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ON THE FREE PROFINITE PRODUCTS OF PROFINITE GROUPS WITH COMMUTING SUBGROUPS

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Abstract. In this paper we introduce the construction of free profinite products of profinite groups with commuting subgroups. We study a particular case: the proper free profinite products of profinite groups with commuting subgroups. We prove some conditions for a free profinite product of profinite groups with commuting subgroups to be proper. We derive some consequences. We also compute profinite completions of free products of (abstract) groups with commuting subgroups.

1. Introduction

Profinite groups are known since 1965 when J. P. Serre introduced them in his book "Cohomologie Galoisienne" [18]. A profinite group G is the inverse limit of a projective system of finite groups i.e. $G = \varprojlim_{i\in I} G_i$, where $(G_i)_{i\in I}$ is a projective system of finite (abstract) groups, I is a directed set. A profinite group G is isomorphic to a closed subgroup of a direct product of finite groups. So, profinite groups are very large. They are very rich since they have algebraic and topological properties. A profinite group is a topological, compact, haussdorff and totally disconnected group. A concrete example of a profinite group is the profinite completion of an abstract group. Given G an abstract group, the profinite completion $\widehat{G}$ of group G is the inverse limit of the projective system $(G/N)_{N\in\mathcal{N}}$ of the (finite) quotient groups G/N, where $\mathcal{N}$ is the collection of all normal subgroups of finite index of G i.e. $\widehat{G} = \varprojlim_{N\in\mathcal{N}} G/N$. Profinite groups are "almost finite", they behave like infinite groups and they can inherit some properties of the finite groups on which they are built.

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In [9], free constructions (free products, amalgamated free products, HNN-extensions and free products with commuting subgroups) of abstract groups were defined. Many properties amongst which residual finiteness were investigated on these constructions. Since Baumslag's study on the residual finiteness of amalgamated free product of groups [1], several authors [2, 4, 5, 6, 7, 8, 10, 11, 12, 16, 17, 19] proved some conditions for the residual finiteness of some free constructions of abstract groups. Residual finiteness and profiniteness are linked. Indeed, a profinite group is residually finite and any residually finite group naturally injects in its profinite completion. Many authors have studied profinite groups in different directions [3, 13, 14, 15, 20]. Luis Ribes and Pavel Zalesskii in [15] have introduced free constructions of profinite groups. They defined free profinite products, amalgamated free profinite products and profinite HNN-extensions of profinite groups. They studied the particular cases of proper amalgamated free profinite products and proper profinite HNN-extensions of profinite groups. They gave examples of amalgamated free profinite product which are not proper and proved some conditions for their properness [13, 14].

In this paper, we carry similar study for the free profinite products of profinite groups with commuting subgroups. We define this construction here and we denote by $A \coprod_{[H,K]} B$ the free profinite product of profinite groups A and B with commuting subgroups H and K, where A and B are two profinite groups, and H is a closed subgroup of the profinite group A and K is a closed subgroup of the profinite group B. It is *proper* if the continuous homomorphisms $A \to A \coprod_{[H,K]} B$ and $B \to A \coprod_{[H,K]} B$ are injective. We give an example of non proper free profinite product of profinite groups with commuting subgroups. We prove some conditions for its properness. We obtain:

Theorem 1.1. *Let G_1 and G_2 be two profinite groups. Let H_1 be a closed subgroup of the profinite group G_1 and let H_2 be a closed subgroup of the profinite group G_2. Let $G = G_1 \coprod_{[H_1,H_2]} G_2$ be the free profinite product of profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2. Then the following conditions are equivalent:*

a. *The natural homomorphism $\theta : G_1 \underset{[H_1,H_2]}{\star} G_2 \longrightarrow G_1 \coprod_{[H_1,H_2]} G_2$ is injective;*
b. *$G = G_1 \coprod_{[H_1,H_2]} G_2$ is proper;*
c. *There exists an indexing set Λ such that for each $i = 1, 2$, there is a familly $\mathcal{U}_i = \{U_{i\lambda} : \lambda \in \Lambda\}$ of open normal subgroups of finite index of G_i with the following properties:*
   (1) *The families $\mathcal{U}_1$ and $\mathcal{U}_2$ are filtrations and*
   (2) *For each $\lambda \in \Lambda$, $U_{1\lambda}$ and $U_{2\lambda}$ are $[H_1, H_2]$-compatible.*

We then derive this consequence.

Corollary 1.2. *Let G be a profinite group. Let H_1 and H_2 be two closed subgroups of G. Let G_1 and G_2 be two copies of G. Then the free profinite product of profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2 is proper.*

Moreover, profinite completions of abstract groups are profinite. It is very interesting and usual to compute profinite completions of abstract groups. L. Ribes and P. A. Zalesskii in [15] proved how to compute the profinite completions of some free constructions of abstract groups. They showed that $\widehat{A\star B}=\widehat{A}\coprod\widehat{B}$ i.e. the profinite completion of the free product of the abstract groups A and B is the free profinite product of the profinite completions $\widehat{A}$ and $\widehat{B}$ of each of these groups. Under some conditions, they also showed that $\widehat{A\underset{H}{\star}B}=\widehat{A}\coprod_{\widehat{H}}\widehat{B}$ i.e. the profinite completion of the amalgamted free product of the abstract groups A and B over a common subgroup H is the amalgamated profinite product of the profinite completions $\widehat{A}$ and $\widehat{B}$ of each of these groups over the profinite completion $\widehat{H}$ of the common subgroup H. We here compute the profinite completion of some free products of abstract groups with commuting subgroups. We prove:

Theorem 1.3. *Let A and B be abstract groups and let $H\leq A$ and $K\leq B$ such that the following two conditions are satisfied:*

a. *The profinite topology on $A\underset{[H,K]}{\star}B$ induces the profinite topologies on A, B, H, K and $[H,K]$.*
b. *There exist an indexing set Λ and the families $\mathcal{U}_A=\{U_{A\lambda}:\lambda\in\Lambda\}$ and $\mathcal{U}_B=\{U_{B\lambda}:\lambda\in\Lambda\}$ of normal subgroups of finite index of A and B respectively, such that $\mathcal{U}_A$ and $\mathcal{U}_B$ are filtrations and for every $\lambda\in\Lambda$, $U_{A\lambda}$ and $U_{B\lambda}$ are $[H,K]$-compatible.*

Then $\widehat{A\underset{[H,K]}{\star}B}=\widehat{A}\coprod_{[\widehat{H},\widehat{K}]}\widehat{B}$.

Finally, one can easily observe that, under the above conditions $a.$ and $b.$ in theorem 1.3, $\widehat{A}\coprod_{[\widehat{H},\widehat{K}]}\widehat{B}$, the free profinite product of the profinite groups $\widehat{A}$ and $\widehat{B}$ with commuting subgroups $\widehat{H}$ and $\widehat{K}$, is proper.

2. Preliminaries

Through out this work, an abstract group is a group with the usual group structure. If G is a profinite group, we will also denote by G the underlying abstract group without the topological structure.

2.1. **Free products of abstract groups with commuting subgroups.** Let A and B be abstract groups and let $H\leq A$ and $K\leq B$ such that H is isomorphic to K through the isomorphism $\varphi:H\to K$. We denote by $A\underset{\underset{\varphi}{H=K}}{\star}B$ the free product of groups A and B amalgamating subgroups H and K via the isomorphism φ. This group is generated by the disjoint union of all the generators of groups A and B, and defined by all the relators of groups A and B, together with all the relations of the form $\varphi(h)=k$, for all $h\in H$ and $k\in K$. Relatively to the isomorphism φ, subgroups H and K can be identified. Then we write $A\underset{H}{\star}B$ the free product of groups A and B over subgroup H, meaning that H is the common subgroup of groups A and B (indeed, $K=\varphi(H)$, where φ is the known isomorphism). See [8, 9] for more details.

