|H| > 16$. □

Corollary 3.2. *For a finite group G the followings are equivalent:*
(a) *G is index stable;*
(b) *G is right index stable;*
(c) *G is left index stable;*
(d) *G is one of the 14 groups mentioned in Theorem 3.1 (up to isomorphism).*

4. Answers and solutions to some other questions, open problems, and conjectures

Since the theory of subfactors and subindices is completely new, it is natural that many questions, open problems, conjectures, and research projects are raised about it. A number of them have been mentioned in the [5, 6] that some of them are fundamental and have special importance. So far we have answered a few of them in whole or in part. The next theorem not only answers question VII from [6] but also improves [5, Corollary 3.18].

Theorem 4.1. *The subset $\{0,1\}$ takes its (relative) maximum upper and minimum lower index in $\mathbb{Z}_n$, for all $n > 2$ (also see the next remark), i.e.,*

$$|\mathbb{Z}_n : \{0,1\}|^- = \left\lceil \frac{n}{3} \right\rceil \ , \ |\mathbb{Z}_n : \{0,1\}|^+ = \left\lfloor \frac{n}{2} \right\rfloor .$$

Proof. Note that $|\{0,1\}| = 2$, $|\{0,1\} - \{0,1\}| = |\{0,1,n-1\}| = 3$ and

$$\left\lceil \frac{n}{3} \right\rceil \leq |\mathbb{Z}_n : \{0,1\}|^- \leq |\mathbb{Z}_n : \{0,1\}|^+ \leq \left\lfloor \frac{n}{2} \right\rfloor .$$

Putting $B = \{0, 2, \ldots, 2(\lfloor n/2 \rfloor - 1)\}$ and

$$B' = \begin{cases} \{0, 3, \ldots, 3\left(\lfloor \frac{n}{3} \rfloor - 1\right)\}, & \text{if } n \not\equiv 1 \pmod 3; \\ \{0, 3, \ldots, 3\left(\lfloor \frac{n}{3} \rfloor - 2\right), 3\left(\lfloor \frac{n}{3} \rfloor - 4\right)\}, & \text{if } n \equiv 1 \pmod 3, \end{cases}$$

we have $|B| = \lfloor \frac{n}{2} \rfloor$ and $|B'| = \lceil \frac{n}{3} \rceil$. Due to the above inequality, it is enough to show that the summations $B + \{0,1\}$ and $B' + \{0,1\}$ are direct, or equivalently

$$(B - B) \cap \{0, 1, n-1\} = \{0\} = (B' - B') \cap \{0, 1, n-1\}.$$

Let x, y are both elements of B or B' and $y \neq 0$. If $x + (n - y) = n - 1$, then we get a contradiction, since $2|x - y$ or $3|x - y$, or one of the relations $3|x$, $y = n - 2$ or $3|y$, $x = n - 2$ be occurred.

Hence, suppose that $x + (n - y) = 1$ and consider the following cases:
(i) $x, y \in B$: we conclude that n is odd, $0 \leq x \leq n-3$, $2 \leq y \leq n-3$, and so $5-n \leq y-x \leq n-3$ that is impossible.
(ii) $x, y \in B'$ and $n \not\equiv 1 \pmod 3$: the equality $y - x = n - 1$ gives a contradiction, since $3|x-y$.
(iii) $x, y \in B'$ and $n \equiv 1 \pmod 3$: then both x and y must be divided by 3 and so $n-1 \leq y-x \leq n-4$ that is a contradiction.
Finally, note that if $y = 0$, then one can see $x - y \neq 1, n-1$. Therefore, the proof is complete. □

Regarding problem [5]II(a), the above theorem, and considering Tables 1 and 2 we have an important conjecture for finite cyclic groups as follows.

Conjecture 4.2 (New). If $n > 11$ then $\mathbb{Z}_n$ is not k-index stable if and only if $2 \leq k \leq \left\lfloor \frac{n}{3} \right\rfloor$.

Remark 4.3. *Since the set of all solutions of the equation* $\left\lceil \frac{n}{3} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor$ *is* $\{2,3,4,5,7\}$, *we deduce that* $\mathbb{Z}_n$ $(n \geq 2)$ *is* 2*-index stable if and only if* $n \in \{2,3,4,5,7\}$ *which agrees with* [5, Lemma 3.17]. *But, if* $n \notin \{1,2,3,4,5,7\}$ *then* $\{0,1\}$ *takes its relatively least* (*resp. largest*) *possible upper* (*resp. lower*) *index which means it is relatively strong index unstable. Note that a subset* A *of a finite group* G *is relatively* (*resp. absolutely*) *strong right index unstable if* $\left\lceil \frac{|G|}{|A^{-1}A|} \right\rceil = |G:A|^- < |G:A|^+ = \left\lfloor \frac{|G|}{|A|} \right\rfloor$ (*resp.* $2 = |G:A|^- < |G:A|^+ = \left\lfloor \frac{|G|}{|2|} \right\rfloor$)

New problem.
(a) Determine all finite abelian (resp. non-abelian) groups that are not k-index stable for all $2 \leq k \leq \left\lfloor \frac{|G|}{3} \right\rfloor$ (resp. $2 \leq k \leq \left\lfloor \frac{|G|}{2} \right\rfloor$). Also, determine all finite non-abelian groups that are not right k-index stable for all $2 \leq k \leq \left\lfloor \frac{|G|}{3} \right\rfloor$.
(b) Characterize or classify all finite groups containing an absolutely strong right index unstable subset.

