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A GENERALIZATION OF TOTALLY IMPRIMITIVE PERMUTATION GROUPS

RÜMEYSA S. ALTINKAYA* AND AYNUR ARIKAN

Abstract. In the present paper, we investigate a general form of totally imprimitive finitary permutation groups. Totally imprimitive groups were originally defined by P. M. Neumann for subgroups of the finitary symmetric group acting on an infinite set. We aim to extend this definition to arbitrary subgroups of symmetric groups, which are not necessarily finitary. To this end, we introduce a property that we shall call the support-block property for subgroups of symmetric groups. This property allows us to generalize several results previously obtained for totally imprimitive finitary permutation groups to a broader class of permutation groups acting on infinite sets. We recover and extend various structural results obtained in earlier studies, with particular emphasis on transitive permutation groups. Moreover, we reconstruct an example arising in previous studies on finitary permutation groups and show that it satisfies the support-block property, illustrating the applicability of our approach. Several consequences and applications of this framework are also discussed.

1. Introduction

Let Ω be a nonempty set. The full permutation groups and finitary permutation groups on Ω are denoted by $\mathrm{Sym}(\Omega)$ and $\mathrm{FSym}(\Omega)$ respectively (see [1]). Throughout the paper, we consider Ω as an infinite set and G is a transitive permutation group on Ω. Hence, there is only one G-orbit. Recall

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*Corresponding author.

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that for every $x \in \mathrm{FSym}(\Omega)$, the support of x $\mathrm{supp}(x)$ is finite. Totally imprimitivity for finitary permutation groups is defined as follows (see [8]):

Suppose $G \leq \mathrm{FSym}(\Omega)$. If G does not have a maximal G-block, and if G has an ascending family of finite G-blocks

$$\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \cdots$$

such that $\Omega = \bigcup_{i\in\omega} \Delta_i$, then G is called a totally imprimitive permutation group. Hence, for every finite subgroup F of G, there is $n \in \omega$ such that $\mathrm{supp}(F) \subset \Delta_n$.

In this paper, our aim is to extend the results obtained for totally imprimitive subgroups of finitary symmetric groups to subgroups of symmetric groups. In the second section, we define the support-block property and present an example of a group that satisfies this property. Therefore, if G is a subgroup of $\mathrm{Sym}(\Omega)$ for an infinite set Ω, we can generalize to G some results that were obtained for totally imprimitive finitary permutation groups. We also demonstrate that our definition is useful in the third section.

Throughout the paper, we mostly rely on the arguments due to Peter M. Neumann in many cases.

2. Support-Block Property

In the present section, we consider the condition of total imprimitivity in a more general context: Let G be a transitive subgroup of $\mathrm{Sym}(\Omega)$. We say that G satisfies the *support-block property* ($\mathcal{SBP}$ in short) if for every finitely generated subgroup F of G, there is a proper G-block Δ (i.e. $\Delta \neq \Omega$) such that

$$\mathrm{supp}(F) \subseteq \Delta.$$

As we shall show in Example 2.1, the context of groups with $\mathcal{SBP}$ is rather large. Now we study certain properties of groups with $\mathcal{SBP}$. By exploiting some of the ideas known for totally imprimitive (finitary permutation) groups, we will obtain certain generalizations to the permutation groups with $\mathcal{SBP}$. The following example shows that the context of groups with $\mathcal{SBP}$ is rather large:

Example 2.1. [4, cf., Section 4] *Let for each $n \in \omega$, $H_n \leq \mathrm{Sym}(\Omega_n)$ be a transitive (possibly infinite) permutation group. We shall consider the infinite iterated wreath product,*

$$W = H_0 wr_{\Omega_1} H_1 wr_{\Omega_2} H_2 wr_{\Omega_3} \cdots wr_{\Omega_i} H_i \cdots$$

We can also consider W as the general wreath product $wr_{i\in\omega} H_i$ (see [2]) on the set

$$\Omega = \{(\alpha_n)_{n\in\omega} \in Dr_{n\in\omega}\Omega_n | \alpha_n = 1_n \text{ for all but finitely many } n \in \omega\}$$

via

$$(\alpha_n)_{n\in\omega} x = \begin{cases} \alpha_n x & \text{if } n = i \text{ and } \alpha_m = 1_m \text{ for all } m > n \\ \alpha_n & \text{else} \end{cases}$$

for $x \in H_i$.