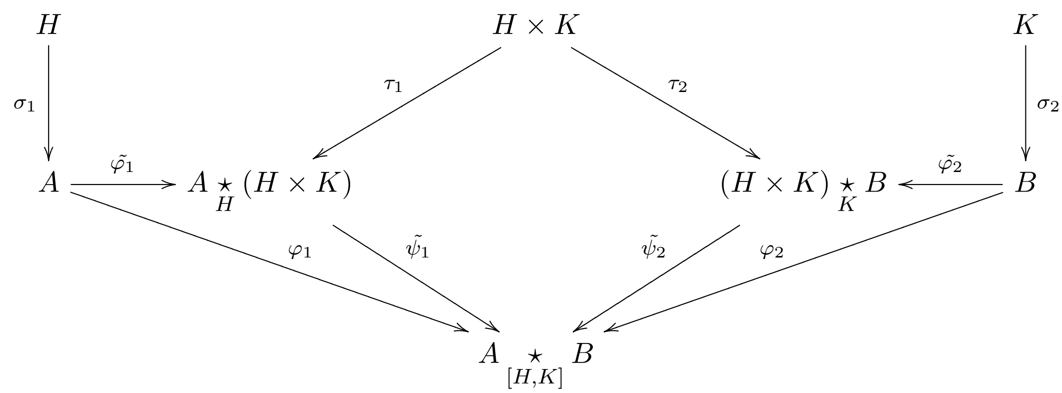
Definition 2.1. *Let H be a subgroup of a group A and let K be a subgroup of a group B. The group $G=(A\star B;[H,K]=1)$ generated by all the generators of groups A and B and defined by all the relators of groups A and B together with all the relations of the form $[h,k]=1$, for all $h\in H$ and*

$k \in K$, is called the free product of groups A and B with commuting subgroups H and K. In order words, G is the free product of groups A and B modulo the normal closure of $([H,K])$, the commutator of the subgroups H and K, in group $A \star B$ i.e. $G = (A \star B)/([H,K])^{(A\star B)}$. We denote this group by $G = A \underset{[H,K]}{\star} B$.

Remark 2.2. *Loginova in [7] studied the residual finiteness of free products of abstract groups with commuting subgroups. She established that the free product $G = A \underset{[H,K]}{\star} B$ of groups A and B with commuting subgroups H and K can be written as*

$$\left(A \underset{H}{\star} (H \times K)\right) \underset{H\times K}{\star} \left((H \times K) \underset{K}{\star} B\right)$$

which we illustrate by the following diagram



where $\sigma_1, \sigma_2, \tau_1, \tau_2, \tilde{\varphi_1}, \tilde{\varphi_2}, \tilde{\psi_1}, \tilde{\psi_2}$ are canonical homomorphisms, and $\varphi_1 = \tilde{\psi_1}\tilde{\varphi_1}$ and $\varphi_2 = \tilde{\psi_2}\tilde{\varphi_2}$.

So G, the free product of groups A and B with commuting subgroups H and K is unique, up to isomorphism. Since it can be written as double amalgamated free product, then groups A and B are canonically embbeded in G. So A and B can be seen as subgroups of group G.

Definition 2.3.

(1) *Let G be an abstract group. G is said to be residually finite if, for any non-identity element g of group G, there is a homomorphism φ from G to a finite group X such that $\varphi(g) \neq 1$ in X.*
(2) *Let G be a group and H a subgroup of G. The subgroup H is said to be finitely separable if for any element g of group G not belonging to the subgroup H, there is a normal subgroup N of finite index of group G, such that $g \notin NH$.*
(3) *Let $G = A \underset{[H,K]}{\star} B$ be the free product of groups A and B with commuting subgroups H and K. Let R and S be normal subgroups of finite index in groups A and B respectively. The subgroups R and S are $[H,K]$-compatible if subgroups $R \cap H$ and $S \cap K$ commute i.e. $[R \cap H, S \cap K] = 1$.*
(4) *Let G be a group and H a subgroup of G. A family $(R_i)_{i\in I}$ of subgroups of group G is called a filtration (respectively a H-filtration) of group G if $\bigcap_{i\in I} R_i = 1$ (respectively $\bigcap_{i\in I} R_i = 1$ and $\bigcap_{i\in I} HR_i = H$).*

2.2. **Free profinite products of profinite groups with commuting subgroups.**

Definition 2.4. *Let H_1 be a closed subgroup of a profinite group G_1 and let H_2 be a closed subgroup of a profinite group G_2. Let $\sigma : H_1 \to G_1$ and $\tau : H_2 \to G_2$ be the inclusion maps. The free profinite product of the profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2 is a family $(G, \varphi_1, \varphi_2)$ where G is a profinite group and $\varphi_1 : G_1 \to G$, $\varphi_2 : G_2 \to G$ are continuous homomorphisms satisfying:*

(1) $[\varphi_1\sigma(H_1), \varphi_2\tau(H_2)] = 1$ *and*

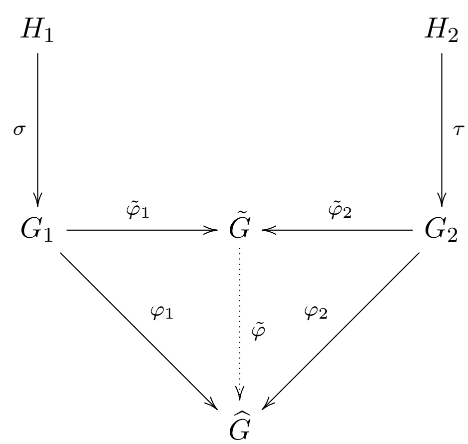
(2) *If G' is a profinite group with continuous homomorphims $\psi_1 : G_1 \to G'$ and $\psi_2 : G_2 \to G'$ such that $[\psi_1\sigma(H_1), \psi_2\tau(H_2)] = 1$, then there exists a unique continuous homomorphism $\psi : G \to G'$ such that $\psi\varphi_1 = \psi_1$ and $\psi\varphi_2 = \psi_2$.*

Remark 2.5. *Since a profinite group is a projective limit of a projective system of finite groups, it is enough to consider G' finite to check the second part of the previous definition.*

A concrete free profinite product of profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2 can be constructed as follow:

Let H_1 be a closed subgroup of a profinite group G_1 and H_2 be a closed subgroup of a profinite group G_2. Let $\sigma : H_1 \to G_1$ and $\tau : H_2 \to G_2$ be continuous monomorphisms. Then one can construct the abstract free product $\tilde{G}$ of abstract groups G_1 and G_2 with commuting subgroups H_1 and H_2 i.e. $\tilde{G} = G_1 \underset{[H_1,H_2]}{\star} G_2$. We have the inclusions $\tilde{\varphi}_i : G_i \to \tilde{G}$, for every $i = 1, 2$. Now any G_i can be identified to its image in the group $\tilde{G}$. Let $\mathcal{N} = \{N \lhd_f \tilde{G} : N \cap G_i \text{ is open in } G_i, i = 1, 2\}$.

If $N_1, N_2 \in \mathcal{N}$, and $N_1 \subseteq N_2$, then a natural epimorphism $\tilde{G}/N_1 \to \tilde{G}/N_2$ can be defined. These maps make the system $\{\tilde{G}/N, N \in \mathcal{N}\}$ projective. Let now $\widehat{G} = \varprojlim_{N\in\mathcal{N}} \tilde{G}/N$ be the profinite completion of the abstract group $\tilde{G}$. Let $\tilde{\varphi} : \tilde{G} \to \widehat{G}$ be the canonical homomorphism. Then for any $i = 1, 2$ we have $\varphi_i = \tilde{\varphi}\tilde{\varphi}_i : G_i \to \widehat{G}$ is a homomorphism. So, the family $(\widehat{G}, \varphi_1, \varphi_2)$ is the free profinite product of the profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2 as we illustrate by the following diagram:



Indeed, we have $[\tilde{\varphi}_1\sigma(H_1), \tilde{\varphi}_2\tau(H_2)] = 1$ from the construction of $\tilde{G}$ and since $\tilde{\varphi}$ is a group homomorphism, then $[\tilde{\varphi}\tilde{\varphi}_1\sigma(H_1), \tilde{\varphi}\tilde{\varphi}_2\tau(H_2)] = 1$. Thus $[\varphi_1\sigma(H_1), \varphi_2\tau(H_2)] = 1$.

