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RATIONAL SUBSETS OF FINITE GROUPS

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Abstract. We characterize the rational subsets of a finite group and discuss the relations to integral Cayley graphs.

1. Introduction

We have investigated the Cayley graphs of finite groups for which all the eigenvalues of the adjacency matrix are integers, [1]; these are called integral Cayley graphs. This study motivated the definition of a rational set (see the definition in Section 2) in a finite group G. One example of a rational set in the finite group G is the set $[a]$ of generators for the cyclic subgroup generated by $a \in G$, called a cyclic rational set. In this article we give a complete description, Theorem 2.1, of the rational subsets of a finite group.

Our results in [1] and [2] are now relatively easy consequences of this new result. The results obtained (in collaboration with B. Peterson) for abelian groups can be summarized: a subset of an abelian finite group is rational iff it is disjoint union of cyclic rational sets. As noted in [1] if the Cayley graph of G on a set S is an integral Cayley graph then S is a rational set. Also, as shown there: for abelian groups a set S is rational if and only if the Cayley graph on S is an integral Cayley graph.

In [1] we introduced the Boolean algebra *generated* by rational subsets, $\mathbb{B}(\mathcal{I}_G)$; let $\mathcal{P}(X)$ denote Boolean algebra of the power set of X. Also $\mathbb{B}(\mathcal{I}_G)$ is a direct sum of $\mathcal{P}(G-Z)$ and $\mathbb{B}(\mathcal{I}_Z)$, where Z is the Center(G), [2]; the atoms of $\mathbb{B}(\mathcal{I}_Z)$ are the cyclic rational sets. In addition $\mathbb{B}(\mathcal{I}_G) = \mathbb{B}(\mathcal{F}_G)$, the Boolean algebra *generated* by the subgroups of G, if and only if G is abelian, [1].

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2. **Rational Subsets**

A set $S \subset G$ is called rational if for every character χ then

$$\chi(S) = \sum_{a \in S} \chi(a) \in \mathbb{Q}.$$

Let $[a] = \{b \mid \langle b \rangle = \langle a \rangle\}$, the set of generators of the cyclic subgroup generated by a. It is easy to see that $[a]$ is a rational set [1].

Consider the conjugacy classes of the elements of $[a]$: these classes, restricted to $\langle a \rangle$, have the same number of elements, some divisor of $\phi(m)$, the Euler function of m, the order of a. To see this, say a is conjugate to a^i via b; now suppose a^j is in a different conjugacy class then a^{ij} also belongs to the class of a^j via conjugation by b. Suppose then that $[a]$ is distributed across r conjugacy classes, each containing s elements, $rs = \phi(m)$. Thus $\chi([a]) = s\sum_{j=1}^{r} \chi(a^{i_j})$, where the a^{i_j} are in distinct classes. Thus $\sum_{j=1}^{r} \chi(a^{i_j})$ is also a rational number.

For each $a \in G$ we choose a fixed set of the distinct conjugates belonging to $[a]$, say $[[a]] = \{a^{i_j} \mid j = 1, \ldots, r\}$. This is also a rational set.

We introduce the relation on *sets* of the same cardinality, where we replace any element with a conjugate element. We call two sets equivalent under this relation *conjugal*. An *elementary* rational set is any set conjugal to one of the sets $[[a]]$ for some $a \in G$.

We prove the following result, thereby generalizing results of [1] for the case of abelian groups.

Theorem 2.1. *Any rational set is a disjoint union of elementary rational sets.*

3. **Character Table**

Let $\mathcal{C}$ denote the conjugacy classes of G. Consider the matrix $M = (m_{i,j}) = (\chi_i(a_j))$ (character table) where $a_j, j = 1, \ldots, n$ are representatives of the distinct classes of $\mathcal{C}$ and $\chi_i, i = 1, \ldots, n$, are the distinct irreducible characters, $n = |\mathcal{C}|$. Consider the conjugate transpose $\bar{M}^t$; the orthogonality formulas for characters is the same as $M\bar{M}^t D = |G|I_n$ where D is the diagonal matrix whose entries d_i are the sizes of the conjugacy classes; hence $M^{-1} = \frac{1}{|G|}\bar{M}^t D$.

For each $[[a]]$ let $v_{[[a]]}$ be the vector with a 1 in those locations which are in $[[a]]$, and 0s elsewhere. Thus $Mv_{[[a]]} \in \mathbb{Q}^n$ since it is a rational set.

Let $V = \{v \in \mathbb{Q}^n \mid Mv \in \mathbb{Q}^n\}$. Let $W = \mathrm{span}_{\mathbb{Q}}\{v_{[[a]]}, a \in G\}$ so $W \subseteq V$. If $v \in V$ then $Mv = w \in \mathbb{Q}^n$; hence $v = \frac{1}{|G|}\bar{M}^t Dw$. Thus its components are $v_i = \frac{1}{|G|}\sum_j d_j \bar{\chi_j}(a_i) w_j$.

The entries of M belong to the field $\mathbb{Q}(\zeta_m)$, where ζ_m is a primitive m-th root of unity, where m is the exponent of G. Say $b = a^r$, where r is relatively prime to $|a|$. The Galois group of $\mathbb{Q}(\zeta_m)$ over $\mathbb{Q}$ acts transitively on roots of unity of the same order. Hence there is an automorphism σ taking $\chi_j(a)$ to $\chi_j(b)$ for every j.

Thus (labelling components by group elements)

$$v_a = \sigma(v_a) = \frac{1}{|G|}\sum_j d_j \sigma(\bar{\chi_j}(a))w_j = \frac{1}{|G|}\sum_j d_j \bar{\chi_j}(b)w_j = v_b.$$

Since the b-component of v is the same as the a-component of v for all such b it now follows that $v \in W$; so $V = W$.

4. **Proof**

Now if $v_{[[a]]}$ and $v_{[[b]]}$ have a non-zero component in common then b is conjugate to a (relatively prime) power of a, so $v_{[[a]]} = v_{[[b]]}$. Thus we can pick a basis for V consisting of $v_{[[a]]}$ with disjoint supports.

Let A be a rational set. The vector $v_A \in \mathbb{Q}^n$ has entry $\#(A \cap c)$ for the component corresponding to conjugacy class c. Hence $Mv_A \in \mathbb{Q}^n$ so $v_A \in V = W$. Thus v_A is a linear combination of certain $v_{[[a]]}$ having disjoint supports. Since the entries of v_A are integers then v_A is a positive linear combination of these $v_{[[a]]}$, say $v_A = \sum r_a v_{[[a]]}$.

It follows that no subset of $[[a]]$ can be a rational set. Also any rational set A is a disjoint union of sets S_a where S_a is a union of r_a sets conjugal to $[[a]]$. The union of these r_a sets, each conjugal to $[[a]]$, is thus a disjoint union of r_a elementary sets. Thus we have proven the Theorem.

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