Such a wreath product W is the direct limit of the iterated wreath products $W_0 = H_0$ and

$$W_n = W_{n-1} wr_{\Omega_n} H_n = H_0 wr_{\Omega_1} H_1 wr_{\Omega_2} H_2 wr_{\Omega_3} \cdots wr_{\Omega_n} H_n$$

for $n \geq 1$, with respect to the canonical embedding of W_{n-1} onto a fixed component of the base group of W_n (corresponding to $1_n \in \Omega_n$).

Let $\Delta_n = \operatorname{supp}_\Omega W_n = \{(\alpha_n)_{n\in\omega} \in \Omega | \alpha_m = 1_m$ for all $m > n\}$. If $x \in W_n$ for an element x, then $\Delta_n^x = \Delta_n$. For $x \notin W_n$, there is an integer $m > n$ such that $x \in W_m$. If $1_m \in \operatorname{supp}(x)$, then $\Delta_n^x \cap \Delta_n = \emptyset$. If $1_m \notin \operatorname{supp}(x)$, then $\Delta_n^x = \Delta_n$. Therefore Δ_n is a W-block in Ω for each n. Since W_n affects nth coordinate of Ω, only $\Delta_n \neq \Omega$.

Let F be a finitely generated subgroup of W, then $F \leq W_n$ for some $n \in \omega$. Hence

$$\operatorname{supp}_\Omega F \subseteq \Delta_n = \operatorname{supp}_\Omega W_n.$$

Since Δ_n is a block for such F, the group W satisfies $\mathcal{SBP}$. We also have

$$\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \cdots$$

such that $\Omega = \bigcup_{n\in\omega} \Delta_n$.

We can use Example 2.1 to give the following embedding theorem.

Theorem 2.2. *Let G be a countable transitive permutation group on a set. Then G can be embedded into a transitive permutation group with $\mathcal{SBP}$.*

Proof. Put $W_0 = G$ and choose any transitive permutation groups W_i on an infinite set for $i \in \omega \backslash \{0\}$. Then we can construct the iterated wreath product

$$W = H_0 wr_{\Omega_1} H_1 wr_{\Omega_2} H_2 wr_{\Omega_3} \cdots wr_{\Omega_i} H_i \cdots$$

Now W is a transitive permutation group with $\mathcal{SBP}$ as in Example 2.1 in which G is embedded. So, the proof is complete. □

Let G be a transitive permutation group on an infinite set with $\mathcal{SBP}$. Suppose that for every finitely generated subgroup F of G there is a unique nontrivial G-block Δ such that $\operatorname{supp}_\Omega F \subseteq \Delta$. Since G is transitive on Ω, there exists a $g \in G$ such that for every $\alpha \in \Omega$, $\alpha \in \operatorname{supp}(\langle g \rangle)$. By G with $\mathcal{SBP}$, $\operatorname{supp}(\langle g \rangle) \subseteq \Delta$. Thus, $\Omega = \Delta$. Consequently, for every finitely generated subgroup F of G, the only block that contains all blocks with $\operatorname{supp}_\Omega F \subseteq \Delta$ is Ω.