Using the computational method presented in this article, we can easily resolve a number of questions, summarized below.

Cases for which counterexamples have been found:

- [5] III(h): $G = C_3 \times C_3$, $A = \{00, 01, 10\}$, $(A^{-1}A)(A^{-1}A) = G$, and $|G:A| = 3 \neq 2$ (the converse has an obvious counterexample: $G = C_2$, $A = \{0\}$). (link)
- [5] IV(a): $G = C_{12}$, $A_1 = \{0,3,10\}$, $A_2 = \{0,3,9,10\}$, $|G:A_1|^- = 2$, and $|G:A_2|^{\pm} = 3$. (link)
- [5] V: $G = S_3$, $A = \{(),(2,3)\}$, $A = A^{-1}$, and $\mathrm{SubF}_r \neq \mathrm{SubF}_\ell$ (The converse of [5]V is also false, e.g., $G = C_3$, $A = \{0,1\}$. Note that the second part of the question [5]V is still open). (link)
- [6] I(1): $G = S_3$, $A = \{(),(2,3),(1,2)\}$ (or $G = D_{10}$, $A = \{1,a,b,a^4b\}$), $|A| > \sqrt{|G|}$, and $|G:A| = 2 \neq 1$. (link)

- [6] I(2): $G = S_3$, $A = \{(), (1,2,3)\}$ (or $G = D_{10}$, $A = \{1, a, a^2\}$), $|G : A| = 2$, and $|A| \leq \frac{|G|}{3}$. (link)
- [6] I(3): $G = C_6, A = \{0, 2\}$ (or $G = C_{10}$, $A = \{0, 1, 3\}$), $|G : A| = 2$, and $|A| \leq \frac{|G|}{3}$. (link)

Further results:

- [6] I(4): $G = C_{10}$, $A = \{0, 1, 2, 5\}$, $|A| = \lceil\sqrt{|G|}\rceil$, and $|G : A| = 1$ (this is a non-obvious example).

  It is also interesting to know about this question that the inequality $\frac{1}{2} + \sqrt{n - \frac{3}{4}} > \lceil\sqrt{n}\rceil$ has the solution set
  $$\bigcup_{k=2}^{\infty}(k^2 - k + 1, k^2] \cap \mathbb{N}.$$
  The explanations in Subsection 2.2 and the associated inequalities imply that no group of order n belonging to this set satisfies the property referenced as [6] I(4). Therefore, a necessary condition for a group G to have property [6]I(4) is that its order belongs to the complement of this set, which includes integers such as 1, 2, 3, 5, 6, 7 and 10. Additionally, if $\lceil\sqrt{|G|}\rceil$ is replaced by $\lfloor\sqrt{|G|}\rfloor$, then the only solution is $G = A = \{1\}$. This is because $\frac{1}{2}+\sqrt{n - \frac{3}{4}} > \lfloor\sqrt{n}\rfloor$ holds for all $n > 1$.
- [5] III(c): No counterexample in groups up to order 24. (link)
- [5] III(i): This is true for finite groups, and no counterexample for the converse in groups up to order 18. (link)
- [6] IV: No counterexample for the first part up to order 23, and none for the second part up to order 18. (link)

Acknowledgments

The authors are grateful to Professor George Bergman for his valuable comments and for providing a theoretical method to prove the non-index stability of $G = C_2 \times C_2 \times C_2 \times C_2 \times C_2$. We also sincerely thank the referee for their valuable suggestions and insightful comments, which have significantly improved the clarity, accuracy, and overall quality of the paper.

References

[1] G. Bergman, A note on factorizations of finite groups, *J. Iran. Math. Soc.*, **1** no. 2 157–161.

[2] R. R. Bildanov, V. A. Goryachenko and A. V. Vasil'ev, Factoring nonabelian finite groups into two subsets, *Sib. Èlektron. Mat. Izv.*, **17** (2020) 683–689.

[3] M. H Hooshmand, Factor subset of finite group, 2014. mathoverflow.net/questions/155986/factor-subset-of-finite-group

[4] M. H. Hooshmand, Basic results on an unsolved problem about factorization of finite groups, *Comm. Algebra*, **49** no. 7 (2021) 2927–2933.

[5] M. H. Hooshmand, Index, sub-index and sub-factor of groups with interactions to number theory, *J. Algebra Appl.*, **19** no. 6 (2020) 23 pp.

[6] M. H. Hooshmand, Subindices and subfactors of finite groups, *Comm. Algebra*, **51** no. 6 (2023) 2644–2657.

[7] M. H. Hooshmand, Groups of orders 7! and $\frac{7!}{2}$, 2022. mathoverflow.net/questions/430529

[8] E. I. Khukhro and V. D. Mazurov, *Unsolved problems in group theory: the kourovka notebook*, **20**, Sobolev Institute of Mathematics, 2022.

Mohammad Hadi Hooshmand

Department of Mathematics, Shi.C., Islamic Azad University, Shiraz, Iran

Email: mh.hooshmand@iau.ac.ir

Moein Yousefian Arani

Department of Mathematical Sciences, Sharif University of Technology, Azadi Avenue, Tehran, 11155-9415, Tehran, Iran

Email: moein.yousefian49@sharif.edu

https://dx.doi.org/10.22108/ijgt.2025.142465.1919