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ON THE TOTAL CHARACTER OF FINITE GROUPS

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Abstract. For a finite group G, we study the total character τ_G afforded by the direct sum of all the non-isomorphic irreducible complex representations of G. We resolve for several classes of groups (the Camina p-groups, the generalized Camina p-groups, the groups which admit $(G, Z(G))$ as a generalized Camina pair), the problem of existence of a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\chi) = \tau_G$ for some irreducible character χ of G. As a consequence, we completely determine the p-groups of order at most p^5 (with p odd) which admit such a polynomial. We deduce the characterization that these are the groups G for which $Z(G)$ is cyclic and $(G, Z(G))$ is a generalized Camina pair and, we conjecture that this holds good for p-groups of any order.

1. Introduction

In this paper, G denotes a finite group. Let $\mathrm{Irr}(G)$ and $\mathrm{nl}(G)$ be the set of all irreducible characters of G and the set of all nonlinear irreducible characters of G respectively. Then $\mathrm{lin}(G) = \mathrm{Irr}(G) \setminus \mathrm{nl}(G)$ is the set of linear characters of G. Suppose ρ is the direct sum of all the non-isomorphic irreducible complex representations of G. The character τ_G afforded by ρ is called the *total character* of G, that is, $\tau_G = \sum_{\chi \in \mathrm{Irr}(G)} \chi$. Since τ_G is stable under the action of the Galois group of the splitting field of G, $\tau_G(g) \in \mathbb{Z}$ for all $g \in G$.

The dimension $\tau_G(1)$ of ρ seems to have remarkable connections with the geometry of the group. For instance, in the case of the symmetric group $G = S_n$, $\tau_G(1)$ is the number of involutions of S_n ([10]) and, in the case of $G = GL(n, q)$, $\tau_G(1)$ is the number of symmetric matrices in $GL(n, q)$ ([5]).

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It is a consequence of a well known theorem due to Burnside and Brauer ([7, Theorem 4.3]) that, the total character of the group G is a constituent of $1+\chi+\cdots+\chi^{m-1}$ if χ is a faithful character which takes exactly m distinct values on G. S. M. Gagola, Jr. & M. L. Lewis classified (in [4]) all the solvable groups for which τ_G equals χ^2, for some $\chi \in \mathrm{Irr}(G)$. A. Mann also studied the decomposition of χ^2 and proved:
"*A nonabelian group G has a faithful irreducible character χ such that $\mathrm{Irr}(\chi^2) \subseteq \mathrm{lin}(G)$ if and only if $|G'| = 2$ and $Z(G)$ is cyclic*".
Here, $\mathrm{Irr}(\chi^2)$ is the set of all irreducible constituents of χ^2 ([1, Theorem 22.7]).

Motivated by this, K. W. Johnson raised the following question:
Does there exist an irreducible character χ of G and a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau_G$? (see [18]).
The aim of the article is to answer a weaker version of this question for several classes of p-groups including all p-groups of order at the most p^5; we examine the existence of a polynomial $f(x) \in \mathbb{Q}[x]$ and $\chi \in \mathrm{Irr}(G)$ such that $f(\chi) = \tau_G$. We call such a polynomial $f(x) \in \mathbb{Q}[x]$, if it exists, *a Johnson polynomial* of G. This problem has been studied for dihedral groups D_{2n} in [18] where it is proved that D_{2n} has a Johnson polynomial if and only if $8 \nmid n$. To describe the classes of groups to which our results apply, we recall some definitions.

A pair (G, N) is said to be a generalized Camina pair (abbreviated GCP) if N is normal in G and, all nonlinear irreducible characters of G vanish outside N ([12]). There are a number of equivalent conditions for $(G, Z(G))$ to be a GCP. An equivalent condition we will refer to is:
A pair $(G, Z(G))$ is a GCP if and only if for all $g \in G \setminus Z(G)$, the conjugacy class of g in G is gG'.
In this case, one can easily observe that $G' \subseteq Z(G)$ and $\chi(1) = |G/Z(G)|^{1/2}$ for all $\chi \in \mathrm{nl}(G)$. For such types of groups, the first author and R. Sarma investigated (in [19]) the existence of a Johnson polynomial. The following theorem was proved in [19].

Theorem 1.1. [19, Theorem 3.2] *Let $(G, Z(G))$ be a GCP. Then G has a Johnson polynomial if and only if $Z(G)$ is cyclic. In fact, if $Z(G)$ is cyclic then a Johnson polynomial of G is given by*

$$f(x) = d^2 \sum_{j=1}^{r} (x/d)^{lj} + d \sum_{\substack{j=1 \\ l \nmid j}}^{m} (x/d)^j,$$

where $d = |G/Z(G)|^{1/2}$, $r = |Z(G)/G'|$, $m = |Z(G)|$ and $l = |G'|$. In particular, $f(x) = d^2(x/d)^m + d\sum_{j=1}^{m-1}(x/d)^j$ when $Z(G)$ is cyclic and $Z(G) = G'$.

Further, the above theorem was used by the authors in [19] to classify all the nonabelian p-groups of order p^4 (p an odd prime) which have a Johnson polynomial. The purpose of this article is to examine the existence of a Johnson polynomial for p-groups of order greater than p^4. In this direction, we examine the family of Camina p-groups and generalized Camina groups. As a consequence, we are able to obtain a complete classification of groups of order p^5 which admit a Johnson polynomial.

A pair (G, N) is said to be a *Camina pair* if $1 < N < G$ is a normal subgroup of G and for every element $g \in G \setminus N$, $gN \subseteq Cl_G(g)$, the conjugacy class of g. In the special case $N = G'$, the group G

is said to be a Camina group. More generally, a group G is said to be a generalized Camina group if $Cl_G(g) = gG'$ for every element $g \in G \setminus G'Z(G)$. It is known (see [13]) that a nilpotent, generalized Camina group G is isoclinic to Camina group which is a p-group; the prime p is said to be associated to G.

Then, our main results can be stated as follows:

Theorem A. *Let G be a Camina p-group. Then G has a Johnson polynomial if and only if the nilpotency class of G is 2 and $Z(G)$ is cyclic.*

Theorem B. *Let $(G, Z(G))$ be a Camina pair and let $(G/Z(G), Z(G/Z(G)))$ be a generalized Camina pair. Then G does not possess a Johnson polynomial.*

Theorem C. *Let G be a nilpotent, generalized Camina group with associated prime p. Then G has a Johnson polynomial if and only if the nilpotency class of G is 2 and $Z(G)$ is cyclic.*

In the last section, we apply the above theorems to obtain the complete list of all groups of order p^5 (with p odd) which admit a Johnson polynomial. This is proved using case-by-case considerations (using a description of all groups of order p^5 by R. James ([8, Section 4.5])) but, in particular, we deduce the following:

Theorem D. *Let G be a nonabelian p-group of order p^5 with p odd. Then G has a Johnson polynomial if $Z(G)$ is cyclic and $(G, Z(G))$ is a GCP.*

In view of Theorem 1.1 and the above theorems, it seems reasonable to pose the following conjecture for p-groups:
Conjecture: A nonabelian p-group (with p odd) admits a Johnson polynomial if and only if $Z(G)$ is cyclic and $G' \leq Z(G)$.

2. Notations and Preliminaries

Throughout, C_n denotes the cyclic group of order n. Suppose G is a finite group. Then $Z(G)$, $G' = G_2$ and $cd(G)$ denote respectively the center, the commutator subgroup and the set of irreducible character degrees of G. If $a, b \in G$, then ${}^b a = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$. For $g \in G$, $Cl_G(g)$ denotes its conjugacy class $\{x^{-1}gx : x \in G\}$. The nilpotency class of a nilpotent group G is the number n such that $G_n \neq 1$ and $G_{n+1} = 1$, where $G_2 = [G, G] = G'$ and $G_{i+1} = [G_i, G]$ for $i \geq 2$. Further, if H is a subgroup of G and χ a character of G, $\chi\downarrow_H$ denotes the restriction of χ to H. Suppose N is a normal subgroup of G. Then we denote by $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) \setminus \mathrm{Irr}(G/N)$.

We start by recalling some basic results that we will need later.

Lemma 2.1. [7, Theorem 2.32]

(1) *If G has a faithful irreducible character, then $Z(G)$ is cyclic.*
(2) *If G is a p-group and $Z(G)$ is cyclic, then G has a faithful irreducible character.*

Proposition 2.2. *An abelian group has a Johnson polynomial if and only if it is cyclic. In fact, if G is a cyclic group of order n then $f(x) = 1+x+\cdots+x^{n-1}$ is a Johnson polynomial of G and $f(\chi) = \tau_G$ for every faithful irreducible character of G.*

Proof. Let $f(x)$ be a Johnson polynomial of G. Suppose, to the contrary, G is non-cyclic. Then by Lemma 2.1, $\ker(\chi) \neq \{1\}$ for all $\chi \in \mathrm{Irr}(G)$. Since G is an abelian group, τ_G is the regular character of G. Hence

$$\tau_G(g) = \begin{cases} |G| & \text{if } g = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $g \neq 1 \in \ker(\chi)$, we have $\tau_G(g) = f(\chi(g)) = f(\chi(1)) = \tau_G(1)$, which is a contradiction.