Lemma 2.3. *Let G be a transitive group with $\mathcal{SBP}$. Then G has no maximal proper G-block.*

Proof. Suppose that Γ is a maximal proper G-block. Choose $\alpha \in \Gamma$ and $\beta \notin \Gamma$. By transitivity, there exists $g \in G$ such that $\alpha^g = \beta$. Since Γ is a G-block and Γ^g contains β, we have $\Gamma^g \cap \Gamma = \emptyset$. Hence, $\Gamma \subseteq \operatorname{supp}(g)$. By $\mathcal{SBP}$, there exists a proper G-block Δ such that $\operatorname{supp}(\langle g \rangle) \subseteq \Delta$. Therefore,

$$\Gamma \subseteq \operatorname{supp}(g) \subseteq \operatorname{supp}(\langle g \rangle) \subseteq \Delta.$$

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Since Γ is maximal, we have $\Delta = \Gamma$. Thus, $\mathrm{supp}(g) \subseteq \Gamma$. Hence $\mathrm{supp}(g) = \Gamma$. Since $\alpha^g = \beta$, we have $\beta^{g^{-1}} = \alpha \neq \beta$. Thus

$$\beta \in \mathrm{supp}(g^{-1}) \subseteq \mathrm{supp}(\langle g \rangle) \subseteq \Gamma,$$

contradicting the choice of $\beta \notin \Gamma$. Therefore, G has no maximal proper G-block. □

3. Applications of Support-Block Property

Let F be a finitely generated subgroup of G and Δ_F be a nontrivial proper G-block with $\mathrm{supp}_\Omega F \subseteq \Delta_F$. Then $\mathcal{X}_{\Delta_F} := \{\Delta_F^g | g \in G\}$ is a system of G-blocks and there is a permutation representation

$$\psi_{\Delta_F} : G \to \mathrm{Sym}(\mathcal{X}_{\Delta_F}),$$

which is defined via

$$(\Delta_F^h)^{g\psi_{\Delta_F}} = \Delta_F^{hg}.$$

We use the notation ψ_{Δ_F} in the sequel to describe the above permutation representation. The following result generalizes the result of [8] known as the lawlessness theorem for totally imprimitive of finitary permutation case to the groups with $\mathcal{SBP}$.

Lemma 3.1. *Let G be a transitive permutation group on an infinite set with $\mathcal{SBP}$. Then $G = \bigcup Ker(\psi_{\Delta_F})$ for every finitely generated subgroup F of G.*

Proof. Clearly $\bigcup Ker(\psi_{\Delta_F}) \leq G$. Let $g \in G$. Then there exists a proper G-block $\Delta_{\langle g \rangle}$ such that $\mathrm{supp}(g) \subseteq \mathrm{supp}(\langle g \rangle) \subseteq \Delta_{\langle g \rangle}$. Since for each $h \in G$, either $\Delta_{\langle g \rangle}^h = \Delta_{\langle g \rangle}$ or $\Delta_{\langle g \rangle}^h \cap \Delta_{\langle g \rangle} = \emptyset$, either $\mathrm{supp}(g) \subseteq \Delta_{\langle g \rangle}^h$ or $\mathrm{supp}(g) \cap \Delta_{\langle g \rangle}^h = \emptyset$. In both cases $(\Delta_{\langle g \rangle}^h)^g = \Delta_{\langle g \rangle}^h$, so $g \in Ker(\psi_{\Delta_{\langle g \rangle}})$. □

Recall that a group G is lawless if it generates the variety of all groups; that is, it does not satisfy any nontrivial law.

Theorem 3.2. *Let G be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$. If $G \in \mathfrak{B}$ for some nontrivial variety $\mathfrak{B}$, then $G/Ker(\psi_{\Delta_F})$ has a finite exponent for some finitely generated subgroup F with $\mathrm{supp}(F) \subseteq \Delta_F$ for a proper nontrivial G-block Δ_F. In other words, there is a positive integer n such that $(\Delta_F^h)^{g^n} = \Delta_F^h$ for every $h \in G$ (i.e. $G^n \leq \bigcap_{h \in G} G_{\{\Delta_F^h\}}$).*

Proof. Assume that $G/Ker(\psi_{\Gamma_Y})$ has infinite exponent for every finitely generated subgroup Y of G with $\mathrm{supp}(Y) \subseteq \Gamma_Y$.