Let now G' be a finite group. Let $\psi_1 : G_1 \to G'$ and $\psi_2 : G_2 \to G'$ be continuous homomorphisms such that $[\psi_1\sigma(H_1), \psi_2\tau(H_2)] = 1$. By the universal property of the free product of the abstract groups G_1 and G_2 with commuting subgroups H_1 and H_2, there exists a unique group homomorphism $\tilde{\psi} : \tilde{G} \to G'$ satisfying $\psi_i = \tilde{\psi}\tilde{\varphi}_i$, $i = 1, 2$. We have $(\tilde{\varphi}_i)^{-1}(ker\tilde{\psi}) = ker\psi_i$, $i = 1, 2$. Since G' is haussdorff, then $\{1_{G'}\}$ is closed. Moreover G' is compact; thus $\{1_{G'}\}$, as closed subgroup of finite index, is open. So, for $i = 1, 2$ $ker\psi_i = (\psi_i)^{-1}(\{1_{G'}\})$ is open i.e. $(\tilde{\varphi}_i)^{-1}(ker\tilde{\psi})$ is open in G_i. Thus $ker\tilde{\psi} \in \mathcal{N}$. Let U be an open normal subgroup of finite index of G'. Then U is an open neighbourhood of $\{1_{G'}\}$, and we trivially have that the image of $ker\tilde{\psi}$ by $\tilde{\psi}$ is contained in U. So $\tilde{\psi}$ is continuous, since it is continuous on $\{1_{\tilde{G}}\}$. Then, by the definition of $\widehat{G}$, there is a continuous homomorphism $\psi : \widehat{G} \to G'$ satisfying $\tilde{\psi} = \psi\tilde{\varphi}$. Hence, we have $\psi\varphi_1 = \psi\tilde{\varphi}\tilde{\varphi}_1 = \tilde{\psi}\tilde{\varphi}_1 = \psi_1$. Similarly, we obtain $\psi\varphi_2 = \psi\tilde{\varphi}\tilde{\varphi}_2 = \tilde{\psi}\tilde{\varphi}_2 = \psi_2$. Since $\widehat{G}$ is the profinite completion of the abstract group $\tilde{G}$ which is generated by groups G_1 and G_2, then $\widehat{G} = \overline{\langle \varphi_1(G_1), \varphi_2(G_2) \rangle}$. Consequently, ψ is unique. Now, $(\widehat{G}, \varphi_1, \varphi_2)$ is the free profinite product of the profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2.

Proposition 2.6. *The free profinite product of profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2 is unique up to a continuous isomorphism.*

So, $G = G_1 \coprod_{[H_1,H_2]} G_2$ will denote the free profinite product of profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2.

In the definition of the free profinite product of profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2, it can happen that the continuous homomorphisms φ_i, $i = 1, 2$ are not injective. We then remind that, when these homomorphisms are injective, then the free profinite product of profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2 is proper. The following result gives an easier characterization of a proper free profinite product of profinite groups with commuting subgroups.

Proposition 2.7. *Let H_1 be a closed subgroup of a profinite group G_1 and H_2 be a closed subgroup of a profinite group G_2. Let $\sigma : H_1 \to G_1$ and $\tau : H_2 \to G_2$ be the inclusions maps. Put $\tilde{G} = G_1 \underset{[H_1,H_2]}{\star} G_2$ the free product of the abstract groups G_1 and G_2 with commuting subgroups H_1 and H_2, and for every $i = 1, 2$ $\tilde{\varphi}_i : G_i \to \tilde{G}$ the canonical inclusion. Let $G = G_1 \coprod_{[H_1,H_2]} G_2$ be the free profinite product of the profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2, and for every $i = 1, 2$, let $\varphi_i : G_i \to G$ the canonical continuous homomorphisms. Consider $\widetilde{\varphi} : \tilde{G} \to G$ the homomorphism such that for every $i = 1, 2$ $\varphi_i = \widetilde{\varphi}\widetilde{\varphi}_i$ and let $P = ker\widetilde{\varphi}$. Then the free profinite product of the profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2 is proper if and only if $G_i \cap P = 1$, for every $i = 1, 2$.*

3. Proofs of theorem 1.1 and corollary 1.2

3.1. **Proof of theorem 1.1.** **a**. ⇒ **b**. Assume that θ is injective. Then $ker\theta \cap G_i = 1$, and the result follows from proposition 2.7.

b. ⇒ **c**. The canonical homomorphisms $\varphi_i : G_i \to G_1 \coprod_{[H_1,H_2]} G_2$ $(i = 1, 2)$ are injective. So G_i can be considered as subgroups of $G_1 \coprod_{[H_1,H_2]} G_2$. Since G_i are profinite groups, let $\mathcal{U}_i = \{U_{i\lambda} : \lambda \in \Lambda_i\}$ be families of open normal subgroups of finite index of G_i $(i = 1, 2)$. Then $\mathcal{U}_i$ is a filtration. For any $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, we have $H_1 \cap U_{1\lambda} \leq H_1$, $H_2 \cap U_{2\lambda} \leq H_2$, and then we obtain $[H_1 \cap U_{1\lambda}, H_2 \cap U_{2\lambda}] = 1$ since $[H_1, H_2] = 1$. Now one can choose Λ such that for any $\lambda \in \Lambda$ there exist $U_{1\lambda} \in \mathcal{U}_1$ and $U_{2\lambda} \in \mathcal{U}_2$ which are $[H_1, H_2]$-compatible.

c. $\Rightarrow$ **a**. Assume that the collections $\mathcal{U}_1$ and $\mathcal{U}_2$ are filtered from below: indeed, if it is not, replace $\mathcal{U}_i$ $(i = 1, 2)$ by the collection of all finite intersections of its elements. It follows from Proposition 2.1.4. in [15] that

$$\bigcap_{\lambda\in\Lambda} H_1U_{1\lambda} = H_1 \quad \text{and} \quad \bigcap_{\lambda\in\Lambda} H_2U_{2\lambda} = H_2 \tag{3.1}$$

So for $i = 1, 2$, the families $\mathcal{U}_i$ are H_i-filtrations.

Let $1 \neq g \in G = G_1 \underset{[H_1,H_2]}{\star} G_2$. We need to show that $\theta(g) \neq 1$. Recall that $G = M \underset{H_1\times H_2}{\star} N$ where $M = G_1 \underset{H_1}{\star} (H_1 \times H_2)$ and $N = G_2 \underset{H_2}{\star} (H_1 \times H_2)$.

Let $g = g_1g_2 \cdots g_n$ be the reduced form of g with respect to the decomposition $M \underset{H_1\times H_2}{\star} N$ of G.

Case 1: Consider $n = 1$; i.e. $g \in M$ or $g \in N$.

If $g \in M$, then let $g = x_1x_2 \cdots x_m$ be the reduced form of g with respect to the decomposition $M = G_1 \underset{H_1}{\star} (H_1 \times H_2)$. Hence if $m = 1$, then $g \in G_1$ or $g \in H_1 \times H_2$.