Conversely, let $G = \langle a \rangle$ be the cyclic group of order n. Set $\zeta_n = e^{\frac{2\pi i}{n}}$ and $f(x) = \sum_{i=0}^{n-1} x^i$. Consider the linear character $\lambda : G \longrightarrow \mathbb{C}^*$ defined by $a \mapsto \zeta_n$. Then λ is a faithful irreducible character and $f(\lambda) = \sum_{i=0}^{n-1} \lambda^i = \tau_G$. □

Lemma 2.3. *Let G be a non-abelian group. Then $\sum_{\chi \in lin(G)} \chi(g) = 0$ for each $g \in G \setminus G_2$.*

In this article, whenever we prove a certain group G does not possess a Johnson polynomial, we use the following simple observation.

Proposition 2.4. *Let χ be an irreducible character of G. If $g_1, g_2 \in G$ are such that $\chi(g_1) = \chi(g_2)$ but $\tau_G(g_1) \neq \tau_G(g_2)$, then there does not exist $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau_G$.*

Proposition 2.5. *Let G be a non-abelian group. Suppose $f(x) \in \mathbb{Q}[x]$ is a Johnson polynomial of G and $\chi \in \mathrm{Irr}(G)$ is such that $f(\chi) = \tau_G$. Then χ is a nonlinear faithful character.*

Proof. Suppose $f(x) \in \mathbb{Q}[x]$ and $\chi \in \mathrm{lin}(G)$. Since G is non-abelian, $\mathrm{nl}(G)$ is non-empty. Pick $\psi \in \mathrm{nl}(G)$. Then the inner product of ψ with $f(\chi)$ is zero but with τ_G is 1. Hence $f(\chi) \neq \tau_G$.

Suppose $f(x) \in \mathbb{Q}[x]$ is a Johnson polynomial of G and $\chi \in \mathrm{nl}(G)$ is such that $f(\chi) = \tau_G$ with $\ker(\chi) \neq \{1\}$. Since $\cap_{\chi \in \mathrm{Irr}(G)} \ker(\chi) = \{1\}$, $\tau_G(1) \neq \tau_G(g)$ for all $g \neq 1 \in G$. Take $g \neq 1 \in \ker(\chi)$. Then $\tau_G(1) = f(\chi(1)) = f(\chi(g)) = \tau_G(g)$, which is a contradiction. □

3. Camina p-Groups

In this section, we investigate the existence question of a Johnson polynomial for Camina p-groups. A. R. Camina in [2] initiated the study of these groups. We start by recalling the definition.

Definition 3.1. ([2]) *Suppose N is a normal subgroup of G. A pair (G, N) is a Camina pair if $1 < N < G$ is a normal subgroup of G and for every element $g \in G \setminus N$, $gN \subseteq Cl_G(g)$.*

It is clear that if (G, N) is a Camina pair and if H is normal in G and $H \leq N$ then $(G/H, N/H)$ is also a Camina pair. The following lemma gives a number of equivalent condition for a pair (G, N) to be a Camina pair.

Lemma 3.2. [17, Lemma 3] *Let N be a normal subgroup of G and let $g \in G \setminus N$. Then following are equivalent:*

(1) $\chi(g) = 0$ *for all* $\chi \in \mathrm{Irr}(G|N)$,
(2) $|C_G(g)| = |C_{G/N}(gN)|$,
(3) $gN \subseteq Cl_G(g)$.

It is easy to see that if (G, N) is a Camina pair, then $Z(G) \leq N \leq G'$.

Camina groups have been studied by many authors [3, 15, 16]. By Lemma 3.2, it is clear that if G is Camina group, then $\chi(g) = 0$ for all $\chi \in \mathrm{nl}(G)$ and $g \in G \setminus G'$. In [3], Dark and Scoppola proved:

Theorem 3.3. ([3]) *If G is a finite Camina p-group, then the nilpotency class of G is at most 3, i.e., $G_4 = \{1\}$.*

Lemma 3.4. [15, Corollary 2.3] *Let G be a p-group of nilpotency class r. If (G, G_k) is a Camina pair, then G_i/G_{i+1} has exponent p for $k-1 \leq i \leq r$.*

Theorem 3.5. [15, Theorem 5.2] *Let G be a Camina p-group of nilpotency class 3 and let $|G/G_2| = p^m$, $|G_2/G_3| = p^n$. Then*
(1) *(G, G_3) is a Camina pair,*
(2) *$m = 2n$ and n is even.*

Corollary 3.6. [15, Corollary 5.3] *If G is a Camina p-group of nilpotency class 3, then $Z_2(G) = G_2$ and $Z(G) = G_3$.*

Remarks on Camina p-groups of class 3.

Suppose G is a Camina p-group of nilpotency class 3. Then by Lemma 3.4, G/G_2, G_2/G_3, and G_3 are elementary abelian p-groups and by Corollary 3.6, we have $G_3 = Z(G)$. Now by Theorem 3.5, we have (G, G_3) is a Camina pair, $|G/G_2| = p^{2n}$, $|G_2/G_3| = p^n$ and $|G/G_3| = p^{3n}$ where n is even. We will show that $\mathrm{nl}(G) = \mathrm{Irr}(G|G_3) \sqcup \mathrm{nl}(G/G_3)$ and $cd(G) = \{1, p^n, p^{3n/2}\}$.

Take $\chi \in \mathrm{Irr}(G|G_3)$. Now $\chi\downarrow_{G_3} = \chi(1)\lambda$ for some $\lambda \in \mathrm{Irr}(G_3)$. Thus

$$\begin{aligned} |G| = \sum_{g \in G} |\chi(g)|^2 &= \sum_{g \in G_3} |\chi(g)|^2 \text{ (since } (G, G_3) \text{ is a Camina pair)} \\ &= \sum_{g \in G_3} |\chi(1)\lambda(g)|^2 \\ &= \chi(1)^2 |G_3|. \end{aligned}$$

Hence $\chi(1)^2 = |G/G_3| = p^{3n}$ for all $\chi \in \mathrm{Irr}(G|G_3)$. Thus we have a bijection

$$\Phi : \mathrm{Irr}(G_3) \setminus \{1_{G_3}\} \longrightarrow \mathrm{Irr}(G|G_3) \text{ defined by}$$

$$\Phi(\lambda)(g) := \begin{cases} p^{3n/2}\lambda(g) & \text{if } g \in G_3, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where 1_{G_3} is the trivial character of G_3. Therefore $|\mathrm{Irr}(G|G_3)| = |G_3| - 1$.

Since (G, G_2) is a Camina pair, $(G/G_3, G_2/G_3)$ is also a Camina pair. By Corollary 3.6, we have $Z(G/G_3) = Z_2(G)/G_3 = G_2/G_3 = [G/G_3, G/G_3]$. Thus G/G_3 is a Camina p-group of nilpotency class 2. Now take $\chi \in \mathrm{nl}(G/G_3)$. Then $\chi\downarrow_{G_2/G_3} = \chi(1)\lambda$ for some $\lambda \in \mathrm{Irr}(G_2/G_3)$. Now

$$\begin{aligned} |G/G_3| = \sum_{gG_3 \in G/G_3} |\chi(gG_3)|^2 &= \sum_{gG_3 \in G_2/G_3} |\chi(gG_3)|^2 \text{ (since } G/G_3 \text{ is a Camina group)} \\ &= \sum_{g \in G_2/G_3} |\chi(1)\lambda(gG_3)|^2 \\ &= \chi(1)^2 |G_2/G_3|. \end{aligned}$$

Hence $\chi(1)^2 = |G/G_2| = p^{2n}$ for all $\chi \in \mathrm{nl}(G/G_3)$. Thus we have a bijection

$$\Psi : \mathrm{Irr}(G_2/G_3) \setminus \{1_{G_2/G_3}\} \longrightarrow \mathrm{nl}(G/G_3) \text{ such that}$$

$$\Psi(\lambda)(g) := \begin{cases} p^{3n/2}(\lambda \circ \eta)(g) & \text{if } g \in G_2, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

where $\eta : G \longrightarrow G/G_3$ is the natural homomorphism and $1_{G_2/G_3}$ is the trivial character of G_2/G_3. Therefore we have $|\mathrm{nl}(G/G_3)| = |G_2/G_3| - 1 = p^n - 1$. Now

$$|G| = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G/G_2| + (|G_3| - 1)|G/G_3| + (|G_2/G_3| - 1)|G/G_2|.$$

This shows that $\mathrm{nl}(G) = \mathrm{Irr}(G|G_3) \sqcup \mathrm{nl}(G/G_3)$ as a disjoint union and $cd(G) = \{1, p^n, p^{3n/2}\}$.

Now, we can compute the total character of a Camina p-group of nilpotency class 3.