Let n be a natural number and F be a finitely generated subgroup of G. Then there is a nontrivial block such that $\mathrm{supp}(F) \subseteq \Delta_F$.

Put $\Delta = \Delta_F$. Note that [8, Lemma 3.1] requires only transitivity, not totally imprimitivity. Since $G/\operatorname{Ker}(\psi_\Delta)$ has infinite exponent, [8, Lemma 3.1] yields an element $g \in G$ such that

$$|\Delta^{\langle g \rangle}| \geq n + 1.$$

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As Δ is a G-block, the blocks

$$\Delta, \Delta^g, \ldots, \Delta^{g^n}$$

are therefore pairwise disjoint. Moreover, the proof of [8, Lemma 3.2] uses total imprimitivity only to obtain a proper block containing the support of the given finite subgroup. In the present setting, this is provided directly by the support-block property. Hence the remainder of Neumann's proof applies without modification. Also

$$\mathrm{supp}(F^{g^i}) = (\mathrm{supp}(F))^{g^i} \subseteq \Delta^{g^i}$$

for $0 \leq i \leq n$ and since

$$\Delta, \Delta^g, \Delta^{g^2}, \ldots, \Delta^{g^n}$$

are pairwise disjoint, we have that

$$\mathrm{supp}(F),\ \mathrm{supp}(F^g),\ \ldots,\ \mathrm{supp}(F^{g^n})$$

are disjoint and $[F, F^{g^i}] = 1$ for every $1 \leq i \leq n$. Hence

$$\langle F^{g^i} | 0 \leq i \leq n \rangle = F \times F^g \times \cdots \times F^{g^n}.$$

Let $\mathfrak{A}$ be the variety of abelian groups. If X is any finitely generated subgroup of G, then $var(X).\mathfrak{A} \leq var(G)$ by [7, Lemma 2.2]. Therefore for

$$\begin{aligned} \mathrm{var}(G).\mathfrak{A} &= (\vee var(X)).\mathfrak{A} \\ &= \vee (var(X).\mathfrak{A}) \\ &\leq var(G) \end{aligned}$$

Now it follows that $\mathfrak{A}^k \leq var(G)$ for every positive integer k. Hence by [6, P.40] we have $var(G) = \mathfrak{O}$ the variety of all groups. This means that G does not satisfy any nontrivial law, contradicting the assumption on the variety $\mathfrak{B}$. Hence $G/Ker(\psi_\Delta)$ has finite exponent for some finitely generated subgroup F of G with $\mathrm{supp}(F) \subseteq \Delta$, as desired. $\square$

Corollary 3.3. *Let G be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$. If G is soluble, then G has a proper subgroup of finite index.*

Proof. Since G is soluble, G satisfies a nontrivial commutator law. So $G \in \mathfrak{B}$ for a nontrivial variety $\mathfrak{B}$. By Theorem 3.2, $G/Ker(\psi_{\Delta_F})$ has finite exponent. Hence $G/Ker(\psi_{\Delta_F})$ is not quasi-radicable. Since $G/Ker(\psi_{\Delta_F})$ is soluble, it has a proper subgroup of finite index by [5, Lemma 4.1]. If $H/Ker(\psi_{\Delta_F})$ is a proper subgroup of finite index of $G/Ker(\psi_{\Delta_F})$, then H is a proper subgroup of finite index of G. $\square$

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Corollary 3.4. *Let G be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$. If $var(G) \neq \mathfrak{D}$, then either G has finite exponent or there is a G-block Δ and a positive integer n such that $g^n \notin G_{(\Delta)}$ for some $g \in G$.*

