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ON FITTING GROUPS WHOSE PROPER SUBGROUPS ARE SOLVABLE

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Abstract. This work is a continuation of [A. O. Asar, On infinitely generated groups whose proper subgroups are solvable, *J. Algebra*, **399** (2014) 870-886.], where it was shown that a perfect infinitely generated group whose proper subgroups are solvable and in whose homomorphic images normal closures of finitely generated subgroups are residually nilpotent is a Fitting p-group for a prime p. Thus this work is a study of a Fitting p-group whose proper subgroups are solvable. New characterizations and some sufficient conditions for the solvability of such a group are obtained.

1. **Introduction**

In [2] it was shown that if G is a Fitting p-group whose proper subgroups are solvable and hypercentral, then G is solvable. In [4] this result was generalized by replacing hypercentrality with a property called $(*)$-triple (see below) as follows. Let G be an infinitely generated group whose proper subgroups are solvable and in whose homomorphic images normal closures of finitely generated subgroups are residually nilpotent. If every homomorphic image of G has a $(*)$-triple for non-central elements, then G is solvable ([4, Theorem 1.1]). If G is not solvable, then it has a homomorphic image H which is a Fitting p-group for a prime p and in every homomorphic image of H there are no $(*)$-triples for non-central elements ([4, Theorem 1.4]). Moreover H has a homomorphic image K such that K has a dominant pair (w_K, V_K) with the property that $W^*(w_K, V_K) = 1$ and $Z(K)$ is non-trivial locally cyclic. However if residually nilpotent is replaced by residually (finite and nilpotent), then G becomes solvable ([4, Corollary 2.5]). An interesting property observed by the referee in [4, p. 874] is that a minimal non-solvable group (MNS**-group** for short) G in whose homomorphic images finitely

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generated subgroups have residually nilpotent normal closers is locally nilpotent if and only if G is periodic.

In the present work it is shown that a Fitting p-group G satisfying the normalizer condition and a condition denoted by $(**)$ (see below) either is not perfect or $\Omega_1(G/M) \neq G/M$ for an $M \lhd G$ (Theorem 1.1), where $p \neq 2$. Hence it follows that a Fitting p-group satisfying the normalizer condition and $(**)$ in which every proper subgroup has finite exponent cannot be perfect and in particular cannot be an MNS-group (Corollary 1.2). (A similar result is contained in ([1, Theorem 1.1]). Using Theorem 1.1 and [4, Theorem 1.4] a further description of the structure of an infinitely generated periodic group whose proper subgroups are solvable is obtained for $p \neq 2$ (Theorem 1.4). Specifically if the group in question is not solvable and thus perfect, then normal solvable subgroups of bounded derived length generate a proper normal subgroup and the group cannot be generated by a subset of finite exponent. In particular there exists a proper subgroup of infinite exponent. As application of these results two solvability criterions are given (Theorems 1.5,1.6).

Theorems 1.1,1.3 and [4, Lemma 2.2] naturally lead one to ask the following: Let G be a perfect Fitting p-group. Is it always true that $\langle A \lhd G : A' = 1, exp(A) \leq n\rangle \neq G$, for a given number $n \geq 1$?

The main results of this work are stated below. These results are special cases of [11, problems 16.5, 16.6]. First some definitions are needed.

Let G be a group, $w \in G \setminus Z(G)$ and V be a finitely generated subgroup of G with $w \notin V$. Then, for brevity, the ordered pair (w, V) is called a Λ-**pair** for G. A subgroup E of G which is maximal with respect to the condition that $w \notin E$ but $V \leq E$ is called a (w, V)**-maximal subgroup** of G. Let

$$E^*(w, V) = \{E : E \text{ is an } (w, V) - \text{maximal subgroup of } G\}$$

and

$$W^*(w, V) = \{Core_G(E) : E \in E^*(w, V)\}$$

We say that (w, V) satisfies the $(**)$-**property** if

(**) $N_G(E) = N_G(E')$ for an $E \in E^*(w, V)$.

Again let (w, V) be a Λ-pair for G. If there exists a proper subgroup L of G such that

$$w \notin V \text{ but } w \in \langle V, y\rangle \text{ for every } y \in G \setminus L$$

then (w, V, L) is called a $(*)$**-triple** for G. Note that the statement "(w, V, L) is a $(*)$-triple" is equivalent to the property that $\bigcap_{y\in G\setminus L}\langle V, y\rangle \neq V$. Furthermore if G is locally solvable then the pair (w, V) is called a **distinguished pair** for G, if there exists no $(*)$-triple (w, U, L) with $V \leq U$ and if

$$d(\langle V, y\rangle) > d(V) \text{ implies that } w \in \langle V, y\rangle \text{ for every } y \in G$$

where $d(V)$ denotes the derived length of V.

Let (w, V) be a distinguished pair for G and let $E \in E^*(w, V)$. Then $d(\langle V, y\rangle) = d(V)$ for $y \in E$, because if $d(\langle V, y\rangle) > d(V)$, then $w \in \langle V, y\rangle$ by the definition of a distinguished pair but $w \notin E$ by the definition of E. We note also that if G is an MNS-group, then for any element $w \in G \setminus Z(G)$ always there exists either a $(*)$-triple or a distinguished pair for G. For convenience a distinguished pair (w, V) for G satisfying the stronger condition that $d(E) = d(V)$ for every $E \in E^*(w, V)$ is called **dominant pair** for G. ([4, Lemmas 3.1, 4.1] for the existence of distinguished pairs and dominant pairs). Finally we note that if (w, V, L) is a $(*)$-triple or (w, V) a distinguished pair, then the corresponding property holds for (wv, V, L) and (wv, V), respectively and $E^*(w, V) = E^*(wv, V)$ for every $v \in V$. The nilpotent class of a nilpotent group G is denoted by $nc(G)$. Finally G is called n-**Engel** for a number $n \geq 1$ if $[x,_n y] = 1$ for all $x, y \in G$.

Theorem 1.1. *Let G be a Fitting p-group satisfying the normalizer condition, where $p \neq 2$. Suppose that in every homomorphic image H of G, $|Z(H)| \neq 3$ and the $(**)$-property is satisfied by every Λ-pair. Then G either is not perfect or has a proper normal subgroup M such that $\Omega_1(G/M)$ is abelian.*

Corollary 1.2. *Let G be a Fitting p-group satisfying the normalizer condition, where $p \neq 2$. Suppose that in every homomorphic image H of G, $|Z(H)| \neq 3$ and the $(**)$-property is satisfied by every Λ-pair. If G is perfect, then G cannot be generated by a subset of finite exponent. In particular G has a proper subgroup of infinite exponent. In other words if every proper subgroup of G has finite exponent, then G cannot be perfect.*

The commutator subgroup of the transitive totally imprimitive p-group G given in [3, Theorem 1.1], cannot satisfy the normalizer condition (it has proper non-FC-subgroups) but satisfies the conclusion of Corollary 1.2. It is perfect and $\Omega_k(G) = \langle g \in G : g^{p^k} = 1\rangle$ is nilpotent of finite exponent for every $k \geq 1$. (e.g. see the explanation following [3, Lemma 2.2]). Also a point stabilizer of the group has infinite exponent. But it is not known whether $(**)$ is satisfied by Λ-pairs of G. However $\langle A \lhd G : A' = 1, exp(A) \leq n\rangle \neq G$, for every number $n \geq 1$.

Theorem 1.3. *Let G be a Fitting p-group which is an MNS-group and satisfies the normalizer condition, where $p \neq 2$. Suppose that in every homomorphic image of G the $(**)$-property is satisfied by every dominant pair. Let*

$$S_t(G) = \{K \lhd G : d(K) \leq t\}$$

where $d(K)$ denotes the derived length of the subgroup K. Then $\langle S_t(G)\rangle \neq G$ for every $t \geq 1$. Furthermore there exists a proper subgroup E of G such that $\langle E^G\rangle = G$.

The above results combined with [4, Theorem 1.4] give the following general result.

Theorem 1.4. *Let G be an infinitely generated periodic group with $2 \notin \pi(G)$ in which every proper subgroup is solvable and in every homomorphic image of G normal closures of finitely generated subgroups are residually nilpotent. Suppose that G satisfies the normalizer condition. Furthermore suppose that if H is a homomorphic image of G, then $|Z(H)| \neq 3$ and every dominant pair for H satisfies $(**)$. Then G either is solvable or there exists a proper normal subgroup M of G such that $\Omega_1(G/M)$ is abelian. In the second case G contains a proper subgroup of infinite exponent, $\langle S_t(G)\rangle \neq G$ for* *every $t \geq 1$ and $G = \langle E^G \rangle$ for a proper subgroup of G. In particular if every proper subgroup of G has finite exponent, then G is solvable.*

Theorem 1.4 holds also in the non-periodic case if the group is locally nilpotent by [4, Corollary 1.7] .

As application of the above results we can state the followings. The first one shows the influence on the group structure of the domination of the exponents of normal abelian subgroups by the exponent of the group center.