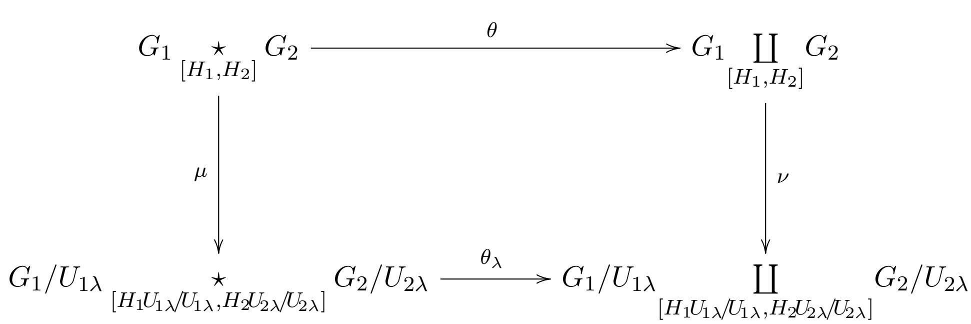
- If $g \in G_1$, and $g \notin (H_1 \times H_2)$, then $g \notin H_1$. From equations (3.1), there exists some $\lambda \in \Lambda$ and $U_{1\lambda} \in \mathcal{U}_1$ such that $g \notin H_1U_{1\lambda}$. We have therefore

$$[H_1 \cap U_{1\lambda}, H_2 \cap U_{2\lambda}] = 1. \tag{3.2}$$

On the other hand, since $H_1U_{1\lambda}/U_{1\lambda} \cong H_1/H_1 \cap U_{1\lambda}$ and $H_2U_{2\lambda}/U_{2\lambda} \cong H_2/H_2 \cap U_{2\lambda}$, then we have from (3.2) that

$$[H_1U_{1\lambda}/U_{1\lambda}, H_2U_{2\lambda}/U_{2\lambda}] = 1.$$

Now consider the following commutative diagram,



where μ and ν are induced by the canonical epimorphisms $G_i \to G_i/U_{i\lambda}$, $(i = 1, 2)$, respectively. By the choice of $U_{1\lambda}$ and $U_{2\lambda}$, one has $\mu(g) \neq 1$. Since $G_1/U_{1\lambda}$ and $G_2/U_{2\lambda}$ are finite, then $G_1/U_{1\lambda} \underset{[H_1U_{1\lambda}/U_{1\lambda}, H_2U_{2\lambda}/U_{2\lambda}]}{\coprod} G_2/U_{2\lambda}$ is residually finite. So $\theta_\lambda(\mu(g)) \neq 1$ i.e. $\nu(\theta(g)) \neq 1$. Consequently $\theta(g) \neq 1$.

- Assume that $g \in H_1 \times H_2$, i.e $g = h_1h_2$, where $h_1 \in H_1$ and $h_2 \in H_2$. Since g is a nonidentity element in $H_1 \times H_2$, at least one of the element h_1 and h_2 differs from 1. Without lost of generality let $h_1 \neq 1$. Since $\mathcal{U}_1$ is a filtration, then there exists $\lambda \in \Lambda$ such that $h_1 \notin U_{1\lambda}$. Hence $1 \neq h_1 \notin U_{1\lambda}$ in the residually finite group $G_1/U_{1\lambda} \coprod_{[H_1U_{1\lambda}/U_{1\lambda}, H_2U_{2\lambda}/U_{2\lambda}]} G_2/U_{2\lambda}$. By a similar method as above, $\theta(g) \neq 1$.
- If $m > 1$, then the elements $x_1, x_2, \ldots, x_m$ belong alternatively to one of the subgroups G_1 and $H_1 \times H_2$ but do not belong to H_1. Let for example $x_1, x_3, \ldots \in G_1 \setminus H_1$ and $x_2, x_4, \ldots \in (H_1 \times H_2) \setminus H_1$. Then, from (3.1), there exists some $\lambda' \in \Lambda$ and $U_{1\lambda'} \in \mathcal{U}_1$ such that $x_1, x_3, \ldots \notin H_1U_{1\lambda'}$. By the H_2-projection $p : H_1 \times H_2 \to H_2$, the images by p of elements $x_2, x_4, \ldots$ differ from 1. Consequently, elements $x_2, x_4, \ldots$ differ from 1 in H_2. Since family $\mathcal{U}_2$ is a filtration, there exists some $\lambda'' \in \Lambda$ such that $x_2, x_4, \ldots \notin U_{2\lambda''}$. Now the collections $\mathcal{U}_1$ and $\mathcal{U}_2$ are filtered from below; so there exists $\lambda \in \Lambda$ such that $x_1, x_3, \ldots \notin H_1U_{1\lambda}$ and $x_2, x_4, \ldots \notin U_{2\lambda}$. Moreover we have $[H_1 \cap U_{1\lambda}, H_2 \cap U_{2\lambda}] = 1$. So $\mu(g) = x_1U_{1\lambda}x_2U_{2\lambda} \cdots \neq 1$ in the residually finite group $G_1/U_{1\lambda} \coprod_{[H_1U_{1\lambda}/U_{1\lambda}, H_2U_{2\lambda}/U_{2\lambda}]} G_2/U_{2\lambda}$. Similarly, $\theta(g) \neq 1$.

A similar method can be used if $g \in N$.

Case 2: Consider $n > 1$. Then the elements $g_1, g_2, \ldots, g_n$ belong alternatively to one of the subgroups M and N but not in $H_1 \times H_2$. If for example $g_1, g_3, \ldots \in M \setminus (H_1 \times H_2)$ and $g_2, g_4, \ldots \in N \setminus (H_1 \times H_2)$, then $g_1, g_3, \ldots \notin H_1$ and $g_2, g_4, \ldots \notin H_2$. Since families $\mathcal{U}_1$ and $\mathcal{U}_2$ are H_1-filtration and H_2-filtration respectively, then there exist $\lambda \in \Lambda$, $U_{1\lambda} \in \mathcal{U}_1$ and $U_{2\lambda} \in \mathcal{U}_2$ such that $g_1, g_3, \ldots \notin H_1U_{1\lambda}$ and $g_2, g_4, \ldots \notin H_2U_{2\lambda}$. Now $\mu(g) = g_1U_{1\lambda}g_2U_{2\lambda} \cdots \neq 1$ in the residually finite group $G_1/U_{1\lambda} \coprod_{[H_1U_{1\lambda}/U_{1\lambda}, H_2U_{2\lambda}/U_{2\lambda}]} G_2/U_{2\lambda}$. As above, $\theta(g) \neq 1$.

3.2. **Proof of Corollary 1.2.** Let G be a profinite group, let H_1 and H_2 two closed subgroups of G. Let G_1 and G_2 be two copies of G. Consider $\sigma : H_1 \to G_1$ and $\tau : H_2 \to G_2$ some continuous monomorphisms. Denote by $\tilde{G} = G_1 \underset{[H_1,H_2]}{\star} G_2$ the free product of abstract groups G_1 and G_2 with commuting subgroups H_1 and H_2. Consider $i_1 : G_1 \to \tilde{G}$ and $i_2 : G_2 \to \tilde{G}$ the canonical injections. We have

$$[i_1\sigma(H_1), i_2\tau(H_2)] = 1. \tag{3.3}$$

Take $\{U_\lambda : \lambda \in \Lambda\}$ the collection of normal subgroups of finite index of G_1 (since G_1 is a profinite group). Let $\varphi : G_1 \to G_2$ be an isomorphism of topological groups and let $\{\varphi(U_\lambda) : \lambda \in \Lambda\}$ the collection of normal subgroups of finite index of G_2. Then from (3.3), $[i_1\sigma(U_\lambda \cap H_1), i_2\tau(\varphi(U_\lambda) \cap H_2)] = 1$. Now the families $\{U_\lambda : \lambda \in \Lambda\}$ and $\{\varphi(U_\lambda) : \lambda \in \Lambda\}$ are filtrations of groups G_1 and G_2 respectively, and for any $\lambda \in \Lambda$, subgroups $U_{1\lambda}$ and $\varphi(U_{1\lambda})$ are $[H_1, H_2]$-compatible. The result then follows from theorem 1.1.