Proof. By the proof of Theorem 3.2, G has a finitely generated subgroup F such that $G/Ker(\psi_{\Delta_F})$ has finite exponent. Hence there exists a positive integer n such that $g^n \in \mathrm{Ker}(\psi_{\Delta_F})$ for every $g \in G$. So $(\Delta_F^h)^{g^n} = \Delta_F^h$ for every $g, h \in G$. Put $\Delta = \Delta_F$ and $G^n := \langle g^n | g \in G\rangle$. Then every element of G^n stabilizes each block Δ^h setwise. For a fixed $h \in G$, consider the induced action of G^n on Δ^h

$$\tau_h : G^n \to (G^n)^{\Delta^h} \text{ via } g^n \mapsto (g^n)^{\Delta^h}.$$

By the first isomorphism theorem $G^n/Ker(\tau_h) \cong (G^n)^{\Delta^h}$. Assume first that $G^n \leq Ker(\tau_h)$ for all $h \in G$. It follows that G^n fixes every point of every block Δ^h. Since the family $\{\Delta^h | h \in G\}$ forms a partition of Ω, it follows that every element of G^n fixes every point of Ω. Thus $G^n = 1$. Consequently, every element of G has order dividing n and therefore G has finite exponent.

Suppose now that G does not have finite exponent. Then there exists some $h \in G$ such that $(G^n)^{\Delta^h} \neq 1$. Hence $G^n \not\leq Ker(\tau_h)$, so there exists an element $x \in G^n$ whose induced action on Δ^h is nontrivial. Since G^n is generated by the elements g^n for $g \in G$, at least one generator g^n must act nontrivially on Δ^h otherwise every generator would belong to $Ker(\tau_h)$, which implies $G^n \leq Ker(\tau_h)$, a contradiction. Therefore, there exists $g \in G$ such that $(g^n)^{\Delta^h} \neq 1$, that is,

$$g^n \notin G_{(\Delta^h)}.$$

Since Δ^h is also a G-block, replacing Δ^h by Δ. This completes the proof. □

Corollary 3.5. *Let G be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$. If G is quasi-radicable and has no proper subgroup of finite index, then G is lawless.*

Proof. Since G is quasi-radicable, we have $G = G^n$ for every positive integer and G has no proper subgroup of finite index. Hence the result follows by Corollary 3.4. □

The following lemma contains analogous results of [3, Lemma 3].

Lemma 3.6. *Let p be a prime and let $G = \langle g_i \mid g_i^p = 1, i \in \omega\rangle$ be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$. Then G contains a subgroup isomorphic to*

$$C_p \wr C_p \wr \cdots \wr C_p \wr \cdots .$$

Proof. Let g be an element of G of order p. Then there exists a proper G-block Δ_{i_1} such that $\mathrm{supp}(g) \subseteq \Delta_{i_1}$. Since G is transitive, there is $i \in \omega$ and $g_{i_1} \in G$ of order p such that

$$\Delta_{i_1}^{g_{i_1}} \neq \Delta_{i_1}.$$

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Now consider $G_{i_1} := \langle g, g_{i_1}\rangle$. The elements g and $g^{g^n_{i_1}}$, $1 \leq n \leq p-1$ commute. Since $\langle g^{g^m_{i_1}}\rangle$ and $\langle g^{g^n_{i_1}}\rangle$, $1 \leq m, n \leq p-1$, $n \neq m$ moves distinct points of Δ_{i_1}, $\langle g^{g^m_{i_1}}\rangle \cap \langle g^{g^n_{i_1}}\rangle = 1$ for all $n \leq m$ and

$$\langle g, g^{g_{i_1}}, g^{g^2_{i_1}}, \dots, g^{g^{p-1}_{i_1}}\rangle = \langle g\rangle \times \langle g^{g_{i_1}}\rangle \times \langle g^{g^2_{i_1}}\rangle \times \cdots \times \langle g^{g^{p-1}_{i_1}}\rangle$$

and

$$\langle g, g_{i_1}\rangle = \langle g\rangle \times \langle g\rangle^{g_{i_1}} \times \langle g\rangle^{g^2_{i_1}} \times \cdots \times \langle g\rangle^{g^{p-1}_{i_1}} \rtimes \langle g_{i_1}\rangle.$$