Theorem 1.5. *Let G be a Fitting p-group satisfying the normalizer condition in which every proper subgroup is solvable, where $p \neq 2$. Suppose that in every homomorphic image H of G, $|Z(H)| \neq 3$ and the $(**)$-property is satisfied by dominant pairs. Furthermore suppose that the following holds. If $B = [B, H]$ is a normal metabelian nilpotent subgroup and A is a normal abelian subgroup of H contained in B, then $exp(A) \leq exp(Z(H))$ whenever $Z(H) \neq 1$. Then G is solvable.*

Theorem 1.6. *Let G be a Fitting p-group satisfying the normalizer condition in which every proper subgroup is solvable, where $p \neq 2$. Suppose that in every homomorphic image H of G, $|Z(H)| \neq 3$ and the $(**)$-property is satisfied by dominant pairs. Furthermore suppose that every proper subgroup of G is n-Engel for an $n \geq 1$. Then G is solvable.*

The proof of Theorem 1.6 depends on the following.

Proposition 1.7. *Let G be a locally finite p-group and let A be a normal elementary abelian subgroup of G. Suppose that every proper subgroup L of G is n-Engel for an $n = n(L) \geq 1$. Then every proper subgroup of $G/C_G(A)$ has finite exponent.*

Let G be a Fitting p-group in which every proper subgroup is solvable and let (w, V) be a dominant pair for G. The present work together with [3] and [4] shows that the elements of $E^*(w, V)$ have considerable influence on the structure of G. For example if every $E \in E^*(w, V)$ is hypercentral, then G is solvable by [4, Lemma 4.6(b)]. Next suppose that any two elements of $E^*(w, V)$ intersect in a subgroup of finite index in each of them. Let M be a normal subgroup of finite exponent of G. It follows from [3, Lemma 2.11] together with [14, Satz(6)] $|M : M \cap E| < \infty$. Now applying the proof of [4, Theorem 1.1] it follows that G is solvable. Another easy observation is the following. If every metabelian subgroup of every $E \in E^*(w, V)$ is nilpotent, then E is nilpotent by [5, Theorem B] and then G must be solvable by [4, Lemma 4.6(b)].

The notations and the definitions are standard and may be found in [6, 7, 8, 12, 13].

2. Proof of Theorem 1.1

We begin by showing that a Λ-pair satisfies many of the properties of a distinguished pair given in [4].

Lemma 2.1. *Let G be a locally finite p-group and let (w, V) be a Λ-pair for G. Then the following hold.*

(a) *Let* $v \in V$. *Then* $E^*(w,V) = E^*(wv,V)$.

(b) $W^*(w,V)$ *contains maximal elements.*

(c) *Suppose that* G *is perfect. Let* M *be a maximal element of* $W^*(w,V)$. *There exists a finite subgroup* U *of* G *containing* V *so that* $w \notin U \not\leq M$ *and if* $wM \in Z(G/M)$, *then* $wuM \notin Z(G/M)$ *for every* $u \in U \setminus M$. *Furthermore* $E^*(wuM, UM/M) = E^*(wM, UM/M)$ *and if* $R/M \in E^*(wuM, UM/M)$, *then* $R \in E^*(w,V)$. *Thus* $W^*(wuM, UM/M) = 1$ *and* $Z(G/M) \neq 1$.

Proof. (a) This is obvious since $E \leq G$ is (w,V)-maximal if and only if it is (wv,V)-maximal.

(b) This follows from the proof of [4, Lemma 3.4].

(c) Suppose that G is perfect. There exists an $E \in E^*(w,V)$ so that $M < E$ by [4, Lemma 4.3]. Hence there exists a finite subgroup U of E satisfying $V \leq U \not\leq M$. Also $w \notin U$ since $w \notin E$. Thus (w,U) is a Λ-pair of G and $E^*(w,U) \subseteq E^*(w,V)$ by the proof of [4, Lemma 3.2]. It is easy to see that if $wM \in Z(G/M)$ and if $u \in U \setminus M$, then $wuM \notin Z(G/M)$. For in the contrary case $\bar{u} \in Z(\bar{G})$ and then $\langle u, M\rangle \leq E$ which contradicts the maximality of M. Thus $(wuM, UM/M)$ is a Λ-pair for G/M and $E^*(wM, UM/M) = E^*(wuM, UM/M)$ by (a). Also if $R/M \in E^*(wuM, UM/M)$, then $R \in E^*(w,V)$ by the proof of [4, Lemma 4.2]. Hence it follows that $W^*(wM, UM/M) = 1$ since M is a maximal element of $W^*(w,V)$. Finally $Z(G/M) \neq 1$ by [4, Lemma 3.5]. □

Remark 1. Lemma 2.1(c) shows that in a locally nilpotent p-group if $W^*(w,V) = 1$, then $Z(G) \neq 1$. This fact is used (without mention) in the rest of this work. This may be shown easily by considering $< w, V > A$, where A is an elementary abelian normal subgroup of G. Assume if possible that $w \in VM$ for every G-invariant subgroup M of A. Let $1 \neq M_1 < A$ be G-invariant. Then $w \in VM_1$ and so $wv_1 \in M_1$ for a $v_1 \in V$ since $W^*(w,V) = 1$. Assume that there exists a $1 \neq M_2 < M_1$ which is G-invariant. Then $wv_2 \in M_2$ for a $v_2 \in V$. Continuing in this way we obtain a descending chain $A > M_1 > M_2 \cdots$ of G-invariant subgroups of A so that for each $i \geq 1$ there exists a $v_i \in V$ with $wv_i \in M_i$. If this stops at a finite step $r \geq 1$, then M_r is a minimal normal subgroup of G and then $M_r \leq Z(G)$ by [13, 12.1.6]. If not then there exists a $v \in V$ so that $wv \in M_i$ for all $i \geq 1$ since V is finite. Let $M^* = \bigcap_{i=}^{\infty} M_i$. Then again M^* is a minimal normal subgroup of G and so $M^* \leq Z(G)$.

Lemma 2.2. *Let* G *be a locally finite* p*-group and let* (w,V) *be a* Λ*-pair for* G. *Let* $E \in E^*(w,V)$. *Then* $N_G(E)/E$ *either is locally cyclic or* $p = 2$ *and isomorphic to a locally quaternion group.*

Proof. Let $N = N_G(E)$ and $\bar{N} = N/E$. Let $\bar{A}$ be a finite abelian subgroup of $\bar{N}$. Assume if possible that $\bar{A}$ is not cyclic. Then $\bar{A}$ contains an elementary abelian subgroup $\langle \bar{a}\rangle \times \langle \bar{b}\rangle$. But since E is (w,V)-maximal we must have $w \in \langle a\rangle E$ and $w \in \langle b\rangle E$. Hence $w \in \langle a\rangle E \cap \langle b\rangle E$ and hence $w \in E$, which is impossible since $w \notin E$. It follows from this that every finite abelian subgroup of $\bar{N}$ is cyclic. In this case every finite subgroup of $\bar{N}$ is cyclic or isomorphic to a generalized quaternion group by [6, Theorem 5.4.10(ii)]. Therefore $\bar{N}$ is locally cyclic or isomorphic to a 2-group which is isomorphic to a locally quaternion group. □

Note that a (generalized) quaternion group Q_n of order 2^n, $(n \geq 3)$, is not abelian and every maximal subgroup is either (generalized) quaternion or cyclic, in particular, $|Z(Q_n)| = 2$ (see [6, Theorem 5.4.3]). Therefore in any case the group N/E of Lemma 2.1 has a unique subgroup of order p for every prime number p.

Lemma 2.3. *Let G be a Fitting p-group and let (w, V) be a Λ-pair for G. Let A be a normal abelian subgroup of G. Let $E \in E^*(w, V)$ such that $E \cap A$ is maximal. Then $A/(A \cap N_G(E)$ is finite.*

Proof. Assume that there exists an $E \in E^*(w, V)$ so that $A \cap E$ is maximal but $A/(A \cap N_G(E)$ is infinite. Put $N = N_G(E)$, $H = NA$, $R = N \cap A$, $D = R \cap E$ and $\bar{H} = H/D$. Let $\bar{B}/\bar{R} = \Omega_1(\bar{A}/\bar{R})$. First suppose that $\bar{B}/\bar{R}$ is finite. Then $\bar{A}/\bar{R}$ is Chernikov by [9, Proposition 1.G.6]. Then also $\bar{A}$ is Chernikow since $\bar{R}$ is Chernikov by Lemma 2.2. Let $\bar{T}$ be the unique radicable abelian subgroup of $\bar{A}$ and let $x \in N$. Then $\bar{T} < \bar{x} >$ is nilpotent since $< \bar{x} >$ is subnormal in G, which implies that $[\bar{T}, \bar{x}] = 1$. Since x is any element of N it follows that $[\bar{T}, \bar{N}] = 1$ and thus $[T, N] \leq D \leq E$. But then $T \leq N$, which is impossible since A/T is finite. Therefore $\bar{B}/\bar{R}$ must be infinite.