We now give an example of non proper free profinite product of profinite groups with commuting subgroups.

Example.

Let A and B be two finite nontrivial groups. Consider $T_1 = A^{\mathbb{Z}}$ and $T_2 = B^{\mathbb{Z}}$, the products of infinite numbers of copies of A and B respectively. Define the automorphisms σ_1 and σ_1' of T_1, σ_2 and σ_2' of T_2 by $\sigma_1(a_n) = (a_{-n})$, $\sigma_1'(a_n) = (a_{1-n})$, $\sigma_2(b_n) = (b_{-n})$ and $\sigma_2'(b_n) = (b_{1-n})$. These automorphisms have order 2, and hence they define continuous actions of $\mathbb{Z}/\mathbb{Z}_2$ on T_1 and of $\mathbb{Z}/\mathbb{Z}_2$ on T_2 respectively. Now consider the two automorphisms φ_1 and φ_2 of $T_1 \times T_2$ defined by $\varphi_1((a_n),(b_n)) = (\sigma_1(a_n), \sigma_2(b_n))$ and $\varphi_2((a_n),(b_n)) = (\sigma_1'(a_n), \sigma_2'(b_n))$. These automorphisms have order 2; they define continuous actions of $\mathbb{Z}/\mathbb{Z}_2$ on $T_1 \times T_2$. Consequently the semidirect products

$$G_1 = (T_1 \times T_2) \times_{\varphi_1} \mathbb{Z}/2\mathbb{Z} \;\text{ and }\; G_2 = (T_1 \times T_2) \times_{\varphi_2} \mathbb{Z}/2\mathbb{Z}$$

are profinite groups. Following Serre through his example in [13], we can say that the free profinite amalgamated product $G_1 \coprod_{T_1\times T_2} G_2$ of the profinite groups G_1 and G_2 over $T_1 \times T_2$ is not proper. Thus by theorem 9.2.4 in [15], the canonical homomorphism $\theta : G_1 \underset{T_1\times T_2}{\star} G_2 \to G_1 \coprod_{T_1\times T_2} G_2$ is not injective. Now consider the inclusion

$$\mu : G_1 \underset{T_1\times T_2}{\star} G_2 \to G_1 \underset{[T_1,T_2]}{\star} G_2,$$

the homomorphism

$$\nu : G_1 \coprod_{T_1\times T_2} G_2 \to G_1 \coprod_{[T_1,T_2]} G_2$$

and the canonical homomorphism

$$\omega : G_1 \underset{[T_1,T_2]}{\star} G_2 \to G_1 \coprod_{[T_1,T_2]} G_2$$

We obtain the following commutative diagram

$$\begin{array}{ccc} G_1 \underset{T_1\times T_2}{\star} G_2 & \xrightarrow{\ \theta\ } & G_1 \coprod_{T_1\times T_2} G_2 \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle \nu} \\ G_1 \underset{[T_1,T_2]}{\star} G_2 & \xrightarrow{\ \omega\ } & G_1 \coprod_{[T_1,T_2]} G_2 \end{array}$$

It is clear that the homomorphism ω is not injective, since θ is not injective. Consequently $G_1 \coprod_{[T_1,T_2]} G_2$ cannot be proper.

4. **Profinite completions of free products of groups with commuting subgroups**

4.1. **Profinite completion of abstract groups.** Let G be an abstract group and let $\mathcal{N}$ be the collection of all normal subgroups of finite index of G. Recall that the profinite completion $\widehat{G}$ of group G is the inverse limit of the projective system $(G/N)_{N\in\mathcal{N}}$ of the groups G/N i.e. $\widehat{G} = \varprojlim_{N\in\mathcal{N}} G/N$. So, $\widehat{G}$ is a profinite group.

The following characterization is obvious:

Proposition 4.1. *Let G be an abstract group. The profinite completion of the group G is a profinite group $\widehat{G}$ together with a continuous homomorphism $\theta : G \to \widehat{G}$ which is onto on a dense subgroup of $\widehat{G}$, where G is endowed with the profinite topology, satisfying the following universal property: for any profinite group G' and any continuous homomorphism $\varphi : G \to G'$, there exists a unique continuous homomorphism $\widehat{\varphi} : \widehat{G} \to G'$ such that $\widehat{\varphi}\theta = \varphi$.*

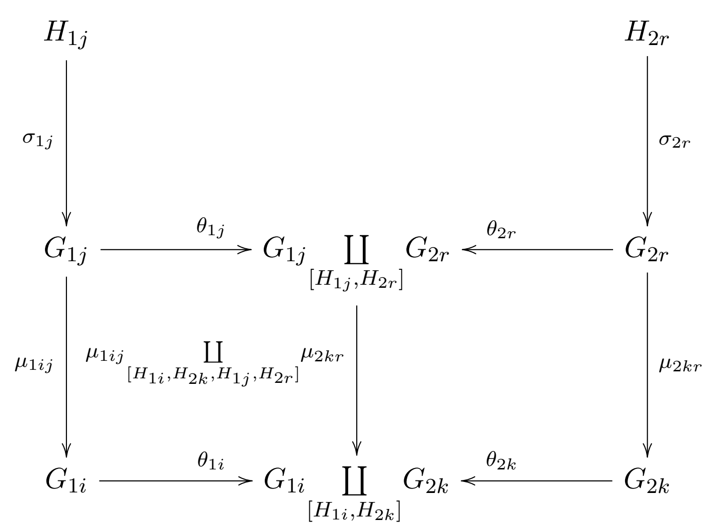
Remark 4.2. *To check the universal property, it suffices to consider G' a finite group.*

The profinite completion of an abstract group is unique, up to a topological groups isomorphism. We write $\widehat{G}$ the profinite completion of the abstract group G when there is no confusion on θ.

4.2. **Lemma.** Let (G_{1i}, μ_{1ij}, I_1) and (G_{2k}, μ_{2kr}, I_2) be projective systems of profinite groups over the directed sets I_1 and I_2 respectively. For any $(i,k) \in I_1 \times I_2$, let H_{1i} be a closed subgroup of the profinite group G_{1i} and H_{2k} a closed subgroup of the profinite group G_{2k}. Consider $\sigma_{1i} : H_{1i} \to G_{1i}$ and $\sigma_{2k} : H_{2k} \to G_{2k}$ the canonical continuous embeddings. Denote by $G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}$ the free profinite product of the profinite groups G_{1i} and G_{2k} with commuting subgroups H_{1i} and H_{2k}, and let $\theta_{1i} : G_{1i} \to G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}$ and
$\theta_{2k} : G_{2k} \to G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}$ be the canonical continuous homomorphisms. Let $I_1 \times I_2$ be endowed with the lexicographical order $\leq$ defined by: for all $(i,k),(j,l) \in I_1 \times I_2$, $(i,k) \leq (j,l)$ if and only if $i \leq j$ and $k \leq l$. Now given $(i,k) \leq (j,r)$ in $I_1 \times I_2$, then

$$[\theta_{1i}\mu_{1ij}\sigma_{1j}(H_{1j}), \theta_{2k}\mu_{2kr}\sigma_{2r}(H_{2r})] = 1.$$

Put $\mu_{1ij} \coprod_{[H_{1i},H_{2k},H_{1j},H_{2r}]} \mu_{2kr} : G_{1j} \coprod_{[H_{1j},H_{2r}]} G_{2r} \to G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}$ the unique continuous homomorphism such that the following diagram is commutative



Then we have:

Lemma 4.3.