Proposition 3.7. *Let G be a Camina p-group of nilpotency class 3. Then the total character τ_G is given by,*

$$\tau_G(g) = \begin{cases} p^{2n} + (|G_3| - 1)p^{3n/2} + (p^n - 1)p^n & \text{if } g = 1, \\ p^{2n} - p^{3n/2} + (p^n - 1)p^n & \text{if } g \in G_3 \setminus \{1\}, \\ p^{2n} - p^n & \text{if } g \in G_2 \setminus G_3, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Proof. By Theorem 3.5, we have $|G/G_2| = p^{2n}$, $|G_2/G_3| = p^n$ and $|G/G_3| = p^{3n}$ where n is even. In view of (3.1) and (3.2), we have all the nonlinear irreducible character of G. Hence, if $g = 1$, then

$$\begin{aligned} \tau_G(1) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(1) + \sum_{\chi \in \mathrm{nl}(G)} \chi(1) \\ &= p^{2n} + (|G_3| - 1)p^{3n/2} + (p^n - 1)p^n. \end{aligned}$$

If $g \in G \setminus G_2$, then by Lemma 2.3 and (3.1), (3.2), we get

$$\tau_G(g) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(g) = \sum_{\chi \in \mathrm{lin}(G)} \chi(g) = 0.$$

If $g \neq 1 \in G_3$, then

$$\begin{aligned}
\tau_G(g) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G)} \chi(g) \\
&= |G/G_2| + \sum_{\chi \in \mathrm{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G/G_3)} \chi(g) \\
&= p^{2n} - p^{3n/2} + (p^n - 1)p^n \quad \text{(by (3.1) and (3.2))}.
\end{aligned}$$

Finally, if $g \in G_2 \setminus G_3$, then

$$\begin{aligned}
\tau_G(g) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G)} \chi(g) \\
&= |G/G_2| + \sum_{\chi \in \mathrm{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G/G_3)} \chi(g) \\
&= p^{2n} - p^n \quad \text{(by (3.1) and (3.2))}.
\end{aligned}$$

This completes the proof. □

Now, we are ready to characterize Camina p-groups which admit a Johnson polynomial (Theorem A).

Proof of Theorem A. By Theorem 3.3, the nilpotency class of G is at most 3, i.e., $G_4 = 1$. Suppose G is of nilpotency class equal to 3. If $Z(G)$ is not cyclic then by Lemma 2.1, G has no faithful irreducible character. Therefore, from Proposition 2.5, G has no Johnson polynomial. Now suppose $Z(G)$ is cyclic and χ is a faithful irreducible character of G. Let $f(x) \in \mathbb{C}[x]$ with $f(\chi) = \tau_G$. From (3.1) and (3.2), it is clear that $\chi \in \mathrm{Irr}(G|G_3)$ and $\chi(g) = 0$ for all $g \in G \setminus G_3$. Now take $h \in G_2 \setminus G_3$. Then from (3.3), we have $f(\chi(h)) = f(0) = \tau_G(h) = p^{2n} - p^n$. If $g \in G \setminus G_2$, then from (3.3), we get $f(\chi(g)) = f(0) = \tau_G(g) = 0$. Therefore, we have a contradiction to the existence of a Johnson polynomial.

Next suppose that nilpotency class of G is 2 i.e., $1 < G_2 \leq Z(G)$. Since G is a Camina group, each nonlinear irreducible character of G vanishes outside G_2. Therefore, $G_2 = Z(G)$. Thus $(G, Z(G))$ is a generalized Camina pair and hence from Theorem 1.1, the proof is complete. □

4. Groups for which $(G, Z(G))$ is a Camina pair

In [14], M. L. Lewis began the study of those groups G for which $(G, Z(G))$ is a Camina pair and, proved that such a group G must be a p-group for some prime p. The next lemma ([15, Lemma 2.1]) was proved by Macdonald in a more general setting where G is a p-group with (G, N) as a Camina pair. In the case $N = Z(G)$, this reduces to the following.

Lemma 4.1. ([15]) *Let G be a p-group of nilpotency class r and let $(G, Z(G))$ be a Camina pair. Then $Z(G) = G_r$.*

Remarks on $\mathrm{Irr}(G|Z(G))$ when $(G, Z(G))$ is a Camina pair.

Suppose $(G, Z(G))$ is a Camina pair. Then by Lemma 3.2, $\chi(g) = 0$ for all $\chi \in \mathrm{Irr}(G|Z(G))$ and for all $g \in G \setminus Z(G)$. Let $1_{Z(G)}$ be the trivial character of $Z(G)$. Now take any $\chi \in \mathrm{Irr}(G|Z(G))$. Then,

$$|G| = \sum_{g\in G} |\chi(g)|^2 = \sum_{g\in Z(G)} |\chi(1)\lambda(g)|^2,$$

where $\lambda \in \mathrm{Irr}(Z(G)) \setminus \{1_{Z(G)}\}$. Therefore, $\chi(1)^2 = |G/Z(G)|$. Hence we have a bijection

$$\Phi : \mathrm{Irr}(Z(G)) \setminus \{1_{Z(G)}\} \longrightarrow \mathrm{Irr}(G|Z(G)) \text{ such that}$$

$$\Phi(\lambda)(g) := \begin{cases} |G/Z(G)|^{1/2}\lambda(g) & \text{if } g \in Z(g) \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Proposition 4.2. *Let $(G, Z(G))$ be a Camina pair and let $(G/Z(G), Z(G/Z(G)))$ be a generalized Camina pair. Then the total character τ_G is given by the following expressions:*
$\tau_G(1) = |G/G_2| + (|Z(G)| - 1)|G/Z(G)|^{1/2} + m|G/Z_2(G)|^{1/2}$, *where* $m = |Z(G/Z(G))| - |Z_2(G)/G_2|$;
$\tau_G(g) = |G/G_2| - |G/Z(G)|^{1/2} + (|Z(G/Z(G))| - |Z_2(G)/G_2|)|G/Z_2(G)|^{1/2}$ *when* $1 \neq g \in Z(G)$;
$\tau_G(g) = |G/G_2| - |Z_2(G)/G_2||G/Z_2(G)|^{1/2}$ *if* $g \in G_2 \setminus Z(G)$;
$\tau_G(g) = 0$ *if* $g \in G \setminus G_2$.

Proof. Since $(G, Z(G))$ is Camina pair, $\mathrm{Irr}(G|Z(G))$ is given by (4.1). Therefore, there are $|Z(G)| - 1$ nonlinear irreducible characters of degree $|G/Z(G)|^{1/2}$. It is given that $(G/Z(G), Z(G/Z(G)))$ is a generalized Camina pair. So,

$$[G/Z(G), G/Z(G)] = G_2Z(G)/Z(G) \subseteq Z(G/Z(G) = Z_2(G)/Z(G).$$

Since $(G, Z(G))$ is a Camina pair, $Z(G) \subseteq G_2$. Hence $G_2Z(G)/Z(G) = G_2/Z(G)$. There is a bijection

$$\Psi : \mathrm{Irr}(Z(G/Z(G)) \,|\, G_2/Z(G)) \longrightarrow \mathrm{nl}(G/Z(G)) \;\text{ such that}$$

$$\Psi(\lambda)(g) := \begin{cases} |G/Z_2(G)|^{1/2}\lambda(g) & \text{if } g \in Z(G) \\ 0 & \text{otherwise} \end{cases} \tag{4.2}$$

(see [19, Theorem 3.1]). Thus G has $|Z(G/Z(G))| - |Z_2(G)/G_2|$ nonlinear irreducible characters with $Z(G)$ in their kernels and, degree of each such character is $|G/Z_2(G)|^{1/2}$. Now

$$\sum_{\chi\in \mathrm{lin}(G)} \chi(1)^2 + \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \chi(1)^2 + \sum_{\chi\in \mathrm{nl}(G/Z(G))} \chi(1)^2$$
$$= |G/G_2| + (|Z(G)| - 1)|G/Z(G)| + (|Z(G/Z(G))| - |Z_2(G)/G_2|)|G/Z_2(G)| = |G|.$$

This shows that $\mathrm{nl}(G) = \mathrm{Irr}(G|Z(G)) \sqcup \mathrm{nl}(G/Z(G))$.

Since $(G/Z(G), Z(G/Z(G)))$ is a generalized Camina pair, use [19, Proposition 3.1] to get,

$$\tau_{G/Z(G)}(g) = \begin{cases} |G/G_2| + m|G/Z_2(G)|^{1/2} & \text{if } g \in Z(G) \\ |G/G_2| - |Z_2(G)/G_2|.|G/Z(G)|^{1/2} & \text{if } g \in G_2 \setminus Z(G) \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

where $m = |Z(G/Z(G))| - |Z_2(G)/G_2|$. We use $\tau_{G/Z(G)}$ to calculate τ_G.