Next let $\bar{B} = \Omega_1(\bar{A})$. Then $\bar{B}$ is infinite since $\bar{A}$ is not Chernikow by the preceding paragraph. Consider $L = BDV$ and $\bar{L} = L/D$. Let $\bar{C} = C_{\bar{B}}(\bar{V})$. Assume that $\bar{C}$ is infinite. Then

$$\bar{C} = < \overline{c_1} > \times \cdots \times < \overline{c_n} > \times \cdots$$

where $c_n \in C \setminus E$ and $(\overline{c_n})^p = 1$ for every $n \geq 1$. Now there exists an $i > 1$ so that $< \overline{c_1} > \bar{V} \cap < \overline{c_i} > \bar{V} = \bar{V}$ since V is finite. Taking the inverse images we get $< c_1 > VD \cap < c_i > VD = VD$. Without loss of generality we may suppose that $w \notin < c_1 > VD$ since $VD \leq E$ but $w \notin E$. Then $< c_1 > VD \leq E_1$ for an $E_1 \in E^*(w, V)$. But since D is a maximal intersection and $< c_1, D > \leq A$, it follows that $c_1 \in D$, which is impossible. Therefore $\bar{C}$ must be finite and so $\bar{B}$ is Chernikov as above. But since $\bar{B}$ is elementary abelian $\bar{B}$ must be finite, which contradicts the assumption that $\bar{B}$ is infinite. Therefore $A/(A \cap N)$ must be finite. □

Lemma 2.4. *Let G be a perfect Fitting p-group p-group and let (w, V) be a Λ-pair for G such that $W^*(w, V) = 1$. Let A be an abelian subgroup of G and let $E \in E^*(w, V)$ such that $E \cap A$ is maximal, $N_G(E)$ normalizes A and $N_G(E) = N_G(E')$. Then $A \leq N_G(E)$.*

Proof. Assume that the assertion is false. Let $N = N_G(E)$, $R = N \cap A$ and $H = NA$. Then $A/(A \cap N)$ is nontrivial finite by Lemma 2.3 and N/E has a unique subgroup of order p by Lemma 2.2 Put $\bar{H} = H/R$ and $\bar{C} = C_{\bar{N}}(\bar{A})$. Then $\bar{H}/\bar{C}$ is finite since $|\bar{A}|$ is finite and hence $\bar{T}/\bar{C} = N_{(\Omega_1(\bar{A})\bar{N})/\bar{C})}(\bar{N}/\bar{C}) > \bar{N}/\bar{C}$. Hence there exists an $a \in N_A(N) \setminus N$ so that $a^p \in N$.

Next let $D = A \cap E$ and put $\bar{H} = H/D$. Now $[\bar{R}, \bar{E}] \leq \bar{R} \cap \bar{E} = 1$. Then also $[\bar{a}^p, \bar{E}] = 1$ since $a^p \in R$. Hence

$$1 = [\bar{a}^p, \bar{E}] = [\bar{a}, \bar{E}]^p$$

by [6, Lemma 2.2.(i)] and so $[\bar{a}, \bar{E}]$ has order equal to p. Also $[\bar{a}, \bar{E}] \leq \bar{A} \cap \bar{N} = \bar{R}$. Thus $\langle [\bar{a}, \bar{E}]\bar{E}/\bar{E} \rangle$ is the unique subgroup of order p of $\bar{N}/\bar{E}$ by Lemma 2.2. But also $1 \neq Z(G) \leq N$ and $Z(G) \cap E = 1$ by Lemma 2.1(c) since $W^*(w, V) = 1$(see also Remark 1). Clearly then $[\bar{a}, \bar{E}]\bar{E}/\bar{E} \leq \overline{Z(G)}\bar{E}/\bar{E}$ which implies that a normalizes $Z(G)E$. In this case also a normalizes E' since $[Z(G)E, Z(G)E] = E'$. But then $a \in N$ since $N = N_G(E')$ by the hypothesis, which is a contradiction and so the proof of the lemma is complete. □

Lemma 2.5. *Let G be a locally finite p-group and let (w, V) be a Λ-pair for G satisfying $(**)$. Suppose that G satisfies the normalizer condition and $W^*(w, V) = 1$. Then $Z(G)$ is cyclic.*

Proof. Assume that $Z(G)$ is infinite. Let $E \in E^*(w, V)$ such that $N_G(E) = N_G(E')$ and put $N = N_G(E)$. Then $Z(G) \leq N$. In this case $Z(G)$ is infinite locally cyclic and so $N = Z(G)E$ by Lemma 2.2 since $Z(G) \cap E = 1$ by the hypothesis. Also there exists a $b \in G \setminus N$ so that $N^b = N$ by the hypothesis. Then b normalizes $Z(G)E$ since $N = Z(G)E$ and then also b normalizes $[Z(G)E, Z(G)E] = E'$. But then $b \in N$ by the hypothesis, which is a contradiction. Therefore the assumption is false and so $Z(G)$ is cyclic. □

Corollary 2.6. *Let G be a locally finite p-group and let (w, V) be a Λ-pair for G satisfying $(**)$. Suppose that G satisfies the normalizer condition and $W^*(w, V) = 1$. Let B be a normal subgroup of G of exponent $\leq exp(Z(G))$. Then B is abelian and $B \leq N_G(E)$ for an $E \in E^*(w, V)$.*

Proof. Assume that B is not abelian. We may suppose that $B \cap Z(G) \neq 1$ and $exp(B) = exp(B \cap Z(G))$. Then $Z(G) \cap B = \langle z \rangle$ for a $1 \neq z \in Z(G)$ by Lemma 2.5 and $exp(B) = |z|$. Let $E \in E^*(w, V)$ such that $N_G(E) = N_G(E')$ and put $N = N_G(E)$, $T = N_B(N)$ and $R = N \cap B$. Clearly $B \nleq N$ since $Core_G(E) = 1$ and $B' \neq 1$ which implies that $T \nleq N$. Also $R \lhd TN$ since $R \lhd N$ and $[T, R] \leq B \cap N = R$. Furthermore $RE/E = \langle z \rangle E/E$ by Lemma 2.2 since $exp(R) = |z|$ and $\langle z \rangle \cap E = 1$. Now T normalizes $\langle z \rangle E$ since $[T, N] \leq B \cap N = R \leq \langle z \rangle E$. Then also T normalizes $(\langle z \rangle E)' = E'$ and so $T \leq N$ by the hypothesis which is impossible since $T \nleq N$. Therefore the assumption is false and so $B \leq N$ and then $B' = 1$ since $B' \leq E$. □

Lemma 2.7. *Let G be a locally finite p-group and let (w, V) be a Λ-pair for G satisfying $(**)$. Suppose that G satisfies the normalizer condition and $W^*(w, V) = 1$. Let A be a normal abelian subgroup of G with $exp(A) \leq p(exp(Z(G)))$. Then $A \leq N_G(E)$ for an $E \in E^*(w, V)$. Let $p \neq 2$. Then any two normal abelian subgroups of G of exponent $\leq p|Z(G)|$ commute element-wise.*

Proof. Let $E \in E^*(w, V)$ such that $N_G(E) = N_G(E')$ and assume that $A \nleq N_G(E)$. Put $N = N_G(E)$, $R = A \cap N$ and $H = NA$. Then $R \lhd H$. There exists an $a \in A \setminus N$ so that $N^a = N$ by the hypothesis. We may suppose that $a^p \in N$ and so $a^p \in R$. Furthermore since RE/E is a normal abelian subgroup of N/E it follows from Lemma 2.2 that RE/E is cyclic and so $R = \langle b \rangle (R \cap E)$ for a $b \in R$. Also $Z(G) \neq 1$ by the hypothesis and cyclic by Lemma 2.5 and so $Z(G) = \langle z \rangle$ for a $1 \neq z \in Z(G)$. Thus $\langle z \rangle E/E \leq \langle b \rangle E/E$ and $|bE/E| \leq p|zE/E| = p|z|$ since $\langle z \rangle \cap E = 1$. Now $a^p E/E \in \langle z \rangle E/E$ since $|a| \leq p|z|$ by the hypothesis. Hence it follows that

$$1 = [a^p E, N/E] = [a^p, N]E/E = [a, N]^p E/E$$

by [6, Lemma 2.2.2((i)] since $[a, N] \leq A$. This implies that $|[a, N]E/E| = p$ and hence $[a, N]E/E \leq \langle z \rangle E/E$. Consequently it follows that $[a, N] \leq \langle z \rangle E$. Clearly then a normalizes $\langle z \rangle E$ and then also

E' is normalized by a. But then a normalizes E and so belongs to N by the hypothesis, which is a contradiction. Therefore the assumption is false and so $A \leq N$.