(1) *$I_1 \times I_2$ is a directed set and*

$$\left(\left(G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}\right), \left(\mu_{1ij} \coprod_{[H_{1i},H_{2k},H_{1j},H_{2r}]} \mu_{2kr}\right), I_1 \times I_2\right)$$

is a projective system over $I_1 \times I_2$, and

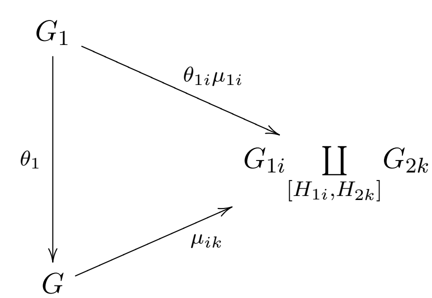
(2) $$\left(\varprojlim_{I_1} G_{1i}\right) \coprod_{\left[\left(\varprojlim_{I_1} H_{1i}\right),\left(\varprojlim_{I_2} H_{2k}\right)\right]} \left(\varprojlim_{I_2} G_{2k}\right) \cong \varprojlim_{I_1\times I_2} \left(G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}\right).$$

Proof. (1) is obvious.

(2) Let $G_1 = \varprojlim_{I_1} G_{1i}$, $G_2 = \varprojlim_{I_2} G_{2k}$, $H_1 = \varprojlim_{I_1} H_{1i}$, $H_2 = \varprojlim_{I_2} H_{2k}$ and $G = \varprojlim_{I_1\times I_2} \left(G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}\right)$. For any $(i,k) \in I_1 \times I_2$, consider the following projections

$$\mu_{1i} : G_1 \to G_{1i},\ \mu_{2k} : G_2 \to G_{2k} \text{ and } \mu_{ik} : G \to G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}.$$

Then the maps $\theta_{1i}\mu_{1i} : G_1 \to G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}$ are compatible and they induce a continuous homomorphism $\theta_1 = \varprojlim_{I_1} \theta_{1i} : G_1 \to G$ such that for any $(i,k) \in I_1 \times I_2$, the following diagram is commutative



Thus $\theta_{1i}\mu_{1i} = \mu_{ik}\theta_1$.

By the same way, we obtain a continuous homomorphism $\theta_2 = \varprojlim_{I_2} \theta_{2k} : G_2 \to G$ such that for any $(i,k) \in I_1 \times I_2$, we have $\theta_{2k}\mu_{2k} = \mu_{ik}\theta_2$.

Now the morphisms $\sigma_{1i}\mu_{1i} : H_1 \to H_{1i} \to G_{1i}$ are also compatible and they induce the continuous homomorphism $\sigma_1 : H_1 \to G_1$ such that, for any $i \in I_1$, $\sigma_{1i}\mu_{1i} = \mu_{1i}\sigma_1$.

We see that σ_1 is injective. Indeed, for all $x, y \in H_1$ and $i \in I_1$, if $\sigma_1(x) = \sigma_1(y)$, then $\mu_{1i}\sigma_1(x) = \mu_{1i}\sigma_1(y)$. Hence $\sigma_{1i}\mu_{1i}(x) = \sigma_{1i}\mu_{1i}(y)$. Since the σ_{1i} are embeddings, we have $\mu_{1i}(x) = \mu_{1i}(y)$ for any $i \in I_1$, where $\mu_{1i}(x)$ is the i-th component of x. Consequently $x = y$. Similarly we can define another continuous and injective homomorphism $\sigma_2 : H_2 \to G_2$ such that $\sigma_{2k}\mu_{2k} = \mu_{2k}\sigma_2$. Then using the definitions of θ_1, θ_2, σ_1 and σ_2, we obtain the following equalities

$$\mu_{ik}\theta_1\sigma_1 = \theta_{1i}\sigma_{1i}\mu_{1i} \text{ and } \mu_{ik}\theta_2\sigma_2 = \theta_{2k}\sigma_{2k}\mu_{2k}, \text{ for all } (i,k) \in I_1 \times I_2 \qquad (\star)$$

Now to prove the lemma, it's enough to prove that (G, θ_1, θ_2) is the free profinite product of the profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2.

First we have $[\theta_1\sigma_1(H_1), \theta_2\sigma_2(H_2)] = 1$. Indeed, for any $(h_1, h_2) \in H_1 \times H_2$ and $(i, k) \in I_1 \times I_2$, since $\mu_{1i}(h_1) \in H_{1i}$ and $\mu_{2k}(h_2) \in H_{2k}$ and since $\left(G_{1i} \coprod_{[H_{1i},H_{2k}]} G_{2k}, \theta_{1i}, \theta_{2k}\right)$ is the free profinite product of the profinite groups G_{1i} and G_{2k} with commuting subgroups H_{1i} and H_{2k}, then using equalities $(\star)$, we have

$$[\mu_{ik}\theta_1\sigma_1(h_1), \mu_{ik}\theta_2\sigma_2(h_2)] = 1, \text{ i.e. } [\theta_{1i}\sigma_{1i}\mu_{1i}(h_1), \theta_{2k}\sigma_{2k}\mu_{2k}(h_2)] = 1.$$

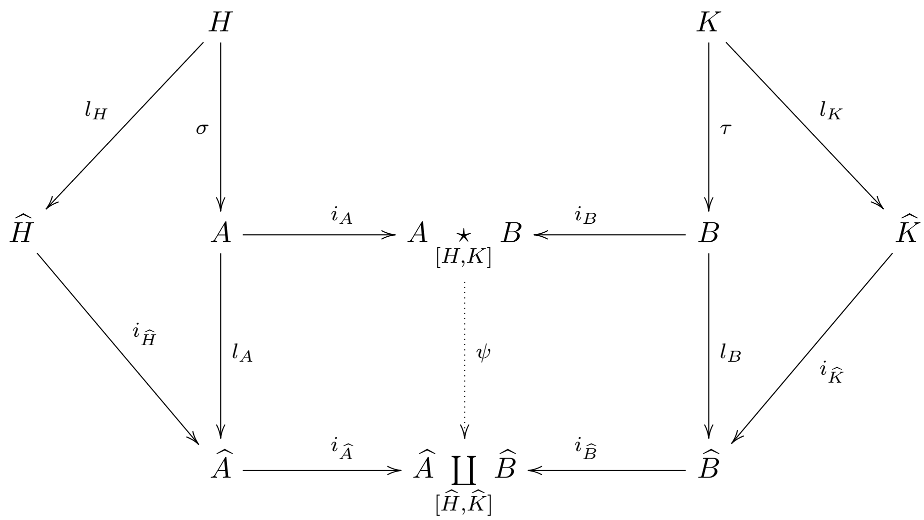
$$\text{thus } [\theta_1\sigma_1(H_1), \theta_2\sigma_2(H_2)] = 1.$$