Next, if $g = 1$, then

$$\begin{aligned} \tau_G(1) &= \sum_{\chi\in \mathrm{lin}(G)} \chi(1) + \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \chi(g) + \sum_{\chi\in \mathrm{nl}(G/Z(G))} \chi(g) \\ &= |G/G_2| + (|Z(G)|-1)|G/Z(G)|^{1/2} + m|G/Z_2(G)|^{1/2}, \end{aligned} \tag{4.4}$$

where $m = |Z(G/Z(G))| - |Z_2(G)/G_2|$.

If $g \neq 1 \in Z(G)$, then by (4.1) and (4.2) we have

$$\begin{aligned} \tau_G(g) &= \sum_{\chi\in \mathrm{lin}(G)} \chi(g) + \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \chi(g) + \sum_{\chi\in \mathrm{nl}(G/Z(G))} \chi(g) \\ &= |G/G_2| - |G/Z(G)|^{1/2} + (|Z(G/Z(G))| - |Z_2(G)/G_2|)|G/Z_2(G)|^{1/2}. \end{aligned} \tag{4.5}$$

If $g \in G_2 \setminus Z(G)$, then then by (4.1), (4.2) and (4.3), we have

$$\begin{aligned} \tau_G(g) &= \sum_{\chi\in \mathrm{lin}(G)} \chi(g) + \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \chi(g) + \sum_{\chi\in \mathrm{nl}(G/Z(G))} \chi(g) \\ &= |G/G_2| + 0 - |Z_2(G)/G_2||G/Z_2(G)|^{1/2}. \end{aligned} \tag{4.6}$$

If $g \in G \setminus G_2$, then then by (4.1), (4.2) and (4.3), one can easily get that

$$\begin{aligned} \tau_G(g) &= \sum_{\chi\in \mathrm{lin}(G)} \chi(g) + \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \chi(g) + \sum_{\chi\in \mathrm{nl}(G/Z(G))} \chi(g) \\ &= 0. \end{aligned} \tag{4.7}$$

This completes the proof. □

Proof of Theorem B. In view of Proposition 2.4 and 4.2, G has no Johnson polynomial. □

5. Generalized Camina groups

In this section, we study the total character for a generalized Camina group and characterize those groups which admit a Johnson polynomial. We begin by recalling the important notion of isoclinism introduced by Philip Hall.

Definition 5.1. *Let G, H be finite groups. G and H are said to be isoclinic if there exist isomorphisms $\theta : G/Z(G) \longrightarrow H/Z(H)$ and $\phi : G_2 \longrightarrow H_2$ such that*

$$[\theta(g_1Z(G)), \theta(g_2Z(G))] = \phi([g_1Z(G), g_2Z(G)]) \text{ for all } g_1, g_2 \in G.$$

The notion of isoclinism was first introduced by P. Hall [6] who proved that two isoclinic nilpotent groups have the same nilpotency class. It is also known that isoclinic groups of the same order have the same character degrees. Recall:

Definition 5.2. ([13]) *A group G is said to be a Generalized Camina group if $Cl_G(g) = gG_2$ for every $g \in G \setminus G_2Z(G)$.*

This generalization of a Camina group was introduced by M. L. Lewis in [13]. It is clear from the definition that if G is a generalized Camina group, then either G has nilpotence class 2 or $G/Z(G)$ is a Camina group. The author proved that G is a nilpotent generalized Camina group if and only if G is isoclinic to a nilpotent Camina group H and H must be p-group ([13]). Lewis also pointed out that a Camina group which is isoclinic to G will be a p-group for the same prime p; one calls p, the prime associated to G.

Definition 5.3. *Let N be a normal subgroup of G and let $\chi \in \mathrm{Irr}(G)$. We say that χ is fully ramified with respect to G/N if $\chi{\downarrow}_N = e\theta$ and $\theta{\uparrow}^G = e\chi$ for some $\theta \in \mathrm{Irr}(N)$ and some integer e.*

In [13], Lewis proved the following theorem:

Theorem 5.4. [13, Theorem 3] *Let G be a nilpotent, generalized Camina group of nilpotency class 3. Then following are true:*

(1) *$G/G_2Z(G)$, $G_2Z(G)/Z(G)$, and $G_3 = G_2 \cap Z(G)$ are elementary abelian p-groups for some prime p.*
(2) *$|G/G_2Z(G)| = p^{2n}$ and $|G_2Z(G)/Z(G)| = |G_2/G_3| = p^n$ for some even integer n.*
(3) *$cd(G) = \{1, p^n, p^{3n/2}\}$.*
(4) *$Z(G/G_3) = G_2Z(G)/G_3$ and $G_2Z(G)/G_3 = G_2/G_3 \times Z(G)/G_3$.*
(5) *Every character in $\mathrm{nl}(G/G_3)$ is fully ramified with respect to $G/G_2Z(G)$ and every character in $\mathrm{Irr}(G|G_3)$ is fully ramified with respect to $G/Z(G)$.*

Remarks on Generalized Camina groups of nilpotency class 3.

Suppose G is a nilpotent, generalized Camina group of nilpotency class 3. Then from the above theorem, we have $|G/Z(G)| = p^{3n}$ and one can observe that there are two bijections namely,

$$\Phi_1 : \mathrm{Irr}(Z(G)|G_3) \longrightarrow \mathrm{Irr}(G|G_3) \text{ such that}$$

$$\Phi_1(\lambda)(g) := \begin{cases} p^{3n/2}\lambda(g) & \text{if } g \in Z(G), \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

and

$$\Psi_1 : \mathrm{Irr}(G_2Z(G)/G_3 \,|\, G_2/G_3) \longrightarrow \mathrm{nl}(G/G_3) \text{ such that}$$

$$\Psi_1(\lambda)(g) := \begin{cases} p^n(\lambda \circ \eta)(g) & \text{if } g \in G_2Z(G), \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

where $\eta : G \longrightarrow G/G_3$ is the natural homomorphism. Therefore G has $|Z(G)| - |Z(G)/G_3|$ nonlinear irreducible characters of degree $p^{3n/2}$ and $(|G_2/G_3| - 1)|Z(G)/G_3|$ nonlinear irreducible characters of degree p^n, and $\mathrm{nl}(G) = \mathrm{Irr}(G|G_3) \sqcup \mathrm{nl}(G/G_3)$.

Lemma 5.5. *Let G be a generalized Camina group of nilpotency class 3. Then*

$$\sum_{\lambda \in \mathrm{Irr}(Z(G)|G_3)} \lambda(g) = \begin{cases} -|Z(G)/G_3| & \text{if } g \in G_3, \\ 0 & \text{if } g \in Z(G) \setminus G_3 \end{cases} \tag{5.3}$$

and

$$\sum_{\lambda \in \mathrm{Irr}(G_2Z(G)/G_3 \,|\, G_2/G_3)} \lambda(g) = \begin{cases} (p^n-1)|Z(G)/G_3| & \text{if } g \in G_3, \\ -|Z(G)/G_3| & \text{if } g \in G_2 \setminus Z(G), \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

where $|G_2/G_3| = p^n$.

Proposition 5.6. *Let G be a generalized Camina group of nilpotency class 3. Then the total character τ_G is given by,*

$$\tau_G(g) = \begin{cases} |G/G_2| + rp^{3n/2} + (p^n-1)|Z(G)/G_3|p^n & \text{if } g = 1, \\ |G/G_2| - |Z(G)/G_3|p^{3n/2} + (p^n-1)|Z(G)/G_3|p^n & \text{if } g \neq 1 \in G_3, \\ |G/G_2| - |Z(G)/G_3|p^n & \text{if } g \in G_2 \setminus Z(G), \\ 0 & \text{otherwise,} \end{cases} \tag{5.5}$$

where $r = |Z(G)| - |Z(G)/G_3|$.

Proof. If $g = 1$, then

$$\begin{aligned} \tau_G(1) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(1) + \sum_{\chi \in \mathrm{nl}(G)} \chi(1) \\ &= |G/G_2| + (|Z(G)| - |Z(G)/G_3|)p^{3n/2} + (|G_2/G_3| - 1)|Z(G)/G_3|p^n \\ &= |G/G_2| + (|Z(G)| - |Z(G)/G_3|)p^{3n/2} + (p^n-1)|Z(G)/G_3|p^n. \end{aligned}$$

If $g \neq 1 \in G_3$, then

$$\begin{aligned} \tau_G(g) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G)} \chi(g) \\ &= |G/G_2| + \sum_{\chi \in \mathrm{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G/G_3)} \chi(g). \end{aligned}$$

Now use (5.1), (5.2) and Lemma 5.5 to get

$$\tau_G(g) = |G/G_2| - |Z(G)/G_3|p^{3n/2} + (|G_2/G_3| - 1)|Z(G)/G_3|p^n.$$

If $g \in Z(G) \setminus G_3$, then

$$\begin{aligned} \tau_G(g) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G)} \chi(g) \\ &= 0 + \sum_{\chi \in \mathrm{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G/G_3)} \chi(g) \quad \text{(by Lemma 2.3)} \\ &= 0 \quad \text{(use (5.1), (5.2) and Lemma 5.5).} \end{aligned}$$

If $g \in G_2 \setminus Z(G)$, then

$$\begin{aligned}\tau_G(g) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G)} \chi(g) \\ &= |G/G_2| + \sum_{\chi \in \mathrm{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G/G_3)} \chi(g) \\ &= |G/G_2| - |Z(G)/G_3|p^n \quad \text{(use (5.1), (5.2) and Lemma 5.5).}\end{aligned}$$

If $g \in G_2Z(G)$ but neither in G_2 nor in $Z(G)$, then

$$\begin{aligned}\tau_G(g) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G)} \chi(g) \\ &= 0 + \sum_{\chi \in \mathrm{Irr}(G|G_3)} \chi(g) + \sum_{\chi \in \mathrm{nl}(G/G_3)} \chi(g) \quad \text{(by Lemma 2.3)} \\ &= 0 \quad \text{(use (5.1), (5.2) and Lemma 5.5).}\end{aligned}$$

Finally, if $g \in G \setminus G_2Z(G)$, then by Lemma 2.3, (5.1) and (5.2), we get

$$\tau_G(g) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(g) = \sum_{\chi \in \mathrm{lin}(G)} \chi(g) = 0.$$

This completes the proof. □

We can now characterize nilpotent, generalized Camina groups (Theorem C).