Next let $p \neq 2$ and let A, B be two normal abelian subgroups of G with exponents $\leq p(exp(Z(G)))$. Then $A, B \leq N_G(E)$ by the first part of the proof. Also N/E is locally cyclic by Lemma 2.2. Hence it follows that $[A, B] \leq E$. But since $Core_G(E) = 1$ by the hypothesis this implies that $[A, B] = 1$ and so AB is abelian. This completes the proof. □

Lemma 2.8. *Let G be a locally finite p-group satisfying the normalizer condition, where $p \neq 2$. Let (w, V) be a Λ-pair for G and let $E \in E^*(w, V)$ such that $N_G(E) = N_G(E')$ and $W^*(w, V) = 1$. Let B be a normal non-abelian subgroup of G and A be a normal abelian subgroup of G contained in B such that $exp(A) = exp(A \cap Z(G))$ and B/A is elementary abelian. Put $N = N_G(E)$, $R = N \cap B$, $D = R \cap E$, $T = N_B(N)$, $H = TN$ and $D^* = Core_H(D)$. Then the following hold.*

(a) *$R = \langle b\rangle D$, $b \notin AD$, $\langle b\rangle \cap D = 1$, $|b| = p|z|$ and $A = \langle z\rangle(A \cap D)$, where $\langle z\rangle = Z(G)$.*
(b)

$$R/D^* \leq Z(N/D^*) \text{ and } C_{T/D^*}(R/D^*) = R/D^*$$

Proof. If $B \leq N$, then $B' \leq E$ since N/E is locally cyclic and then $B' = 1$ since $W^*(w, V) = 1$, which is impossible since B is not abelian. Therefore $B \nleq N$ and then also $T \nleq N$ but $T \lhd H$. Now $Z(G) \neq 1$ by the hypothesis and cyclic by Lemma 2.5. Thus $A \cap Z(G) = \langle z\rangle$ for a $1 \neq z \in Z(G)$. Furthermore $A \leq R$ by Lemma 2.7. Clearly $R \lhd NB$ since $A \leq R$. Now $R = \langle b\rangle D$ for a $b \in R \setminus D$ and $A = \langle z\rangle(A \cap D)$ since N/E is locally cyclic, $|z| = exp(A)$ and $E \cap Z(G) = 1$ by the hypothesis. Note that if $\langle b\rangle D = \langle z\rangle D$, then $\langle b\rangle E = \langle z\rangle E$. Hence $[T, E] \leq N \cap B = R = \langle z\rangle D \leq \langle z\rangle E$ and so $\langle z\rangle E$ and then also E' is normalized by T. But then $T \leq N$ by the hypothesis, a contradiction. Therefore $\langle b\rangle D > \langle z\rangle D \geq A$ which implies that $b \notin AD$. Since R/A is elementary abelian, $exp(A) = |z|$ and $\langle z\rangle \cap D = 1$ we have $(bD)^p \in \langle zD\rangle$ since $\langle zD\rangle \leq \langle bD\rangle$ and hence $|b^pD| = |zD| = |z|$. Thus $|bD| = p|z|$ which implies that $p|z| \leq |b|$. However as $b^p \in A$ we have $|b^p| \leq exp(A) = |z|$ and hence $|b| \leq p|z|$. Comparison gives $|b| = p|z|$ and so $\langle b\rangle \cap D = 1$. Thus (a) follows.

Put $\bar{H} = H/D^*$. Clearly $[N, R] \lhd H$ and $[N, R] \leq E \cap R \leq D$ since N/E is abelian which implies that $[N, R] \leq D^*$. Therefore $[\bar{N}, \bar{R}] = 1$ and so $\bar{R} \leq Z(\bar{N})$. Next assume if possible that there exists a $t \in T \setminus N$ so that $[\bar{t}, \bar{R}] = 1$. We may suppose that $t^p \in N$. Then

$$1 = [\bar{t}^p, \bar{N}] = [\bar{t}, \bar{N}]^p$$

by [6, Lemma 2.2.2(i)] since $\bar{t}^p \in \bar{B} \cap \bar{N} = \bar{R}$ and $\bar{R} \leq Z(\bar{N})$. Hence $[\bar{t}, \bar{N}]$ has order equal to p and since $\langle\bar{z}\rangle$ is the only cyclic subgroup of $\bar{N}/\bar{E}$ of order $|z| \geq p$ it follows that $[\bar{t}, \bar{N}] \leq \langle\bar{z}\rangle$ and this implies that $[t, N] \leq \langle z\rangle D \leq \langle z\rangle E$. This shows that t normalizes $\langle z\rangle E$ and then also E' is normalized by t. But then $t \in N$ by the hypothesis, which is a contradiction. Therefore $C_{\bar{T}}(\bar{R}) = \bar{R}$ and so (b) follows. This completes the proof of the lemma. □

Lemma 2.9. *Let B be a group and A be a normal abelian subgroup of B. Let $a \in A$, $b \in B$ and $n \geq 1$. Then*

$$(ab)^n = b^n a^n \prod_{k=1}^{n} [a,_k b]^{m_k}$$

where $m_k = \binom{n}{k} + \binom{n-1}{k} + \cdots + \binom{k}{k}$. If $n = p^s$ and $1 \leq k < p-1$, then $n | m_k$.

Proof. The following hold.

$$(ab)^n = b^n a^n [a, b^n] \cdots [a, b]$$

$$[a, b^n] = \prod_{k=1}^{n} [a,_k b]^{\binom{n}{k}}$$

for $n \geq 1$. Using these equalities we get

$$(ab)^n = b^n a^n \prod_{k=1}^{n} [a,_k b]^{m_k}$$

and $m_k = \binom{n}{k} + \cdots + \binom{k}{k}$, where $1 \leq k \leq n$. Also it is well known that $m_k = \binom{n+1}{k+1}$ for $1 \leq k \leq n$. Now let $n = p^s$ and $1 \leq k < p-1$. We have

$$m_k = \binom{n+1}{k+1} = \frac{(n+1)!}{(k+1)!(n+1-k-1)!} = \frac{(n+1)!}{(k+1)!(n-k)!}$$

Since $n - k < n$ the last equality takes the form

$$\frac{(n+1)!}{(k+1)!(n-k)!} = \frac{(n-k+1)\cdots n(n+1)!}{(k+1)!}$$

Also $p \nmid (k+1)!$ since $k < p-1$. Therefore

$$n | \frac{(n-k+1)\cdots n(n+1)!}{(k+1)!}$$

which completes the proof of the lemma. □

Lemma 2.10. *Let K be a locally finite p-group such that $K = \Omega_1(K)$ and $nc(K) = 3$, where $p \geq 3$. Then $exp(K/\gamma_3(K)) = p$ and $exp(K) \leq p^2$.*

Proof. Put $\bar{K} = K/\gamma_3(K)$. Since $nc(\bar{K}) = 2$ and $p \neq 2$ it follows that $\Omega_1(\bar{K}) = p$ by [6, 5.3.9(i)]. Since $K = \Omega_1(K)$ it follows that $exp(\bar{K}) = p$. Since $exp(K/K') = p$ it follows that $K'/\gamma_3(K)$ and $\gamma_3(K)$ have exponents p by [10, 1.2.14(i)]. Therefore $exp(K) \leq p^2$, which completes the proof of the lemma. □

Lemma 2.11. *Let G be a locally finite p-group satisfying the normalizer condition, where $p \neq 2$. Let (w, V) be a Λ-pair for G satisfying $(**)$ such that $W^*(w, V) = 1$. Let B be a normal nilpotent subgroup of G with $nc(B) = c$ and A be a normal abelian subgroup of G such that $A \leq B$ and B/A is elementary abelian. Then B is abelian if one of the following conditions is satisfied.*

(a) *$c < p$ and $exp(A) = exp(A \cap Z(G))$.*
(b) *$B = \Omega_1(B)$, $c \leq 3$ and $|Z(G)| \neq 3$.*
(c) *$c \leq 3$, $exp(A) = exp(A \cap Z(G))$ and $|Z(G)| \neq 3$.*

Proof. (a) Let $c < p$. Assume that B is not abelian. Then $c \geq 2$ and $p \geq 3$. Let $E \in E^*(w, V)$ such that $N_G(E) = N_G(E')$. Then $B \nleq N_G(E)$ since $N_G(E)/E$ is abelian and $W^*(w, V) = 1$. Also $Z(G)$ is cyclic by Lemma 2.5 and non-trivial by the hypothesis. Let $A \cap Z(G) = \langle z \rangle$. Then $exp(A) = |z|$ by the hypothesis. Put $N = N_G(E)$, $R = N \cap B$, $D = R \cap E$, $T = N_B(N)$, $Z = Z(B)$ and $H = TN$. Then $AZ \leq N$ by Lemma 2.7 and $T \nleq N$ since G satisfies the normalizer condition. Furthermore $R = \langle b \rangle D$, $b \notin AD$, $\langle b \rangle \cap D = 1$, $|b| = p|z|$, $A = \langle z \rangle (A \cap D)$ and $\langle b \rangle D > \langle z \rangle D$ by Lemma 2.8(a). Next let $D^* = Core_H(D)$ and put $\bar{H} = H/D^*$. Then $\bar{R} \leq Z(\bar{N})$ and $C_{\bar{T}}(\bar{R}) = \bar{R}$ by Lemma 2.8(b). Then also $Z(\bar{T}) \leq \bar{R}$.