Hence $\langle g, g_{i_1}\rangle = \langle g\rangle \wr \langle g_{i_1}\rangle \cong C_p \wr C_p$. Now there exists a proper G-block Δ_{i_2} such that $\mathrm{supp}(G_{i_1}) \subseteq \Delta_{i_2}$ and there exists $g_{i_2} \in G$ of order p such that

$$\Delta_{i_2}^{g_{i_2}} \neq \Delta_{i_2}.$$

Then the elements of G_{i_1} and $\langle g_{i_2}\rangle$ do not commute but, for any $x \in G_{i_1}$, the elements x and $x^{g^n_{i_2}}$, $1 \leq n \leq p-1$ commute, since $\Delta_{i_2}, \Delta_{i_2}^{g_{i_2}}, \dots, \Delta_{i_2}^{g^{p-1}_{i_2}}$ are pairwise disjoint G-block, the supports of $G_{i_1}, G_{i_1}^{g_{i_2}}, \dots, G_{i_1}^{g^{p-1}_{i_2}}$ are pairwise disjoint. Hence these subgroups centralize one another and

$$\langle G_{i_1}, g_{i_2}\rangle = G_{i_1} \times G_{i_1}^{g_{i_2}} \times \cdots \times G_{i_1}^{g^{p-1}_{i_2}} \rtimes \langle g_{i_2}\rangle.$$

Thus

$$G_{i_2} := \langle g, g_{i_1}, g_{i_2}\rangle = G_{i_1} \wr \langle g_{i_2}\rangle.$$

We can continue this process inductively. If G_{i_j} has already been constructed, then G_{i_j} is finite and finitely generated. By $\mathcal{SBP}$, there exists a proper G-block $\Delta_{i_{j+1}}$ such that $\mathrm{supp}(G_{i_j}) \subseteq \Delta_{i_{j+1}}$. Choose $g_{i_{j+1}} \in G$ of order p with

$$\Delta_{i_{j+1}}^{g_{i_{j+1}}} \neq \Delta_{i_{j+1}}.$$

As above, the conjugates

$$G_{i_j}, G_{i_j}^{g_{i_{j+1}}}, \dots, G_{i_j}^{g^{p-1}_{i_{j+1}}}$$

have pairwise disjoint supports and therefore centralize one another. Consequently,

$$G_{i_{j+1}} = \langle G_{i_j}, g_{i_{j+1}}\rangle = G_{i_j} \wr \langle g_{i_{j+1}}\rangle.$$

Hence we obtain $G_{i_1} < G_{i_2} < \cdots$, the union of this ascending chain is therefore isomorphic to the infinite iterated wreath product

$$C_p \wr C_p \wr \cdots \wr C_p \wr \cdots.$$

This completes the proof. □

The rest of this section contains analogous results of [9, Lemma 3.2, Theorem 3.3, Corollary 3.4, Theorem 1].

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Lemma 3.7. *Let G be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$ and N be a transitive normal subgroup of G. If $g \in G$ and Δ is a nontrivial proper G-block such that* $\mathrm{supp}(g) \subseteq \Delta$ *then there exists $h \in N$ such that $\alpha^g = \alpha^h$ for all $\alpha \in \Delta$.*

Proof. Since N is a transitive subgroup of G on Ω, there is $x \in N$ such that $\Delta \cap \Delta^x = \emptyset$. Now $h := [x, g] = x^{-1}x^g = (g^{-1})^x g \in N$. Since $N \trianglelefteq G$, we have $h \in N$. Since $\mathrm{supp}((g^{-1})) \subseteq \Delta$,

$$\mathrm{supp}((g^{-1})^x) = (\mathrm{supp}(g^{-1}))^x = (\mathrm{supp}(g))^x \subseteq \Delta^x.$$

Because $\Delta \cap \Delta^x = \emptyset$, every point of Δ lies outside the support of $(g^{-1})^x$. Therefore $\alpha^{(g^{-1})^x} = \alpha$ for all $\alpha \in \Delta$. Finally, for every $\alpha \in \Delta$

$$\alpha^h = \alpha^{(g^{-1})^x g} = (\alpha^{(g^{-1})^x})^g = \alpha^g,$$

since $(g^{-1})^x$ fixes every point of Δ. This completes the proof. □

Theorem 3.8. *Let G be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$ and N be a transitive normal subgroup of G. If M is a normal subgroup of N, then M is normal in G.*