Proof of Theorem C. Let G be a nilpotent, generalized Camina group with associated prime p. If $p = 2$, then G has nilpotency class 2 and for p odd, G has nilpotency class at most 3 (see [13, Theorem 2]). Now if nilpotency class is 2, then $(G, Z(G))$ is a generalized Camina pair and hence the result follows from Theorem 1.1.

Next suppose G has nilpotency class 3. If $Z(G)$ is not cyclic then by Lemma 2.1, G has no faithful irreducible character. Therefore from Proposition 2.5, G has no Johnson polynomial. Now suppose $Z(G)$ is cyclic. Therefore G has a faithful irreducible character χ (say). Let $f(x)$ be a Johnson polynomial and let $f(\chi) = \tau_G$. From (5.1) and (5.2), it is clear that $\chi \in \mathrm{Irr}(G|G_3)$. Then, in view of Proposition 2.4 and 5.6, G has no Johnson polynomial.
This completes the proof. □

6. p-groups of order p^5

In this final section, we completely classify the groups of order p^5 (for p odd) which admit a Johnson polynomial. Throughout this section p always denotes an odd prime. We will use not only the results of the previous sections but, more crucially, also use the classification of groups of order p^5 by R. James ([8, Section 4.5]).

We begin by recalling some well known results which we will use.

Theorem 6.1. [1, Theorem 22.5] *If G is a nonabelian p-group with $cd(G) = \{1, p\}$, then exactly one of the following holds:*

(1) *G has an abelian subgroup of index p,*
(2) *$G/Z(G)$ is of order p^3 and exponent p.*

Lemma 6.2. [7, Lemma 2.9] *Let H be a subgroup of G. Suppose χ is a character of G. Then*

$$\langle \chi{\downarrow_H}, \chi{\downarrow_H} \rangle \leq |G/H| \langle \chi, \chi \rangle$$

with equality if and only if $\chi(g) = 0$ for all $g \in G \setminus H$.

Lemma 6.3. [1, Theorem 20] *If G is a p-group, then for each $\chi \in \mathrm{Irr}(G)$, $\chi(1)^2$ divides $|G/Z(G)|$.*

Here is an easy consequence of the above lemma.

Lemma 6.4. *Let G be a non-abelian group of order p^4. Then $cd(G) = \{1, p\}$.*

Proof. Since $Z(G) \neq 1$, $|Z(G)| = p$ or p^2. Therefore $|G/Z(G)| = p^3$ or p^2. So by Lemma 6.3, the result follows. □

Theorem 6.5. [7, Theorem 6.15] *Let H be an abelian normal subgroup of G. Then $\chi(1)$ divides $|G/H|$ for all $\chi \in \mathrm{Irr}(G)$.*

As mentioned at the outset of this section, we will use the classification of groups of order p^5 by R. James ([8, Section 4.5]). More particularly, we will use the list of polycyclic presentations of these groups that the author compiled, and divided the non-abelian ones into families denoted by $\Phi_1, \cdots, \Phi_{10}$, according to isoclinism.

Lemma 6.6. *If $G \in X = \{\Phi_2(41), \Phi_2(311)b, \Phi_5(2111), \Phi_5(1^5)\}$* (see [8, Section 4.5])*, then G has a Johnson polynomial.*

Proof. First we consider the isoclinism family Φ_2. There are two type of groups in this family with $Z(G)$ cyclic namely,

$$\begin{aligned} G = \Phi_2(41) &= \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^3} = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle \quad \text{and} \\ H = \Phi_2(311)b &= \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^{p^2} = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle. \end{aligned}$$

Here $|Z(G)| = |\langle \alpha^p \rangle| = p^3$, $|G_2| = |\langle \alpha^{p^3} \rangle| = p$,$|Z(H)| = |\langle \gamma^p \rangle| = p^3$, $|H_2| = |\langle \gamma^{p^2} \rangle| = p$. By Lemma 6.3, we have $cd(G) = \{1, p\}$ and $cd(H) = \{1, p\}$. Now by Lemma 6.2 it is clear that $(G, Z(G))$ and $(H, Z(H))$ are generalized Camina pair. Hence by Theorem 1.1, G and H have a Johnson polynomial.

Now we discuss the isoclinism family Φ_5. There are only two type of groups in this family and both have cyclic center. Here are the groups:

(1) $\Phi_5(2111) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \alpha_1^p = \beta, \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$;
(2) $\Phi_5(1^5) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta, \alpha_1^p = \alpha_2^p = \alpha_3^p = \alpha_4^p = \beta^p = 1 \rangle$.

Note that both $\Phi_5(2111)$ and $\Phi_5(1^5)$ are extra-special p-groups. Therefore for these two groups $(G, Z(G))$ is a GCP (see [9, Theorem 2.18]). Since G is an extra-special p-group, $Z(G) = G_2$ and $|Z(G)| = p$. Therefore by Theorem 1.1, the polynomial

$$f(x) = p^n \sum_{j=1}^{p-1} (x/p^n)^j + p^{2n}(x/p^n)^p$$

is a Johnson polynomial of G and $f(\chi) = \tau_G$ for every $\chi \in \mathrm{nl}(G)$, where $G \in \Phi_5$. $\square$

Lemma 6.7. *If G in the isoclinism family Φ_3, then G has no Johnson polynomial.*

Proof. Let $G \in \Phi_3$. There are two type of groups in this family with $Z(G)$ cyclic namely, $\Phi_3(2111)c$ and $\Phi_3(311)b_r$ (see [8, Section 4.5]). For $p = 3$ and $p \geq 5$, we define these groups separately.

(1) $G = \Phi_3(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^p = \alpha_1^p \alpha_3 = \alpha_2^p = \alpha_3^p = 1 \rangle$ for $p = 3$;

(2) $H = \Phi_3(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p = \alpha_3, \alpha^p = \alpha_i^p = 1 \ (i = 1, 2, 3) \rangle$ for $p \geq 5$;

(3) $K = \Phi_3(311)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha]^r = \alpha_1^{p^2} = \alpha_3, \alpha^p = \alpha_2^p = \alpha_3^p = 1 \rangle$ for $r = 1$ or ν, where ν is a fixed quadratic non-residue $\mod p$, and $p \geq 3$.

Observe that $|Z(G)| = |\langle \gamma \rangle| = p^2$, $|Z(H)| = |\langle \gamma \rangle| = p^2$ and $|Z(K)| = |\langle \alpha_1^p \rangle| = p^2$.

First we will deal with H. Consider a normal abelian subgroup

$$N = \langle \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \gamma^p, \alpha_i^p = \gamma^{p^2} = 1 \ (i = 1, 2) \rangle$$

of H of index p. By Theorem 6.5, $cd(H) = \{1, p\}$. Since N is a normal abelian subgroup of index p, every nonlinear irreducible characters of H must be induced from N and hence $\chi(H \setminus N) = 0$ for all $\chi \in \mathrm{nl}(H)$. Now

$$\overline{H} := H/Z(H) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_i^p = 1 \ (i = 1, 2) \rangle$$

is an extra-special p-group of order p^3. Therefore, $\overline{H}$ has $p - 1$ nonlinear irreducible characters of each of degree p which vanish out side $Z(\overline{H})$ in $\overline{H}$ and on $Z(\overline{H})$ it is $p\lambda$, where $\lambda \in \mathrm{Irr}(Z(\overline{H})) \setminus \{1_{Z(\overline{H})}\}$. In particular, H has $p - 1$ nonlinear irreducible characters which contains $Z(H)$ in their kernel.