Finally, consider a finite and discrete group G'. Let $\varphi_1 : G_1 \to G'$ and $\varphi_2 : G_2 \to G'$ be continuous homomorphisms such that $[\varphi_1\sigma_1(H_1), \varphi_2\sigma_2(H_2)] = 1$. Since $G_1 = \varprojlim_{I_1} G_{1i}$, from lemma 1.1.16 in [15], there exist $j \in I_1$ and a continuous homomorphism $\varphi_{1j} : G_{1j} \to G'$ such that $\varphi_{1j}\mu_{1j} = \varphi_1$. Similarly since $G_2 = \varprojlim_{I_2} G_{2k}$, there exist $r \in I_2$ and a continuous homomorphism $\varphi_{2r} : G_{2r} \to G'$ such that $\varphi_{2r}\mu_{2r} = \varphi_2$. Recall that $[\varphi_1\sigma_1(H_1), \varphi_2\sigma_2(H_2)] = 1$. So $[\varphi_{1j}\mu_{1j}\sigma_1(H_1), \varphi_{2r}\mu_{2r}\sigma_2(H_2)] = 1$. Since $\mu_{1j}\sigma_1 = \sigma_{1j}\mu_{1j}$ and $\mu_{2r}\sigma_2 = \sigma_{2r}\mu_{2r}$, we obtain $[\varphi_{1j}\sigma_{1j}\mu_{1j}(H_1), \varphi_{2r}\sigma_{2r}\mu_{2r}(H_2)] = 1$. Recall also that $\mu_{1j}(H_1) = H_{1j}$ and $\mu_{2r}(H_2) = H_{2r}$. Then $[\varphi_{1j}\sigma_{1j}(H_{1j}), \varphi_{2r}\sigma_{2r}(H_{2r})] = 1$. By the definition of $G_{1j} \coprod_{[H_{1j},H_{2r}]} G_{2r}$, there exists a continuous homomorphism $\varphi_{jr} : G_{1j} \coprod_{[H_{1j},H_{2r}]} G_{2r} \to G'$ such that $\varphi_{1j} = \varphi_{jr}\theta_{1j}$ and $\varphi_{2r} = \varphi_{jr}\theta_{2r}$. Let $\varphi = \varphi_{jr}\mu_{jr} : G \to G'$. We have $\varphi\theta_1 = \varphi_{jr}\mu_{jr}\theta_1 = \varphi_{jr}\theta_{1j}\mu_{1j} = \varphi_{1j}\mu_{1j} = \varphi_1$. Similarly we have $\varphi\theta_2 = \varphi_2$. Consequently (G, θ_1, θ_2) is the free profinite product of the profinite groups G_1 and G_2 with commuting subgroups H_1 and H_2.

□

4.3. **Proof of Theorem 1.3.** We suppose conditions $a.$ and $b.$ are satisfied. We need to prove that $\widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B}$ is the profinite completion of the group $A \coprod_{[H,K]} B$. We use proposition 4.1.

So, by the definition of $\widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B}$, put $i_{\widehat{H}} : \widehat{H} \to \widehat{A}$, $i_{\widehat{K}} : \widehat{K} \to \widehat{B}$, $i_{\widehat{A}} : \widehat{A} \to \widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B}$ and $i_{\widehat{B}} : \widehat{B} \to \widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B}$ the canonical continuous homomorphisms. Then $[i_{\widehat{A}}i_{\widehat{H}}(\widehat{H}), i_{\widehat{B}}i_{\widehat{K}}(\widehat{K})] = 1$. Let groups A, B, H, K, $[H, K]$ and $A \underset{[H,K]}{\star} B$ be endowed with the profinite topology. Define the following canonical continuous homomorphisms: $\sigma : H \to A$, $\tau : K \to B$, $i_A : A \to A \coprod_{[H,K]} B$ and $i_B : B \to A \coprod_{[H,K]} B$. We then have $[i_A\sigma(H), i_B\tau(K)] = 1$. Now consider the following canonical continuous homomorphisms $l_H : H \to \widehat{H}$, $l_K : K \to \widehat{K}$, $l_A : A \to \widehat{A}$ and $l_B : B \to \widehat{B}$. Since $\widehat{H}$ and $\widehat{K}$ are the profinite completions of H and K respectively, by the universal property, $i_{\widehat{H}}$ and $i_{\widehat{K}}$ are the unique continuous homomorphisms such that $i_{\widehat{H}}l_H = l_A\sigma$ and $i_{\widehat{K}}l_K = l_B\tau$. Hence $[i_{\widehat{A}}l_A\sigma(H), i_{\widehat{B}}l_B\tau(K)] = [i_{\widehat{A}}i_{\widehat{H}}l_H(H), i_{\widehat{B}}i_{\widehat{K}}l_K(K)] = 1$, since $[i_{\widehat{A}}i_{\widehat{H}}(\widehat{H}), i_{\widehat{B}}i_{\widehat{K}}(\widehat{K})] = 1$, $l_H(H) \leq \widehat{H}$ and $l_K(K) \leq \widehat{K}$. Thus by the definition of $A \underset{[H,K]}{\star} B$, there exists a group homomorphism $\psi : A \underset{[H,K]}{\star} B \to \widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B}$ such that $i_{\widehat{A}} l_A = \psi i_A$ and $i_{\widehat{B}} l_B = \psi i_B$. Consequently the following diagram commutes



Recall that the maps $i_{\widehat{A}}$, l_A, $i_{\widehat{B}}$, l_B, i_A and i_B are continuous. So, the homomorphism ψ is continuous.

1. Let us prove that $\psi\left(A \underset{[H,K]}{\star} B\right)$ is a dense subgroup of $\widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B}$. From the previous lemma 4.3, we have

$$\widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B} \cong \varprojlim_{\mathcal{N}_\mathcal{A} \times \mathcal{N}_\mathcal{B}} \left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k \right)$$

where $\widehat{A} = \varprojlim_{U_i \in \mathcal{N}_\mathcal{A}} A/U_i$, $\widehat{B} = \varprojlim_{V_k \in \mathcal{N}_\mathcal{B}} B/V_k$, $\widehat{H} = \varprojlim_{U_i \in \mathcal{N}_\mathcal{A}} H/(U_i \cap H)$, $\widehat{K} = \varprojlim_{V_k \in \mathcal{N}_\mathcal{B}} K/(V_k \cap K)$ and $\left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k, \psi_{ik,jr}, \leq \right)$ is a projective system of profinite groups, $\mathcal{N}_\mathcal{A}$ and $\mathcal{N}_\mathcal{B}$ are the collection of all normal subgroups of finite index of groups A and B respectively, and the order $\leq$ is defined on $\mathcal{N}_\mathcal{A} \times \mathcal{N}_\mathcal{B}$ by: $(U_i, V_k) \leq (U_j, V_r)$ if and only if $U_j \leq U_i$ in $\mathcal{N}_\mathcal{A}$ and $V_r \leq V_k$ in $\mathcal{N}_\mathcal{B}$ (i, k, j, r are natural numbers), and $\psi_{ik,jr} : A/U_j \coprod_{[HU_j/U_j, KV_r/V_r]} B/V_r \to A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k$ is a continuous homomorphism for $(U_i, V_k) \leq (U_j, V_r)$ in $\mathcal{N}_\mathcal{A} \times \mathcal{N}_\mathcal{B}$.

Now for any $(U_i, V_k) \in \mathcal{N}_\mathcal{A} \times \mathcal{N}_\mathcal{B}$, $A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k$ is the free profinite product of the finite (thus profinite) groups A/U_i and B/V_k with commuting subgroups HU_i/U_i and KV_k/V_k. In the cartesian product

$$P = \prod_{(U_i,V_k) \in \mathcal{N}_\mathcal{A} \times \mathcal{N}_\mathcal{B}} \left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k \right)$$

the open sets are unions of intersections of sets of the form

$$\left(\prod_{(s.t) \in I_0 \times K_0} (S_{i_s,k_t}) \right) \times \left(\prod_{(i,k) \notin I_0 \times K_0} \left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k \right) \right)$$

where S_{i_s,k_t} is an open subset of $A/U_{i_s} \coprod_{[HU_{i_s}/U_{i_s}, KV_{k_t}/V_{k_t}]} B/V_{k_t}$ and, I_0 and K_0 are finite sets of positive integers. Now since $\varprojlim_{\mathcal{N}_{\mathcal{A}}\times\mathcal{N}_{\mathcal{B}}} \left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k \right)$ is a subset of P, then by the induced topology, every open subset in $\varprojlim_{\mathcal{N}_{\mathcal{A}}\times\mathcal{N}_{\mathcal{B}}} \left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k \right)$ has the form