Choose a $\bar{t} \in \bar{T} \setminus \bar{R}$. Then $|\bar{t}| \leq p|\bar{z}| = p|z|$. We claim that $[\bar{t}, \bar{E}] \leq \langle \bar{z} \rangle \bar{D}$. Let $\bar{e} \in \bar{E}$. Then $[\bar{e}, \bar{t}^{-1}] \leq \bar{T} \cap \bar{N} = \bar{R}$. Hence $[\bar{e}, \bar{t}^{-1}] = \bar{b}^i \bar{y}$ for a $\bar{y} \in \bar{D}$ and $i \geq 1$ and hence $\overline{t^e} = (\bar{b}^i \bar{y})\bar{t}$. Taking the $|z|th$ power of each side and applying Lemma 2.9 we get

$$(\overline{t^e})^{|z|} = \bar{t}^{|z|} (\bar{b}^i \bar{y})^{|z|} \prod_{k=1}^{|z|} [\bar{b}^i \bar{y},_k \bar{t}]^{\binom{|z|+1}{k+1}}$$

by Lemma 2.9. Since $p||z|$ we have $\bar{t}^{|z|} \in \bar{R} \leq Z(\bar{N})$ which implies that $(\overline{t^e})^{|z|} = \bar{t}^{|z|}$. Substituting this above and simplifying gives

$$1 = \bar{b}^{i|z|} \bar{y}^{|z|} \prod_{k=1}^{|z|} [\bar{b}^i \bar{y},_k \bar{t}]^{\binom{|z|+1}{k+1}}$$

Consider a factor $[\bar{b}^i \bar{y},_k \bar{t}]^{\binom{|z|+1}{k+1}}$. If $k \geq c$, then this factor is equal to 1 since $nc(B) = c$. If $k < c$ then again this factor is equal to 1 by Lemma 2.9 since $k \leq c - 1 < p - 1$. Consequently it follows that

$$\prod_{k=1}^{|z|} [\bar{b}^i \bar{y},_k \bar{t}]^{\binom{|z|+1}{k+1}} = 1$$

and substituting this value above gives $\bar{b}^{i|z|} \bar{y}^{|z|} = 1$. Thus we get $\bar{b}^{i|z|} = \bar{y}^{-|z|} \in \bar{D}$. This forces $\bar{b}^{i|z|} = 1 = \bar{y}^{-|z|}$ since $\langle b \rangle \cap D = 1$. Since $|b| = p|z|$ we see that $p|i$ and so $\bar{b}^i \in \langle \bar{z} \rangle \bar{D}$. Clearly then $[\bar{e}, \bar{t}^{-1}] \in \langle \bar{z} \rangle \bar{D} \leq \langle \bar{z} \rangle \bar{E}$ since $y \in D \leq E$. This is equivalent to $[e, t^{-1}] \in \langle z \rangle E$. Since e is any element of E it follows that $[E, t^{-1}] \in \langle z \rangle E$ and so t normalizes $\langle z \rangle E$. But then t normalizes E' and so $t \in N$, which is a contradiction. Therefore the assumption is false and so B must be abelian.

(b) Assume that B is not abelian. By the hypothesis B/B' is elementary abelian, $nc(B) \leq 3$, $p \geq 3$ and $|Z(G)| \neq 3$ by the hypothesis. Since $exp(B/B') = p$ it follows that $exp(B'/\gamma_3(B)) = exp(\gamma_3(B)) = p$ by [10, 1.2.14(i)]. Thus $exp(B/\gamma_3(B)) = p$ and $exp(B) \leq p^2$ by Lemma 2.10. Define $A = B'$. Then B/A is elementary abelian.

We use the same notation as above. Now if $|Z(G)| \geq p^2$, then considering $BZ(G)$ and $AZ(G)$ it follows that B is abelian by (a) since $exp(B) \leq p^2$. Therefore we may suppose that $|Z(G)| = p$. Let $Z(G) = \langle z \rangle$. We may suppose that $z \in A$.

Clearly $p > 3$ since $|Z(G)| \neq 3$. Thus $p \geq 5$. Since $|z| = p$ the simplified equality in (a) can be written as

$$1 = \bar{b}^{ip} \bar{y}^p [\bar{b}^i \bar{y}, \bar{t}]^{\binom{p+1}{2}} [\bar{b}^i \bar{y}, \bar{t}, \bar{t}]^{\binom{p+1}{3}}$$

It is easy to see that p divides $\binom{p+1}{2}$ and $\binom{p+1}{3}$ since $p \geq 5$. Hence $\binom{p+1}{2} = pu$ and $\binom{p+1}{3} = pv$ for some $u, v \geq 1$. Also $[b^i y, t, t] \in Z(B)$ since $c \leq 3$. Using these we get

$$[\bar{b}^i\bar{y}, \bar{t}, \bar{t}]^{\binom{p+1}{3}} = [\bar{b}^i\bar{y}, \bar{t}, \bar{t}]^{pv} = [\bar{b}^i\bar{y}, \bar{t}, \bar{t}^p]^v = 1$$

since $t^p \in \gamma_3(B)$(or $\bar{t}^p \in \bar{R} \leq Z(\bar{N})$). In the same way

$$[\bar{b}^i\bar{y}, \bar{t}]^{\binom{p+1}{2}} = [\bar{b}^i\bar{y}, \bar{t}]^{pu} = [(\bar{b}^i\bar{y})^p, \bar{t}]^u = 1$$

since $\bar{b}^i, \bar{y} \in \bar{R}$ and $\bar{R} \leq Z(\bar{N})$. Substituting these values above we obtain $1 = \bar{b}^{ip}\bar{y}^p$ which gives a contradiction as in (a). Therefore B must be abelian.

(c) If $p > 3$, then B abelian by (a) since $c \leq 3$. So suppose that $p = 3$. In this case $|Z(G)| \geq 3^2$. Again we use the same notation which is used in (a). Thus we have

$$\begin{aligned} 1 &= \bar{b}^{i|z|}\bar{y}^{|z|} \prod_{k=1}^{|z|} [\bar{b}^i\bar{y},_k \bar{t}]^{\binom{|z|+1}{k+1}} \\ &= \bar{b}^{i|z|}\bar{y}^{|z|}[\bar{b}^i\bar{y}, \bar{t}]^{\binom{|z|+1}{2}}[\bar{b}^i\bar{y}, \bar{t}, \bar{t}]^{\binom{|z|+1}{3}} \end{aligned}$$

Obviously $\binom{|z|+1}{2} = |z|u$ for a $u \geq 1$, which implies that $[\bar{b}^i\bar{y}, \bar{t}]^{\binom{|z|+1}{2}} = 1$ since $\bar{b}^i, \bar{y} \in \bar{R}$. Furthermore $\binom{|z|+1}{3} = 3v$ for a $v \geq 1$ since $|z| \geq 3^2$. Also $[\bar{b}^i\bar{y}, \bar{t}, \bar{t}] \in Z(\bar{T})$. Hence, as in (b),

$$[\bar{b}^i\bar{y}, \bar{t}, \bar{t}]^{\binom{|z|+1}{3}} = [\bar{b}^i\bar{y}, \bar{t}, \bar{t}]^{3v} = [\bar{b}^i\bar{y}, \bar{t}, \bar{t}^3]^v = 1$$

Substituting these values above we get $1 = \bar{b}^{i|z|}\bar{y}^{|z|}$ which gives a contradiction as in (a). Therefore B must be abelian in this case as well. This completes the proof of the lemma. □

Proof of Theorem 1.1. Assume that G is perfect. We claim that there exists a proper normal subgroup M of G so that $\Omega_1(G/M) = \langle aM \in G : |aM| = p\rangle$ is abelian. From Lemma 2.1(c) we know that if (w, V) is a Λ-pair for G and if $1 \neq M$ is a maximal element of $W^*(w, V)$, then there exists a finite subgroup U of G with $U \not\leq M$ and containing V so that $(wuM/M, UM/M)$ is a Λ-pair for G/M with $W^*(wuM/M, UM/M) = 1$ for a $u \in U$. This shows that every homomorphic image H of G has a homomorphic image K with a Λ-pair (w_K, V_K) such that $W^*(w_K, V_K) = 1$. Let $I_G = \{a \in G : |a| = p\}$. For each $a \in I_G$ put $N_a = \langle a^g : g \in G\rangle$. Then each N_a is nilpotent since G is a Fitting group. Let $n(G)$ be the minimum of all the $nc(N_a) > 1$ as a ranges over I_G.

First we show the following. G has a homomorphic image H with the following property. H has a Λ-pair (w_H, V_H) satisfying $(**)$ and the condition $W^*(w_H, V_H) = 1$ such that for every $a \in I_H$ the subgroup N_a is abelian, that is $n(H) = 1$. Assume that there exists no such H. Among all the homomorphic images X of G having a Λ-pair (w_X, V_X) satisfying $(**)$ and the condition $W^*(w_X, V_X) = 1$ there is a homomorphic image H such that $1 < n(H) \leq n(X)$ for all such X. Without loss of generality we may suppose that $H = G$. Thus G has a Λ-pair (w, V) such that $(**)$ and the condition $W^*(w, V) = 1$ are satisfied. In particular $1 \neq |Z(G)| \neq 3$ by Lemma 2.1(c) and by the hypothesis. Also $n(G)$ is minimal in the above sense and $n(G) > 1$ by the assumption. Let $a \in I_G$ so that $nc(N_a) = n(G)$. Put $K = N_a$. Then $K = \Omega_1(K)$. First suppose that $nc(K) = 2$.

Then $K/\gamma_2(K)$ is elementary abelian. Also $exp(\gamma_2(K)) = p$ by [10, 1.2.14(i)] since $exp(K/\gamma_2(K)) = p$ which implies that K is abelian by Lemma 2.11(a). Therefore $nc(K) > 2$. Let $nc(K) = c$ and put $\bar{G} = G/\gamma_c(K)$. Then $nc(\bar{K}) = c - 1$.