Take $Q = \langle \gamma^p \rangle$. Then $\mathrm{Irr}(H|Z(H)) = \mathrm{Irr}(H/Q|Z(H)/Q) \sqcup \mathrm{Irr}(H|Q)$. Now, suppose $\chi \in \mathrm{Irr}(H|Q)$. Then χ is faithful. Let ϕ be an irreducible constituent of $\chi\downarrow_M^H$, where $M = \langle \alpha_2, \gamma \rangle$. Since χ is faithful, ϕ is not H-invariant. Therefore, by Clifford's theorem $\chi\downarrow_M^H = \sum_1^p \phi_i$, where $\phi_1 = \phi$ and p is the index of the inertia group N of ϕ in H. Now $\phi_i\downarrow_{Z(H)}^M = \lambda$, where $\lambda \in \mathrm{Irr}(Z(H)) \setminus \{1_{Z(H)}\}$ for each $1 \leq i \leq p$. Therefore, by [7, Corollary 6.17], we have

$$\chi\downarrow_M^H = \sum_{\sigma \in \mathrm{Irr}(M/Z(H))} \sigma\phi_1 = \rho_{M/Z(H)}\phi_1, \tag{6.1}$$

where $\rho_{M/Z(H)}$ is the regular character of $M/Z(H)$. Hence for each $\chi \in \mathrm{Irr}(H|Q)$, we have $\chi(M \setminus Z(H)) = 0$.

Next, we consider $\chi \in \mathrm{Irr}(H/Q|Z(H)/Q)$, where $H/Q = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \gamma^p = \alpha_i^p = 1 \ (i = 1, 2)\rangle$ and $Z(H/Q) = M/Q$. Since $(H/Q, Z(H/Q))$ is a generalized Camina pair, $\chi(\alpha_2) = p\lambda(\alpha_2)$, where $\lambda \in \mathrm{Irr}(Z(H/Q)) \setminus \mathrm{Irr}(Z(H/Q)/(H/Q)_2)$ (see [19, Theorem 3.1]).

But then

$$\begin{aligned}
\tau_H(\alpha_2) &= \sum_{\chi\in\mathrm{lin}(H)} \chi(\alpha_2) + \sum_{\chi\in\mathrm{Irr}(H/Z(H))} \chi(\alpha_2) + \sum_{\chi\in\mathrm{nl}(H|Z(H))} \chi(\alpha_2) \\
&= |H/H_2| + \sum_{\lambda\in Z(\overline{H})\setminus\{1_{Z(\overline{H})}\}} p\lambda(\alpha_2) + \sum_{\chi\in\mathrm{Irr}(H|Q)} \chi(\alpha_2) + \sum_{\chi\in\mathrm{Irr}(H/Q|Z(H)/Q)} \chi(\alpha_2) \\
&= p^3 - p + 0 - p^2 + p \\
&= p^3 - p^2. \qquad (6.2)
\end{aligned}$$

Now suppose H has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_H$, where $\chi \in \mathrm{nl}(H)$. Therefore χ is faithful and $\chi \in \mathrm{Irr}(H|Q)$. Now $f(0) = f(\chi(\alpha_2)) = \tau_H(\alpha_2) = p^3-p^2$ and $f(0) = f(\chi(\alpha)) = \tau_H(\alpha) = 0$. The resultant contradiction proves that H can have no Johnson polynomial.
One can use a very similar argument to show that neither G nor K can have a Johnson polynomial. □

Lemma 6.8. *If G in the isoclinism family Φ_7 or Φ_8, then G has no Johnson polynomial.*

Proof. Suppose G is in the isoclinism family Φ_7. For $p = 3$ and $p \geq 5$, we will define these groups separately.

For $p = 3$:

(1) $G = \Phi_7(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \alpha^3, \alpha_1^3\alpha_3 = \alpha_{i+1}^3 = \beta^3 = 1 \ (i = 1, 2)\rangle$;
(2) $G = \Phi_7(2111)b_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha_1^3 = \alpha^3 = \alpha_{i+1}^3 = \beta^3 = 1 \ (i = 1, 2)\rangle$;
(3) $G = \Phi_7(2111)b_2 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^2 = \alpha_3^2, \alpha_1^3 = \alpha_3, \alpha^3 = \alpha_{i+1}^3 = \beta^3 = 1 \ (i = 1, 2)\rangle$;
(4) $G = \Phi_7(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \beta^3, \alpha^3 = \alpha_1^3\alpha_3 = \alpha_{i+1}^3 = 1 \ (i = 1, 2)\rangle$;
(5) $G = \Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^3 = \alpha_1^3\alpha_3 = \alpha_{i+1}^3 = \beta^3 = 1 \ (i = 1, 2)\rangle$.

For $p \geq 5$:

(1) $G = \Phi_7(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \alpha^p, \alpha_1^p = \alpha_{i+1}^p = \beta^p = 1 \ (i = 1, 2)\rangle$;
(2) $G = \Phi_7(2111)b_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \alpha_1^p, \alpha^p = \alpha_{i+1}^p = \beta^p = 1 \ (i = 1, 2)\rangle$;
(3) $G = \Phi_7(2111)b_\nu = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta]^\nu = \alpha_3^\nu = \alpha_1^p, \alpha^p = \alpha_{i+1}^p = \beta^p = 1 \ (i = 1, 2)\rangle$ where ν is a fixed quadratic non-residue mod p and $2 \leq \nu \leq p-1$;

(4) $G = \Phi_7(2111)c = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3 = \beta^p, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1 \ (i = 1, 2)\rangle$;

(5) $G = \Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = \beta^p = 1 \ (i = 1, 2)\rangle$.

It is clear that $|Z(G)| = |\langle \alpha_3 \rangle| = p$ and

$$G/Z(G) = H \times K = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1\rangle \times \langle \beta \rangle$$

is of order p^4 for all $G \in \Phi_7$, where H is extra-special p-group of order p^3 and K is a cyclic group of order p. Hence, by Lemma 6.4, we have $cd(G/Z(G)) = \{1, p\} \subseteq cd(G)$. Since G has no abelian subgroup of index p for all $G \in \Phi_7$, from Theorem 6.1 and Lemma 6.3 we get $cd(G) = \{1, p, p^2\}$. From Lemma 6.2 it is easy to observe that if $\chi(1) = p^2$, then $\chi(g) = 0$ for all $g \in G \setminus Z(G)$. Hence $(G, Z(G))$ is a Camina pair. Since H is a extra-special p-group, every nonlinear irreducible character ϕ of H vanishes outside $Z(H) = \langle \alpha_2 \rangle$ in H and $\phi\downarrow^H_{Z(H)} = p\lambda$ for some $\lambda \in \mathrm{Irr}(Z(H)) \setminus \{1_{Z(H)}\}$, where $1_{Z(H)}$ is the trivial character of $Z(H)$. Hence

$$\mathrm{nl}(G/Z(G)) = \{\phi \times \psi \mid \phi \in \mathrm{nl}(H), \psi \in \mathrm{Irr}(K)\}.$$

Now if $g = \alpha_2$, then

$$\begin{aligned}
\tau_G(\alpha_2) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(g) + \sum_{\chi \in \mathrm{Irr}(G|Z(G))} \chi(g) + \sum_{\chi \in \mathrm{nl}(G/Z(G))} \chi(g) \\
&= |G/G_2| + 0 + \sum_{\substack{\phi \in \mathrm{nl}(H) \\ \psi \in \mathrm{Irr}(K)}} (\phi \times \psi)(\alpha_2) \quad \text{((G,Z(G)) is a Camina pair)} \\
&= p^3 + \sum_{\substack{\lambda \in \mathrm{Irr}(Z(H)) \setminus \{1_{Z(H)}\}, \\ \psi \in \mathrm{Irr}(K)}} (p\lambda \times \psi)(\alpha_2) \quad (H \text{ is a extra-special group}) \\
&= p^3 + p \sum_{\lambda \in \mathrm{Irr}(Z(H)) \setminus \{1_{Z(H)}\}} p\lambda(\alpha_2) \\
&= p^3 - p^2. \qquad (6.3)
\end{aligned}$$

Since $(G, Z(G))$ is a Camina pair and H is a extra-special group,

$$\tau_G(g) = \sum_{\chi \in \mathrm{lin}(G)} \chi(g) + \sum_{\chi \in \mathrm{Irr}(G|Z(G))} \chi(g) + \sum_{\chi \in \mathrm{nl}(G/Z(G))} \chi(g) = 0 \qquad (6.4)$$

for all $g \in H \setminus Z(H)$. Now suppose G has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \mathrm{nl}(G)$. Therefore χ is faithful and $\chi \in \mathrm{Irr}(G|Z(G))$. Since $\chi \in \mathrm{Irr}(G|Z(G))$, $\chi(g) = 0$ for all $g \in G \setminus Z(G)$. In particular, $\chi(\alpha_1) = \chi(\alpha_2) = 0$. Now $f(0) = f(\chi(\alpha_2)) = \tau_G(\alpha_2) = p^3 - p^2$ whereas $f(0) = f(\chi(\alpha_1)) = \tau_G(\alpha_1) = 0$, which is a contradiction. Thus, G cannot have a Johnson polynomial.