Proof. If $m \in M$ and $g \in G$, then there is a nontrivial proper G-block Δ containing $\mathrm{supp}(\langle m, g\rangle)$. By Lemma 3.7, there exists $h \in N$ such that $\alpha^g = \alpha^h$ for all $\alpha \in \Delta$. Since

$$h = [x, g] = (g^{-1})^x g,$$

and

$$\mathrm{supp}(m) \subseteq \Delta, \qquad \mathrm{supp}((g^{-1})^x) \subseteq \Delta^x,$$

where $\Delta \cap \Delta^x = \emptyset$, the elements m and $(g^{-1})^x$ commute. Hence

$$m^h = m^{(g^{-1})^x g} = (m^{(g^{-1})^x})^g = m^g.$$

Since $h \in N$ and $M \trianglelefteq N$, we have $m^h \in M$, and therefore $m^g \in M$. Hence M is normal in G. □

Corollary 3.9. *Let G be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$. Then transitive subnormal subgroups are normal in G.*

Theorem 3.10. *Let G be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$. If N is a transitive normal subgroup of G, then $G' \leq N$.*

Proof. Suppose that $g, h \in G$. Then there is a nontrivial proper G-block Δ such that $\mathrm{supp}(\langle g, h\rangle) \subseteq \Delta$. Since N is transitive on Ω, there is $x \in N$ so that $\Delta \cap \Delta^x = \emptyset$. Put

$$c := g^{-1}x^{-1}gxh^{-1}x^{-1}hx(gh)x^{-1}(gh)^{-1}x.$$

Since

$$\mathrm{supp}(\langle x^{-1}gx\rangle), \mathrm{supp}(\langle x^{-1}hx\rangle), \mathrm{supp}(\langle x^{-1}(gh)^{-1}x\rangle) \subseteq \Delta^x,$$

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these subgroups commute with all of g, h and gh. Therefore c can be rewritten as

$$c := g^{-1}h^{-1}ghx^{-1}gxx^{-1}hxx^{-1}(gh)^{-1}x.$$

Thus c is just the commutator $[g, h]$. On the other hand, c is a product of conjugates of x and x^{-1}, and so $c \in N$. Hence all commutators lie in N and $G' \leq N$. □

Lemma 3.11. *Let G be a group which acts transitively on an infinite set Ω and N be a proper normal subgroup of G, then G/N acts transitively on the set of all orbits of N.*

Proof. Put $\Gamma = \{\alpha^N \mid \alpha \in \Omega\}$. We would show that $\rho : G/N \to \mathrm{Sym}(\Gamma), gN \mapsto (\alpha^N)^{gN} = (\alpha^g)^N$ is a homomorphism. Hence $\Gamma \times G/N \to \Gamma, (\alpha^N, gN) \mapsto (\alpha^N)^{gN} = (\alpha^g)^N$ defines an action of G/N on Γ.

Let $\alpha^N, \beta^N \in \Gamma$, then $\alpha^g = \beta$ for some $g \in G$ and so $(\alpha^N)^{gN} = \beta^N$ which implies that G/N acts transitively on Γ. □

Remark 3.12. *Let G be an abelian transitive permutation group on Ω, then G is regular. In particular $\Omega = \mathrm{supp}(g)$ for all nontrivial $g \in G$.*

Proof. It is clear that $g^{-1}G_\alpha g = G_{\alpha^g}$ by the transitivity of G. Since G is abelian, $G_{\alpha^g} = g^{-1}G_\alpha g = G_\alpha$ for all $\alpha \in \Omega$. This implies that every element of G_α fixes every element of Ω. But the only element that satisfies this condition is trivial permutation. Hence $G_\alpha = 1$ for all $\alpha \in \Omega$. So, the proof is complete. □