First assume if possible that $\bar{K}' \leq Z(\bar{G})$. Then $[\bar{K}', \bar{G}] = 1$ and hence $[K', G] \leq \gamma_c(K)$ which implies that $[K', G, K] = 1$. This implies that $[K, K, K, K] = 1$ and so $c = 3$ since $c > 2$. Furthermore it follows as above that $K'/\gamma_3(K)$ and $\gamma_3(K)$ have exponents equal to p since $exp(K/K') = p$. Since $nc(K') \leq 2$ if we put $A = \gamma_3(K)$ in Lemma 2.11(a), then it follows that K' is abelian. Thus $K = \Omega_1(K)$, K' is abelian, K/K' is elementary abelian and $|Z(G)| \neq 3$. But then applying Lemma 2.11(b) shows that K must be abelian which is a contradiction. Therefore $\overline{K'} \nleq Z(\bar{G})$ and so there exists an $\bar{s} \in \overline{K'} \setminus Z(\bar{G})$.

Let T be a finite subgroup of G so that $\bar{s} \notin \bar{T}$. Then $(\bar{s}, \bar{T})$ is a Λ-pair for $\bar{G}$. Let $\bar{M}$ be a maximal element of $W^*(\bar{s}, \bar{T})$. If $\bar{M} = 1$, then $\bar{K}$ is abelian by the induction hypothesis. But then $nc(K) = 2$, which gives a contradiction as above. Therefore $\bar{M} \neq 1$. Now consider $\bar{G}/\bar{M}$. By Lemma 2.1(c) there exists a finite subgroup U of G such that $\bar{s} \notin \bar{U}$ and $\bar{T} \leq \bar{U} \nleq \bar{M}$. Then there exists a $u \in U$ so that $(\bar{s}\bar{u}\bar{M}, \bar{U}\bar{M}/\bar{M})$ is a Λ-pair for $\bar{G}/\bar{M}$. Also $(\bar{s}\bar{u}\bar{M}, \bar{U}\bar{M}/\bar{M})$ satisfies the hypothesis and $W^*(\bar{s}\bar{u}\bar{M}, \bar{U}\bar{M}/\bar{M}) = 1$. In this case $\bar{K}\bar{M}/\bar{M}$ is abelian by the induction hypothesis and this implies that $\bar{K}' \leq \bar{M}$. However since $\bar{s} \in \bar{K}'$ but $\bar{s} \notin \bar{M}$ this is another contradiction. Therefore the assumption is false and so it follows that N_a is abelian for every $a \in I_G$. In particular each N_a has exponent p.

Put $N = \langle N_a : a \in I_a \rangle$. Then N is abelian by Lemma 2.7 since N_a is abelian and $exp(N_a) = p$ for every $a \in I_G$, and G has a Λ-pair (w, V) such that $(**)$ and the condition $W^*(w, V) = 1$ are satisfied. Since $N = \Omega_1(G)$ the proof of the theorem is complete. □

Proof of Corollary 1.2. Suppose that G is a Fitting p-group satisfying the normalizer condition, where $p \neq 2$. Furthermore suppose that in every homomorphic image H of G every Λ-pair satisfies (**) and $Z(H) \neq 3$. Suppose also that G is perfect. Assume if possible that $G = \Omega_k(G)$ for a $k \geq 1$. Without loss of generality we may suppose that $k = 1$. Then $G = \Omega_1(G)$. By Theorem 1.1 there exists a proper normal subgroup M of G so that $\Omega_1(G/M)$ is abelian. But since $\Omega_1(G)M/M \leq \Omega_1(G/M)$ it follows that $\Omega_1(G)M/M \neq G/M$ since G is perfect which is a contradiction since $G = \Omega_1(G)$. Therefore $\Omega_k(G) \neq G$ for every $k \geq 1$.

Next suppose that G is perfect and assume that every proper subgroup of G has finite exponent. Then there exists a smallest $k \geq 1$ so that $G = \Omega_k(G)$ by [1, Theorem 1.1]. But since G cannot be generated by a subset of finite exponent by the first paragraph this gives a contradiction. Hence it follows that if every proper subgroup of G has finite exponent, then G cannot be perfect. This completes the proof of the corollary. □

3. Proof of Theorems 1.3, 1.4

Lemma 3.1. *Let G be an infinitely generated MNS-group such that G has no $(*)$-triples for non-central elements. If the $(**)$-property is satisfied by dominant (distinguished) pairs, then the same property is satisfied for Λ-pairs.*

Proof. Let (w, V) be a Λ-pair for G. By [4, Lemmas 3.1, 4.1(a)] there exist a finite subgroup U of G containing V so that (w, U) is a dominant pair for G. By the hypothesis there exists an $E \in E^*(w, U)$ so that $N_G(E) = N_G(E')$. But since $E^*(w, U) \subseteq E^*(w, V)$ by [4, Lemma 3.2] it follows that $E \in E^*(w, V)$ and so $(**)$ is satisfied by (w, V). □

Lemma 3.2. *Let G be a Fitting p-group which is an MNS-group satisfying the normalizer condition, where $p \neq 2$. Suppose that G has no homomorphic images having $(*)$-triples for non-central elements. Let (w, V) be a dominant pair for G with $W^*(w, V) = 1$. Finally suppose that in every homomorphic image of G dominant pairs satisfy $(**)$. Then $\langle S_1(G)\rangle \neq G$.*

Proof. Note that $Z(G) \neq 1$ by Lemma 2.1(c) since $W^*(w, V) = 1$. First we show that $S_1(G)^e = \{A \in S_1(G) : exp(A) \leq p^e\}$ generates a proper subgroup of G for every $e \geq 1$. We may use induction on $e \geq 1$. Let $e = 1$. Then $\langle S_1(G)^1\rangle$ is abelian by Lemma 2.7 since $p \neq 2$ and $W^*(w, V) = 1$. Now suppose that the assertion holds for an $e \geq 1$. We show that it holds for $e+1$. Let $K = \langle S_1(G)^1\rangle$ and put $\bar{G} = G/K$. Clearly then $\{\bar{A} : A \in S_1(G)^{e+1}\} \subseteq S_1(\bar{G})^e$. Let $\bar{u} \in \bar{G} \setminus Z(\bar{G})$. By the hypothesis and by [4, Lemmas 3.1, 4.1(a)] there exists a finite subgroup S of G so that $(\bar{u}, \bar{S})$ is a dominant pair for $\bar{G}$. If $W^*(\bar{u}, \bar{S}) = 1$, then $\langle S_1(\bar{G})^e\rangle \neq \bar{G}$ by the induction hypothesis which implies that $\langle S_1(G)^{e+1}\rangle \neq G$. Therefore we may suppose that $W^*(\bar{u}, \bar{S}) \neq \{1\}$. Let $\bar{M}$ be a maximal element of $W^*(\bar{u}, \bar{S})$ and consider the group $\bar{G}/\bar{M}$. By Lemma 2.1(c) there exists a finite subgroup $\bar{T}$ of $\bar{G}$ with $\bar{S} \leq \bar{T} \nleq \bar{M}$ and a $t \in T$ so that $(\bar{u}\bar{t}\bar{M}, \bar{T}\bar{M}/\bar{M})$ is a Λ- pair for $\bar{G}/\bar{M}$ such that $W^*(\bar{u}\bar{t}\bar{M}, \bar{T}\bar{M}/\bar{M}) = 1$. By [4, Lemmas 3.1, 4.1(a)] we may suppose that $(\bar{u}\bar{t}\bar{M}, \bar{T}\bar{M}/\bar{M})$ is a dominant pair for $\bar{G}/\bar{M}$. Without loss of generality we may assume that $\bar{u}\bar{M} \notin Z(\bar{G}/\bar{M})$ and so consider $(\bar{u}\bar{M}, \bar{T}\bar{M}/\bar{M})$. Also $(\bar{u}\bar{M}, \bar{T}\bar{M}/\bar{M})$ satisfies $(**)$ by the hypothesis. Therefore $\langle S_1(\bar{G})^e\rangle \neq \bar{G}$ by the induction hypothesis which implies that $\langle S_1(G)^{e+1}\rangle \neq G$ as before. This completes the proof of the assertion.