Next suppose G is in the isoclinism family Φ_8;
$G := \Phi_8(32) = \langle \alpha_1, \alpha_2, \beta \mid [\alpha_1, \alpha_2] = \beta = \alpha_1^p, \beta^{p^2} = \alpha_2^{p^2} = 1\rangle$. Here $|Z(G)| = |\langle \alpha_1^{p^2} \rangle| = p$ and

$$G/Z(G) = \langle \alpha_1, \alpha_2 \mid [\alpha_1, \alpha_2] = \alpha_1^p, \alpha_1^{p^2} = \alpha_2^{p^2} = 1\rangle$$

is of order p^4. To show that $cd(G) = \{1, p, p^2\}$, we may use the same argument as we do for the groups in the family Φ_7; hence we skip the details. Now one can observe that $(G, Z(G))$ is a Camina pair and $(G/Z(G), Z(G/Z(G)))$ is a generalized Camina pair. Therefore, by Theorem B, G has no Johnson polynomial. □

Lemma 6.9. *Let*

$$H = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^p = \alpha_1^p = \alpha_2^p = \alpha_3^p = 1 \rangle.$$

Then, H is a group of order p^4 for an odd prime $p \geq 5$ and

$$\sum_{\chi \in \mathrm{nl}(H)} \chi(\alpha_2) = \sum_{\chi \in \mathrm{nl}(H)} \chi(\alpha_3) = -p.$$

Proof. Observe that $Z(H) = \langle \alpha_3 \rangle$ and $H_2 = \langle \alpha_2, \alpha_3 \rangle$. Since H has a normal abelian subgroup $N = \langle \alpha_1, \alpha_2, \alpha_3 \rangle = \langle \alpha_1 \rangle \times \langle \alpha_2 \rangle \times \langle \alpha_3 \rangle$ of order p^3, by Theorem 6.5, $cd(H) = \{1, p\}$. Now, if we consider the group

$$\overline{H} := H/Z(H) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^p = \alpha_1^p = \alpha_2^p = 1 \rangle,$$

we see that it is an extra-special p-group of order p^3. Therefore, $\overline{H}$ has $p-1$ nonlinear irreducible characters of degree p which vanish outside $Z(\overline{H}) = \langle \alpha_2 \rangle$ and, for $\chi \in \mathrm{nl}(\overline{H})$, we have

$$\chi\downarrow_{Z(\overline{H})} = p\lambda \tag{6.5}$$

for some $\lambda \in \mathrm{Irr}(Z(\overline{H})) \setminus \{1_{Z(\overline{H})}\}$. In particular, we have all the nonlinear irreducible characters of H having $Z(H)$ in their kernel. Now, let $\psi \in \mathrm{Irr}(H|Z(H))$. Since $|Z(H)| = p$, ψ is faithful and hence ϕ is not H-invariant, where ϕ is an irreducible constituent of $\psi\downarrow_{H_2}^H$. Therefore, by Clifford's theorem $\psi\downarrow_{H_2}^H = \sum_1^p \phi_i$, where $\phi_1 = \phi$ and p is the index of the inertia group N of ϕ in H. Now $\phi_i\downarrow_{Z(H)}^{H_2} = \lambda$, where $\lambda \in \mathrm{Irr}(Z(H)) \setminus \{1_{Z(H)}\}$ for each $1 \leq i \leq p$. Therefore, by [7, Corollary 6.17], we have

$$\psi\downarrow_{H_2}^H = \sum_{\beta \in \mathrm{Irr}(H_2/Z(H))} \beta\phi_1 = \rho_{H_2/Z(H)}\phi_1, \tag{6.6}$$

where $\rho_{H_2/Z(H)}$ is the regular character of $H_2/Z(H)$. Hence for each $\psi \in \mathrm{Irr}(H|Z(H))$, we have $\psi(H_2 \setminus Z(H)) = 0$.

Now

$$\begin{aligned}
\sum_{\chi \in \mathrm{nl}(H)} \chi(\alpha_2) &= \sum_{\chi \in \mathrm{nl}(H/Z(H))} \chi(\alpha_2) + \sum_{\chi \in \mathrm{Irr}(H|Z(H))} \chi(\alpha_2) \\
&= \sum_{\lambda \in \mathrm{Irr}(Z(K)) \setminus \{1_{Z(K)}\}} p\lambda(\alpha_2) + \sum_{\chi \in \mathrm{Irr}(H|Z(H))} \chi(\alpha_2) \quad \text{(Use (6.5))} \\
&= -p + 0 = -p \qquad (6.7)
\end{aligned}$$

and

$$\begin{aligned}\sum_{\chi\in \mathrm{nl}(H)} \chi(\alpha_3) &= \sum_{\chi\in \mathrm{nl}(H/Z(H))} \chi(\alpha_3) + \sum_{\chi\in \mathrm{Irr}(H|Z(H))} \chi(\alpha_3)\\ &= p(p-1) + \sum_{\chi\in \mathrm{Irr}(H|Z(H))} \chi(\alpha_3) \text{ (Since } \chi(\alpha_3)=p \text{ for all } \chi\in \mathrm{nl}(\overline{H}))\\ &= p(p-1) - p^2 = -p \qquad (6.8)\end{aligned}$$

This completes the proof of the lemma. □

Lemma 6.10. *If $G \in \Phi_9$, then G has no Johnson polynomial.*

Proof. Suppose G is in the isoclinism family Φ_9; these are defined as follows. **For** $p = 3$:

(1) $G = \Phi_9(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^3 = \alpha_4, \alpha_1^3\alpha_3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1\ (i = 1, 2, 3)\rangle$;
(2) $G = \Phi_9(2111)b_0 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_1^3\alpha_3 = \alpha_4, \alpha^3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1\ (i = 1, 2, 3)\rangle$;
(3) $G = \Phi_9(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^3 = \alpha_1^3\alpha_3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1\ (i = 1, 2, 3)\rangle$.

For $p \geq 5$:

(1) $G = \Phi_9(2111)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_4, \alpha_1^p = \alpha_{i+1}^p = 1\ (i = 1, 2, 3)\rangle$;
(2) $G = \Phi_9(2111)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha_1^p = \alpha_4^k, \alpha^p = \alpha_{i+1}^p = 1\ (i = 1, 2, 3)\rangle$ where $k = g^r$ for $r + 1 = 1, 2, \cdots, (p-1, 3)$;
(3) $G = \Phi_9(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1\ (i = 1, 2, 3)\rangle$.

Here $|Z(G)| = |\langle \alpha_4 \rangle| = p$ and

$$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1\ (i = 1, 2)\rangle$$

is of order p^4 for all $G \in \Phi_9$. Note that G has an abelian normal subgroup $N = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ of index p for all $G \in \Phi_9$. Therefore, by Theorem 6.5, we have $cd(G) = \{1, p\}$ for all $G \in \Phi_9$.

Now consider $p \geq 5$. In this case

$$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^p = \alpha_3^p = 1\ (i = 1, 2)\rangle.$$

Since $N = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is a normal abelian subgroup of index p, every nonlinear irreducible character of G must be induced from N and hence $\chi(G \setminus N) = 0$ for all $\chi \in \mathrm{nl}(G)$. Let $K = \langle \alpha_3, \alpha_4 \rangle$. Now, let $\chi \in \mathrm{Irr}(G|Z(G))$. Since $|Z(G)| = p$, χ is faithful. Let ϕ be an irreducible constituent of $\chi\downarrow_K^G$. Since χ is faithful, ϕ is not G-invariant. And hence by Clifford's theorem, we have $\chi\downarrow_K^G = \sum_1^p \phi_i$, where $\phi_1 = \phi$ and p is the index of the inertia group N of ϕ in G. Now $\phi_i\downarrow_{Z(G)}^K = \lambda$, where $\lambda \in \mathrm{Irr}(Z(G)) \setminus \{1_{Z(G)}\}$ for each $1 \leq i \leq p$. Therefore, by [7, Corollary 6.17], we have

$$\chi\downarrow_K^G = \sum_{\gamma\in \mathrm{Irr}(K/Z(G))} \gamma\phi_1 = \rho_{K/Z(G)}\phi_1, \qquad (6.9)$$

where $\rho_{K/Z(G)}$ is the regular character of $K/Z(G)$. Hence for each $\chi \in \mathrm{Irr}(G|Z(G))$, we have $\chi(K \setminus Z(G)) = 0$. Therefore,

$$\begin{aligned}
\tau_G(\alpha_3) &= \sum_{\chi\in \mathrm{lin}(G)} \chi(\alpha_3) + \sum_{\chi \in \mathrm{nl}(G)} \chi(\alpha_3) \\
&= |G/G_2| + \sum_{\chi\in \mathrm{Irr}(G|Z(G))} \chi(\alpha_3) + \sum_{\chi\in \mathrm{nl}(G/Z(G))} \chi(\alpha_3) \quad (\text{Since } \alpha_3 \in G_2) \\
&= p^2 + 0 - p \quad (\text{Use Lemma 6.9}) \\
&= p^2 - p \qquad (6.10)
\end{aligned}$$

Now suppose G has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \mathrm{nl}(G)$. Therefore χ is faithful and $\chi \in \mathrm{Irr}(G|Z(G))$. By (6.9), we have $\chi(\alpha_3) = 0$ for all $\chi \in \mathrm{Irr}(G|Z(G))$. Now by (6.10), we have $f(0) = f(\chi(\alpha_3)) = \tau_G(\alpha_3) = p^2 - p$. and since $\chi(\alpha) = 0$ for all $\chi \in \mathrm{nl}(G)$, $f(0) = f(\chi(\alpha)) = \tau_G(\alpha) = 0$, which is a contradiction. Hence in the case of $p \geq 5$, G has no Johnson polynomial.