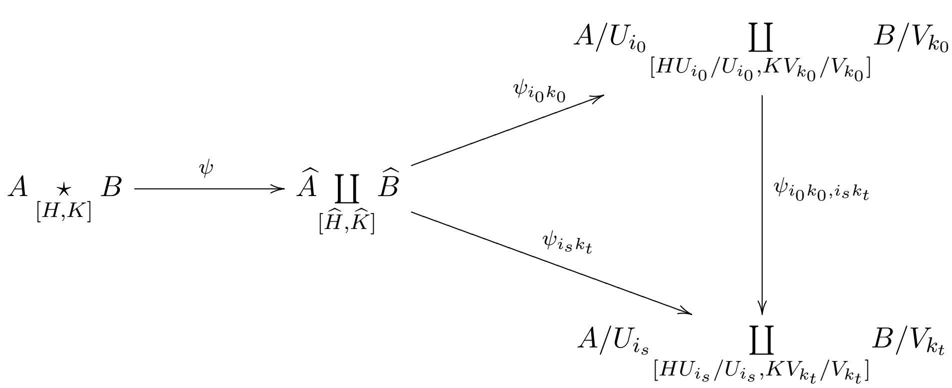
$$\left[\varprojlim_{\mathcal{N}_{\mathcal{A}}\times\mathcal{N}_{\mathcal{B}}} \left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k \right) \right] \bigcap O, \text{ where } O \text{ is an open set in } P.$$

Now take $I_0 = \{i_1, \ldots, i_n\}$ and $K_0 = \{k_1, \ldots, k_m\}$ be finite subsets of $\mathbb{N}$ and let V be a non empty open subset in $\varprojlim_{\mathcal{N}_{\mathcal{A}}\times\mathcal{N}_{\mathcal{B}}} \left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k \right)$. Then V has the form

$$V = \varprojlim_{\mathcal{N}_{\mathcal{A}}\times\mathcal{N}_{\mathcal{B}}} \left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k \right) \bigcap$$

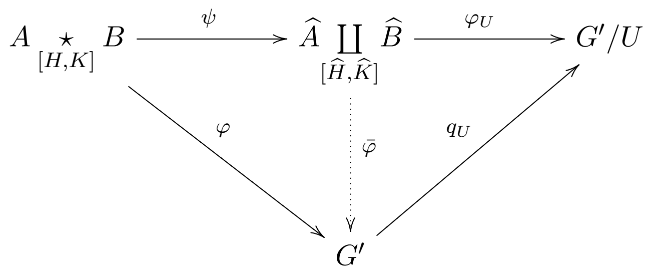
$$\left[\left(\prod_{(s.t)\in\{1,\ldots,n\}\times\{1,\ldots,m\}} (S_{i_s,k_t}) \right) \times \left(\prod_{(i,k)\notin I_0\times K_0} \left(A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k \right) \right) \right]$$

where the S_{i_s,k_t} are finite subsets of $A/U_{i_s} \coprod_{[HU_{i_s}/U_{i_s}, KV_{k_t}/V_{k_t}]} B/V_{k_t}$ with $(s,t) \in \{1,\ldots,n\} \times \{1,\ldots,m\}$. Then $\psi \left(A \underset{[H,K]}{\star} B \right) \bigcap V \neq \emptyset$. Indeed, let $x \in A \underset{[H,K]}{\star} B$. Choose $i_0 \geq i_1, \ldots, i_n$ and $k_0 \geq k_1, \ldots, k_m$ such that $\psi_{i_0k_0}(\psi(x)) = (x_{i_0}, y_{k_0})$ where the $\psi_{ik} : \widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B} \to A/U_i \coprod_{[HU_i/U_i, KV_k/V_k]} B/V_k$ are projections. Then, $\psi(x) \in V$, since the following diagram commutes



2. Let us prove the universal property. Let $\varphi : A \underset{[H,K]}{\star} B \to G'$ be a continuous homomorphism, where G' is a finite and discrete group. Consider $\mathcal{U} = \{U, U \lhd_0 G'\}$, the collection of all normal open subgroups of group G'. For each $U \in \mathcal{U}$, let $N_U = \varphi^{-1}(U)$; then the morphisms $\varphi_U : \widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B} \to A \underset{[H,K]}{\star} B/N_U \to G'/U$ are compatible. They induce a continuous homomorphism

$\bar{\varphi} : \widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B} \to \varprojlim_{U \in \mathcal{U}} G'/U = G'$ such that for any $U \in \mathcal{U}$, we have $\varphi_U = q_U \bar{\varphi}$, and $q_U \varphi = \varphi_U \psi$ as shown on the following commutative diagram



Now $\varphi = \bar{\varphi}\psi$. Indeed, take $x \in A \underset{[H,K]}{\star} B$. Then $q_U \varphi(x) = \varphi_U \psi(x)$. Therefore $q_U \varphi(x) = q_U \bar{\varphi}\psi(x)$ for all $x \in A \underset{[H,K]}{\star} B$ and any $U \in \mathcal{U}$. Thus $\varphi = \bar{\varphi}\psi$. The uniqueness of $\bar{\varphi}$ is obtained from the fact that $\widehat{A} \coprod_{[\widehat{H},\widehat{K}]} \widehat{B} = \overline{\langle i_{\widehat{A}} l_A(A), i_{\widehat{B}} l_B(B) \rangle}$.

References

[1] G. Baumslag, On the residual finiteness of generalised free products of nilpotent groups, *Trans. Amer. Math. Soc.*, **106** (1963) 193–209.

[2] B. Baumslag and M. Tretkoff, Residually finite HNN-extensions, *Comm. Algebra*, **6** (1978) 179–194.

[3] T. Coulbois, *Propriétés de Ribes-Zalesskii, topologie profinie, produit libre et généralisations*, Thèse de Doctorat, Université Paris VII, 2000.

[4] G. Kim and J. McCarron, On amalgamated free products of residually p-finite groups, *J. Algebra*, **162** (1993) 1–11.

[5] G. Kim and J. McCarron, Some residually p-finite one relator groups, *J. Algebra*, **169** (1994) 817–826.

[6] G. Kim and C. Y. Tang, On generalized free products of residually finite p-groups, *J. Algebra*, **201** (1998) 317–327.

[7] E. D. Loginova, Residual finiteness of the free product of two groups with commuting subgroups, translation in *Siberian Math. J.*, **40** (1999) 341–350.

[8] R. C. Lyndon and P. E. Schupp, *Combinatorial Group Theory*, Reprint of the 1977 edition. Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[9] W. Magnus, A. Karass and D. Solitar, *Combinatorial Group Theory*, New York, London, Sydney: John Wiley and Sons, Inc., 1966.

[10] S. Meskin, Nonresidually finite one-relator groups, *Trans. Amer. Math. Soc.*, **164** (1972) 105–114.

[11] D. I. Moldavanskii, p-finite residuality of HNN-extensions, (in Russian), *Viesnik Ivanov. Gos. Univ.*, **3** (2000) 129–140.

[12] E. Raptis and D. Varsos, Residual properties of HNN-extensions with base group an abelian group, *J. Pure Appl. Algebra*, **59** (1989) 285–290.

[13] L. Ribes, Amalgamated products of profinite groups: counterexamples, *Proc. Amer. Math. Soc.*, **37** (1973) 413–416.

[14] L. Ribes, On amalgamated products of profinite groups, *Math. Z.*, **123** (1971) 357–364.

[15] L. Ribes and P. A. Zalesskii, *Profinite Groups*, Springer-Verlag, Berlin, 2010.

[16] M. Shirvani, On residually finite HNN-extensions, *Arch. Math. (Basel)*, **44** (1985) 110–115.

[17] M. Shirvani, A converse to a residual finiteness theorem of G. Baumslag, *Proc. Amer. Math. Soc.*, **104** (1988) 703–706.

[18] J. P. Serre, *Cohomologie Galoisienne*, 3ieme ed., Lecture Notes in Math., no. 5, Springer-Verlag, Berlin and New York, 1965.

[19] D. Tieudjo, Root-class residuality of some free constructions, *JP J. Algebra Number Theory Appl.*, **18** (2010) 125–143.

[20] J. S. Wilson, *Profinite Groups*, Clarendon Press, Oxford,1998.

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