Now consider the general case of $S_1(G)$. Let $g \in G \setminus Z(G)$. Let $|g| = n$ and let the subnormal index of $\langle g\rangle$ in G be m. Then applying [2, Corollary 2.2] we see that $K = \langle A^{m^n} : A \in S_1(G)\rangle \leq C_G(g)$. Also $K \neq G$ by the choice of g. Put $e = m^n$ and $\bar{G} = G/K$. Choose an $\bar{s} \in \bar{G} \setminus Z(\bar{G})$. Then there exists a finite subgroup $\bar{T}$ of $\bar{G}$ so that $(\bar{s}, \bar{T})$ is a dominant pair for $\bar{G}$. Moreover, as in the proof of Theorem 1.1, there exists a normal subgroup $\bar{M}$, a finite subgroup $\bar{R}$ with $\bar{T} \leq \bar{R} \leq \bar{M}$ and $\bar{s} \notin \bar{R}$ such that $(\bar{s}\bar{r}\bar{M}, \bar{R}\bar{M}/\bar{M})$ is a dominant pair for $\bar{G}/\bar{M}$ for an $r \in R$, and $W^*(\bar{s}\bar{M}, \bar{R}\bar{M}/\bar{M}) = 1$. Then $\bar{G}/\bar{M}$ satisfies the hypothesis of the lemma and so $\langle S_1(\bar{G}/\bar{M})^e\rangle \neq \bar{G}/\bar{M}$ by the first part of the lemma. Also $\bar{A}\bar{M}/\bar{M} \in S_1(\bar{G}/\bar{M})^e$ since $A^e \leq K$ for all $A \in S_1(G)$ which implies that $\langle \bar{A}\bar{M}/\bar{M} : A \in S_1(G)\rangle \neq \bar{G}$ and hence it follows that $\langle A : A \in S_1(G)\rangle \neq G$ which completes the proof of the lemma. □

Lemma 3.3. *Let G be an infinitely generated MNS-group and in every homomorphic image of G normal closures of finitely generated subgroups are residually nilpotent. Suppose that $S_t(G) \neq G$ for every $t \geq 1$. Then there exists a homomorphic image H of G and a subgroup E of H which is maximal with respect to a dominant pair for H so that $\langle E^H \rangle = H$.*

Proof. Since G is an infinitely generated perfect group whose proper subgroups are solvable there exists a homomorphic image H of G so that H and its homomorphic images cannot satisfy the $(*)$-property for non-central elements by [4, Theorem 1.4(b)]. Without loss of generality we may suppose that $H = G$. Let (w, V) be a dominant pair for G and assume if possible that $\langle E^G \rangle \neq G$ for every $E \in E^*(w, V)$. Choose an $E_1 \in E^*(w, V)$ and let $L_1 = \langle M \lhd G : d(M) \leq d(\langle E_1^G \rangle)\ \rangle$. Then $L_1 \neq G$ by the hypothesis. Hence there exists an $E_2 \in E^*(w, V)$ so that $E_2 \nleq L_1$ by [4, Lemma 3.3]. Choose a $y_1 \in E_2 \setminus L_1$. Then $d(\langle y_1^g : g \in G \rangle) > d(\langle E_1^g : g \in G \rangle)$. Also $w \notin \langle V, y_1 \rangle$. So there exists a finite subgroup V_1 of G containing V and y_1 so that (w, V_1) is a dominant pair for G and $E^*(w, V_1) \subseteq E^*(w, V)$ by [4, Lemmas 4.1(a), 3.2]. Put $L_2 = \langle M \lhd G : d(M) \leq d(\langle y_1^g : g \in G \rangle) \rangle$. Then $L_2 \neq G$ and so as in the first case there exists a $y_2 \in G \setminus L_2$ so that $w \notin \langle V_1, y_2 \rangle$ and $d(\langle y_2^g : g \in G \rangle) > d(\langle y_1^g : g \in G \rangle)$. Continuing in this way we obtain an infinite sequence $y_1, y_2, \ldots$ of elements of G so that $d(\langle y_{i+1}^g : g \in G \rangle) > d(\langle y_i^g : g \in G \rangle)$ for all $i \geq 1$ and $w \notin \bigcup_{i=1}^{\infty} \langle V, y_1, \ldots, y_i \rangle$. Hence there exists a (w, V)-maximal subgroup R of G so that $\bigcup_{i=1}^{\infty} \langle V, y_1, \ldots, y_i \rangle \leq R$. Also $d(R) = \infty$ by the choice of the y_i and this implies that $R = G$, which is a contradiction since $w \notin R$ and so the proof of the lemma is complete. □

Proof of Theorem 1.3. Let G be a Fitting p-group which is an MNS-group and satisfies the normalizer condition, where $p \neq 2$. Suppose that in every homomorphic image of G dominant pairs satisfy $(**)$. Let (w, V) be a dominant pair for G. First we show that $\langle S_t(G) \rangle \neq G$ for every $t \geq 1$. By Lemma 2.1(c) we may suppose that $W^*(w, V) = 1$. If $t = 1$, then the assertion follows by Lemma 3.2. Next assume that $t > 1$ and the assertion holds for $t - 1$. Put $K = \langle S_{t-1}(G) \rangle$ and $\bar{G} = G/K$. Then $\bar{G} \neq 1$ since $K \neq G$. By the hypothesis $\bar{G}$ contains no $(*)$-triples for non-central elements. Therefore there exists an $\bar{s} \in \bar{G}$, a finite subgroup $\bar{T}$ and a normal subgroup $\bar{M}$ of $\bar{G}$ so that $(\bar{s}\bar{M}, \bar{T}\bar{M}/\bar{M})$ is a dominant pair for $\bar{G}$ and $W^*(\bar{s}\bar{M}, \bar{T}\bar{M}/\bar{M}) = 1$. Then again $\langle S_1(\bar{G}/\bar{M}) \rangle \neq \bar{G}/\bar{M}$ by Lemma 3.2. Since $\bar{L}\bar{M}/\bar{M} \in S_1(\bar{G}/\bar{M})$ for every $L \in S_1(G)$ it follows that $\langle \bar{L}\bar{M}/\bar{M} : L \in S_t(G) \rangle \neq \bar{G}/\bar{M}$ which implies that $\langle S_t(G) \rangle \neq G$ and so the induction is complete. The second part follows from Lemma 3.3 and so the proof of the theorem is complete. □

Proof of Theorem 1.4. Assume that G is not solvable. Then G is perfect and applying [4, Theorem 1.4(b)] we see that G is a Fitting p-group and has a homomorphic image H such that in every homomorphic of H there are no $(*)$-triples for non-central elements. Also $p \neq 2$. Furthermore if K is a homomorphic image of H, then every dominant pair for K satisfies $(**)$ and $|Z(K)| \neq 3$. Finally G satisfies the normalizer condition and without loss of generality we may suppose that $H = G$. Then in every homomorphic image of G there are no $(*)$-triples for non-central elements and the dominant pairs for G satisfy $(**)$.

Let H be a homomorphic image of G. Let $w_H \in H \setminus Z(H)$ and V_H be a finite subgroup of H with $w_H \notin V_H$. Thus (w_H, V_H) is a Λ-pair for H. Also (w_H, V_H) satisfies $(**)$ by Lemma 3.1. Consequently it follows that G satisfies the hypothesis of Theorem 1.1. Therefore there exists a proper normal subgroup M of G such that $\Omega_1(G/M)$ is abelian. Furthermore G contains a proper subgroup of infinite exponent, $S_t(G) \neq G$ for every $t \geq 1$ and $G = \langle E^G \rangle$ for a proper subgroup of G by Corollary 1.2 and Theorem 1.3.

Finally if every proper subgroup of G has finite exponent, then G cannot be perfect by Corollary 1.2 and then G must be solvable. This completes the proof of the theorem. □

4. **Applications**

In this last section we apply the above results to obtain two results on solvability. The first result also shows the influence on the group structure of the bounding of the exponents of the normal abelian subgroups of a group by the exponent of the group center.

Proof of Theorem 1.5. Let G be a Fitting p-group satisfying the normalizer condition in which every proper subgroup is solvable, where $p \neq 2$. Furthermore suppose that every homomorphic image H of G satisfies $(**)$ and $Z(H) \neq 3$. Assume if possible that G is not solvable. Then G is perfect. As in the proof of Theorem 1.4 we may suppose that in every homomorphic image of G there are no $(*)$-triples for non-central elements. First we show the following. G has a homomorphic image H with the following property. H has a dominant pair (w_H, V_H) satisfying $W^*(w_H, V_H) = 1$ such that every normal metabelian nilpotent subgroup $B = [B, H]$ of H is abelian. Assume not. We may suppose that G has a dominant pair (w, V) with the property $W^*(w, V) = 1$. Let $n(G)$ be the minimum of the nilpotent classes of all the normal metabelian but not abelian nilpotent subgroups $B = [B, G]$ of G. Then $n(G) > 1$. As in the proof of Theorem 1.1 we may suppose that $n(G) \leq n(X)$ for all homomorphic images X of G having a dominant pair (w_X, V_X) and satisfying $W^*(w_X, V_X) = 1$.

Let $B = [B, G]$ be a normal metabelian nilpotent subgroup of G such that $n(G) = nc(B)$. First suppose that $nc(B) = 2$. Then B is already abelian by Lemma 2.11(a) since $p > 2$ and $exp(A) \leq exp(Z(G))$ for every normal abelian subgroup A of G contained in B by the hypothesis. Therefore we may suppose that $nc(B) \geq 3$. Put $\bar{G} = G/Z(B)$. Let $\bar{t} \in \bar{B} \setminus Z(\bar{G})$. By [4, Lemmas 3.1, 4.1(a)] there exists a finite subgroup $\bar{U}$ of $\bar{G}$ such that $(\bar{t}, \bar{U})$ is a dominant pair for $\bar{G}$ since $\bar{G}$ has no $(*)$-triples for non-central elements and satisfies $(**)$ by the hypothesis. First suppose that $W^*(\bar{t}, \bar{U}) = 1$. Then $\bar{B}$ is abelian since $nc(\bar{B}) < nc(B) = n(G)$. Hence since $\bar{B} = B/Z(B)$ it follows that $nc(B)$=2, which is impossible since $nc(B) \geq 3$. Therefore $W^*(\bar{t}, \bar{U}) \neq 1$.