Lemma 3.13. *Let G be a transitive permutation group with $\mathcal{SBP}$ on an infinite set Ω. Then G' is also transitive on Ω.*

Proof. Suppose that $G \neq G'$. Assume that G' is not transitive on Ω. So $\alpha^{G'} \neq \Omega$ for any $\alpha \in \Omega$. By Lemma 3.11 G/G' acts transitively on the set of all orbits of G' such that $(\beta^{G'})^{xG'} = (\beta^x)^{G'}$ for all $\beta \in \Omega$. Put $X = \{\alpha^{G'} \mid \alpha \in \Omega\}$. So there is a homomorphism

$$\rho : G/G' \to \mathrm{Sym}(X),\ gG' \mapsto (\beta^{G'})^{gG'} = (\beta^g)^{G'}.$$

Since G/G' is abelian, $\rho(G/G')$ is regular by Remark 3.12 i.e. each point stabilizer $\rho(G/G')_{\alpha^{G'}} = 1$ for all $\alpha^{G'} \in X$. If $G/G' = \mathrm{Ker}\rho$, then, for all $g \in G$, $\alpha^g \in \alpha^{G'}$ for some $\alpha \in \Omega$. Hence $\alpha^{G'} = \Omega$ which contradicts assumption. If $G/G' \neq \mathrm{Ker}\rho$ and hence, by the regularity of $\rho(G/G') \neq 1$, there is a nontrivial element $(gG')^\rho \in \rho(G/G')$ such $\alpha^{G'} \in \mathrm{supp}((gG')^\rho)$ for all $\alpha \in \Omega$. Hence $(\alpha^g)^{G'} \neq \alpha^{G'}$ for all $\alpha \in \Omega$. Since X is a partition of Ω, $(\alpha^g)^{G'} \cap \alpha^{G'} = \emptyset$ for all $\alpha \in \Omega$. So $\Omega \subseteq \mathrm{supp}(g)$ since G is transitive and there is a proper G-block Γ such that $\mathrm{supp}(\langle g \rangle) \subseteq \Gamma$. Hence $\Omega \subseteq \Gamma$, which is a contradiction. Therefore, the assertion is true. □

Lemma 3.14. *Let G be a transitive permutation group with $\mathcal{SBP}$ on an infinite set Ω and $H \leq G$. If H is transitive on Ω, then H also satisfies the $\mathcal{SBP}$.*

Proof. Let F be a finitely generated subgroup of H. Since $F \leq G$ and G satisfies $\mathcal{SBP}$, there exists a proper G-block Δ such that $\mathrm{supp}(F) \subseteq \Delta$. Since $H \leq G$, for every $h \in H$

$$\Delta^h = \Delta \text{ or } \Delta^h \cap \Delta = \emptyset$$

because the same property already holds for every element of G. Hence Δ is also H-block. Therefore H satisfies the $\mathcal{SBP}$. $\square$

Corollary 3.15. *If G is a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$, then G' is perfect.*

Proof. By Lemma 3.13, the derived subgroup G' is transitive on Ω. Since $G' \leq G$, the Lemma 3.14 shows that G' also satisfies the $\mathcal{SBP}$. Applying Lemma 3.13 to the transitive group G', we conclude that $(G')' = G''$ is transitive on Ω. Moreover G'' is a transitive normal subgroup of G. Since $G'' \leq G'$, Theorem 3.10 yields

$$G' \leq G'' \leq G',$$

and hence

$$G' = G''.$$

$\square$

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Rümeysa S. Altınkaya

Department of Mathematics, Faculty of Science, Gazi University, P.O.Box 06500, Ankara, Türkiye

Email: `rsgolcu@gazi.edu.tr`

Aynur Arıkan

Department of Mathematics, Faculty of Science, Gazi University, P.O.Box 06500, Ankara, Türkiye

Email: `yalincak@gazi.edu.tr`

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