Very similarly, for $p = 3$, one can show that G has no Johnson polynomial. This completes the proof of this lemma. □

Lemma 6.11. *If $G \in \Phi_{10}$, then G has no Johnson polynomial.*

Proof. Suppose that $G \in \Phi_{10}$; these are defined as follows.

For $p = 3$:

(1) $\Phi_{10}(2111)a_0 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4 = \alpha^3, \alpha_1^3\alpha_3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1\ (i = 1, 2, 3)\rangle$;

(2) $\Phi_{10}(2111)a_1 = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^2 = \alpha_4^2 = \alpha^3, \alpha_1^3\alpha_3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1\ (i = 1, 2, 3)\rangle$;

(3) $\Phi_{10}(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^3 = \alpha_1^3\alpha_3 = \alpha_2^3\alpha_4 = \alpha_3^3 = \alpha_4^3 = 1\ (i = 1, 2, 3)\rangle$.

For $p \geq 5$:

(1) $\Phi_{10}(2111)a_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha^p, \alpha_1^p = \alpha_{i+1}^p = 1\ (i = 1, 2, 3)\rangle$ where $k = g^r$ for $r + 1 = 1, 2, \cdots, (p-1, 4)$ and g is the smallest positive integer which is primitive root $\mod p$;

(2) $\Phi_{10}(2111)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2]^k = \alpha_4^k = \alpha_1^p, \alpha^p = \alpha_{i+1}^p = 1\ (i = 1, 2, 3)\rangle$ where $k = g^r$ for $r + 1 = 1, 2, \cdots, (p-1, 3)$ and g is the smallest positive integer which is primitive root $\mod p$;

(3) $\Phi_{10}(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_1^p = \alpha_{i+1}^p = 1\ (i = 1, 2, 3)\rangle$.

Here $|Z(G)| = |\langle \alpha_4 \rangle| = p$ and

$$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1\ (i = 1, 2)\rangle$$

is of order p^4 for all $G \in \Phi_{10}$. Note that G has no abelian subgroup of index p for all $G \in \Phi_{10}$. By Lemma 6.4, we have $cd(G/Z(G)) = \{1, p\} \subseteq cd(G)$. Therefore from Theorem 6.1 and Lemma 6.3 we get $cd(G) = \{1, p, p^2\}$. If $\chi(1) = p^2$, then by Lemma 6.2, χ vanish outside $Z(G)$ in G for all G in Φ_{10}. This shows that $(G, Z(G))$ is a Camina pair for all G in Φ_{10}. Now consider the group $G/Z(G)$ for $p \geq 5$,

$$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^p = \alpha_1^p = \alpha_2^p = \alpha_3^p = 1 \rangle.$$

Then

$$\begin{aligned}
\tau_G(\alpha_2) &= \sum_{\chi \in \mathrm{lin}(G)} \chi(\alpha_2) + \sum_{\chi \in \mathrm{nl}(G)} \chi(\alpha_2) \\
&= |G/G_2| + \sum_{\chi \in \mathrm{Irr}(G|Z(G))} \chi(\alpha_2) + \sum_{\chi \in \mathrm{nl}(G/Z(G))} \chi(\alpha_2) \quad (\text{Since } \alpha_2 \in G_2) \\
&= p^2 + 0 + \sum_{\chi \in \mathrm{nl}(G/Z(G))} \chi(\alpha_2) \quad (\text{Since } (G, Z(G)) \text{ is a Camina pair}) \\
&= p^2 - p \quad (\text{Use Lemma 6.9}) \qquad (6.11)
\end{aligned}$$

Next suppose that G has a Johnson polynomial $f(x)$ such that $f(\chi) = \tau_G$, where $\chi \in \mathrm{nl}(G)$. Therefore χ is faithful and $\chi \in \mathrm{Irr}(G|Z(G))$. Since $(G, Z(G))$ is a Camina pair, $\chi(g) = 0$ for all $g \in G \setminus Z(G)$ and $\chi \in \mathrm{Irr}(G|Z(G))$. In particular, $\chi(\alpha) = \chi(\alpha_2) = 0$. Now $f(0) = f(\chi(\alpha_2)) = \tau_G(\alpha_2) = p^2 - p$ and $f(0) = f(\chi(\alpha)) = \tau_G(\alpha) = 0$, which is a contradiction. Hence in this case, G has no Johnson polynomial.

Next, for $p = 3$, the group $G/Z(G)$ is

$$G/Z(G) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, \alpha^3 = \alpha_1^3\alpha_3 = \alpha_2^3 = \alpha_3^3 = 1 \rangle,$$

and one can use a very similar argument to show that G has no Johnson polynomial in this case also. This completes the proof of the lemma. ☐

Finally, we may summarize the results of this section in the form of Theorem D.

Proof of Theorem D. As above, we use the list of nonabelian p-group of order p^5 given by R. James [8, Section 4.5]. From Lemma 2.1 and Proposition 2.5, it is clear that if G has a Johnson polynomial then $Z(G)$ must be cyclic. The nonabelian p-groups of order p^5 with $Z(G)$ cyclic occur in the isoclinism family $\Phi_2, \Phi_3, \Phi_5, \Phi_7, \Phi_8, \Phi_9$, and Φ_{10} (see [8, pages 620-621]). Therefore, the result follows from Lemmata 6.6, 6.7, 6.8, 6.10, and 6.11. ☐

In view of the above results, it seems reasonable to pose the following conjecture for p-groups:
Conjecture: A p-group (with p odd) admits a Johnson polynomial if and only if $Z(G)$ is cyclic and $G' \leq Z(G)$.

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References

[1] Y. Berkovich, *Groups of prime power order. Vol. 1*, **46** of de Gruyter Expositions in Mathematics, With a foreword by Zvonimir Janko, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[2] A. R. Camina, Some conditions which almost characterize Frobenius group, *Israel J. Math.*, **31** (1978) 153–160.

[3] R. Dark and C. M. Scoppola, On Camina groups of prime power order, *J. Algebra*, **181** (1996) 787–802.

[4] S. M. Gagola, Jr. and M. L. Lewis, Squares of characters that are the sum of all irreducible characters, *Illinois J. Math.*, **42** no. 4 (1998) 655–672.

[5] R. Gow, Properties of the finite linear group related to the transpose-inverse involution, *Proc. London Math. Soc.*, **47** no. 3 (1983) 493–506.

[6] P. Hall, The classification of prime-power groups, *J. Reine Angew. Math.*, **182** (1940) 130–141.

[7] I. M. Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Academic Press, New York, 2000.

[8] R. James, The groups of order p^6 (p an odd prime), *Math. Comp.*, **34** no. 150 (1980) 613–637.

[9] G. Karpilovsky, *Group representations. Vol. 1. Part B*, North-Holland Mathematics Studies, **175** of North-Holland Publishing Co., Amsterdam, Introduction to group representations and characters, i-xiv, 1992, pp. 621-1274.

[10] V. Kodiyalam and D. N. Verma, A natural representation model for symmetric groups, http://arxiv.org/pdf/math/0402216v1.pdf.

[11] M. L. Lewis, A. Moreto and T. R. Wolf, Non-divisibility among character degrees, *J. Group Theory*, **8** (2005) 561–588.

[12] M. L. Lewis, The vanishing-off subgroup, *J. Algebra*, **321** no. 4 (2009) 1313–1325.

[13] M. L. Lewis, Generalizing Camina groups and their character tables, *J. Group Theory*, **12** (2009) 209-218.

[14] M. L. Lewis, On p-group Camina pairs, *J. Group Theory*, **15** (2012) 469–483.

[15] I. D. Macdonald, Some p-groups of Frobenius and extra-special type, *Israel J. Math.*, **40** (1981) 350–364.

[16] I. D. Macdonald, More on p-groups of Frobenius type, *Israel J. Math.*, **56** (1986) 335–344.

[17] S. Mattarei, Character tables and Metabelian groups, *J. London Math. Soc.*, **46** no. 2 (1992) 92–100.

[18] E. Poimenidou and H. Wolfe, Total characters and Chebyshev polynomials, *Int. J. Math. Math. Sci.*, **38** (2003) 2447–2453.

[19] S. K. Prajapati and R. Sarma, On the Gel'fand Character, submitted for publication.

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