Assume if possible that $\bar{B}' \leq Z(\bar{G})$. Then B' is abelian by Lemma 2.11(a) as above since $nc(B') \leq 2$. Also $[\bar{B}', \bar{G}] = 1$ and hence $[B', G] \leq Z(B)$ which implies that $[B, B, B, B] = 1$ and so $nc(B) = 3$ since $nc(B) \geq 3$. Let A be a maximal normal abelian subgroup of G containing B' and contained in B. Let

$T/A = \Omega_1(B/A)$. Then T is abelian by Lemma 2.11(c) since $exp(A) \leq exp(Z(G))$ and $|Z(G)| \neq 3$. But this contradicts the maximality of A. Therefore $\bar{B}' \nleq Z(\bar{G})$ and so there exists an $\bar{s} \in \bar{B}' \setminus Z(\bar{G})$. As above there exists a finite subgroup $\bar{T}$ of $\bar{G}$ so that $(\bar{s}, \bar{T})$ is a dominant pair for $\bar{G}$. Let $\bar{M}$ be a maximal element of $W^*(\bar{s}, \bar{T})$. If $\bar{M} = 1$, then $\bar{B}$ is abelian by the induction hypothesis. But then $nc(B) = 2$ which is impossible. Therefore $\bar{M} \neq 1$. Consider $\bar{G}/\bar{M}$. By Lemma 2.1(c) there exists a finite subgroup R of G such that $\bar{s} \notin \bar{R}$ and $\bar{T} \leq \bar{R} \nleq \bar{M}$. As before there exists an $r \in R$ so that $(\bar{s}\bar{r}\bar{M}, \bar{R}\bar{M}/\bar{M})$ is a Λ-pair for $\bar{G}/\bar{M}$. Furthermore using [4, Lemmas 3.1, 4.1(a)], as before, we may suppose that $(\bar{s}\bar{r}\bar{M}, \bar{R}\bar{M}/\bar{M})$ is a dominant pair and so satisfies $(**)$. Also $W^*(\bar{s}\bar{r}\bar{M}, \bar{U}\bar{M}/\bar{M}) = 1$. This implies that $\bar{B}\bar{M}/\bar{M}$ is abelian by the induction hypothesis which means that $\bar{B}' \leq \bar{M}$. However since $\bar{t} \in \bar{B}'$ but $\bar{t} \notin \bar{M}$ this is another contradiction. Consequently it follows that B is abelian.

Let F be a finite non-abelian subgroup of G and put $N = \langle F^G \rangle$. Then N is a normal nilpotent subgroup of G. Let A be a maximal normal abelian subgroup of G contained in N. Put $Z/A = Z(N/A)$. Then Z is a normal metabelian nilpotent subgroup of G and $Z > A$. Clearly then $[Z, G]$ is abelian by the first part of the proof. Furthermore G has a dominant pair (w, V) satisfying $(**)$ and the equality $W^*(w, V) = 1$ holds by our supposition above. Hence there exists an $E \in E^*(w, V)$ so that $N_G(E) = N_G(E')$. Then $[Z, G]Z(G) \leq N_G(E)$ by Lemma 2.7 since $exp([Z, G]) \leq exp(Z(G))$ by the hypothesis. Moreover if $Z(G) = \langle z \rangle$, then $[Z, G] \leq \langle z \rangle E$ by Lemma 2.8(a) which implies that Z normalizes $\langle z \rangle E$ and then $Z \leq N_G(E)$ as before. But then $1 \neq Z' \leq E$, which is a contradiction since $W^*(w, V) = 1$. Therefore the assumption that G is perfect is false and so G must be solvable. This completes the proof of the theorem. □

Lemma 4.1. *Let L be a locally finite p-group which is n-Engel for an $n \geq 1$ and let A be an elementary abelian normal subgroup of L. Then $L/C_L(A)$ has exponent $\leq p^n$*

Proof. Let $x \in L$ and $y \in A$ and put $H = \langle x, [x, y] \rangle$. Then

$$[x^{p^n}, y] \equiv [x, y]^{p^n} \pmod{\gamma_2(H)^{p^n} \prod_{r=1}^{n} \gamma_{p^r}(H)^{p^{n-r}}}$$

by [8, VIII.1.1 Lemma (b)]. Clearly $\langle [x, y]^H \rangle \leq H \cap A$ and hence $H = (H \cap A)\langle x \rangle$ since $A \lhd L$. Thus $\gamma_i(H) \leq A$ and so $\gamma_i(H)$ is elementary abelian for every $i \geq 2$. By using this property above we get

$$[x^{p^n}, y] \equiv 1 \pmod{\gamma_{p^n}(H)}$$

Clearly $p^n \geq 2n$ for all $n \geq 1$. Furthermore $\gamma_{p^n}(H) = \langle [h_1, h_2, \ldots, h_{p^n}] : h_i \in H \rangle$ by [7, III.1.11 Hilfsatz]. We claim that $\gamma_{p^n}(H) = 1$. Assume that $[h_1, h_2, \ldots, h_{p^n}] \neq 1$. Now each $h_i \in H$ has the form $h_i = a_i x^{m_i}$ for an $a_i \in H \cap A$ and $i, m_i \geq 1$. Substituting these values above gives

$$[h_1, h_2, \ldots, h_{p^n}] = [a_1 x^{m_1}, a_2 x^{m_2}, \ldots, a_{p^n} x^{m_{p^n}}]$$

The following identities are well-known. Let $x, y, z \in L$ and $a, b \in A$. Then

$$[xy, z] = [x, z][x, z, y][y, z], \ [x, yz] = [x, z][x, y][x, y, z], \ [ab, x] = [a, x][b, x]$$

$$[a, xb] = [a, x] = [a, bx], [x, a] = [a^{-1}, x]$$

First of all

$$[a_1x^{m_1}, a_2x^{m_2}, \dots, a_{p^n}x^{m_{p^n}}] = [a_1x^{m_1}, a_2x^{m_2}, x^{m_3}, \dots, x^{m_{p^n}}]$$

since $[a_1x^{m_1}, a_2x^{m_2}] \in A$. Also $[a_1x^{m_1}, a_2x^{m_2}]$ is a product of elements of the form $[a_1, {}_ix]$ and $[a_2^{-1}, {}_jx]$ for $i, j \geq 1$. Combining the above results we see that

$$[h_1, h_2, \dots, h_{p^n}] = \prod_{i\geq 1}[b_i, {}_{j_i}x]$$

as a product of non-trivial factors, where $b_i \in \langle a_1\rangle \cup \langle a_2\rangle$, and $j_i \geq n$ since $p^n - 1 \geq n$ for every $i \geq 1$. But since L is n-Engel each factor $[b_i, {}_{j_i}x] = 1$ and then $[h_1, h_2, \dots, h_{p^n}] = 1$, which is a contradiction. Therefore $\gamma_{p^n}(H) = 1$ and substituting this value above we get $[x^{p^n}, y] = 1$. Since x is any element of L and y is any element of A it follows that $L/C_L(A)$ has exponent $\leq p^n$. □

Proof of Proposition 1.7. Let G be a locally finite p-group such that every proper subgroup of G is n-Engel for an $n \geq 1$. Let A be an elementary abelian normal subgroup of G and put $C = C_G(A)$. Then $C \lhd G$. Let L/C be a proper subgroup of G/C. Then L is a proper subgroup of G and so is n-Engel for an $n \geq 1$ by the hypothesis. Therefore L/C has finite exponent $\leq p^n$ by Lemma 4.1. Since L/C is any proper subgroup of G/C the assertion follows. □

Proof of Theorem 1.6. Let G be a Fitting p-group satisfying the normalizer condition, where $p \neq 2$. Suppose that G satisfies the hypothesis of the theorem. Assume that G is perfect. As in the proof of Theorem 1.4, we may suppose that in every homomorphic image H of G there are no $(*)$-triples for non-central elements and every dominant pair of H satisfies the $(**)$-property. In particular G itself satisfies this property. Let (w, V) be a dominant pair for G. Let M be a maximal element of $W^*(w, V)$. First suppose that $M = 1$. Then $Z(G)$ is cyclic by Lemma 2.5. Let A be a maximal normal elementary abelian subgroup of G. Then $A \neq Z(G)$ since A is infinite by [9, 1.G.6 Proposition]. Hence $C_G(A) \neq G$. By Theorem 1.3 there exists a proper subgroup E of G such that $\langle E^G\rangle = G$. Also AE is n-Engel for an $n \geq 1$ by the hypothesis. Therefore $EC_G(A)/C_G(A)$ has finite exponent in $G/C_G(A)$ by Proposition 1.7 and generates $G/C_G(A)$ by the choice of E. But since $G/C_G(A)$ cannot be generated by a subset of finite exponent by Corollary 1.2 this gives a contradiction.

Next suppose that $M \neq 1$ and consider G/M. By Lemma 2.1(c) there exists a finite subgroup U and a $u \in U$ so that $(wuM, UM/M)$ is a Λ-pair for G/M. Also $W^*(wuM, UM/M) = 1$ by the choice of M. Clearly then we get another contradiction as in the first case. Therefore the assumption is false and so G must be solvable, which completes the proof of the theorem. □

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