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GENERALIZING QUASINORMALITY

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Abstract. Quasinormal subgroups have been studied for nearly 80 years. In finite groups, questions concerning them invariably reduce to p-groups, and here they have the added interest of being invariant under projectivities, unlike normal subgroups. However, it has been shown recently that certain groups, constructed by Berger and Gross in 1982, of an important universal nature with regard to the existence of core-free quasinormal subgroups generally, have remarkably few such subgroups. Therefore in order to overcome this misfortune, a generalization of the concept of quasinormality will be defined. It could be the beginning of a lengthy undertaking. But some of the initial findings are encouraging, in particular the fact that this larger class of subgroups also remains invariant under projectivities of finite p-groups, thus connecting group and subgroup lattice structures.

1. Introduction

In 1937 Ore introduced the concept of quasinormal subgroups (see [7]):- a subgroup H of a group G is *quasinormal* if $HK = KH$ for all subgroups K of G, i.e. $\langle H, K\rangle = HK$. We write H *qn* G. Sometimes the term *permutable* is used (see [8]). In finite groups, *quasinormal subgroups are always subnormal*. In fact they are always ascendant in infinite groups, indeed in at most $\omega + 1$ steps. (Napolitani and Stonehewer proved this independently many years ago, but neither published the result, possibly because all known examples were ascendant in at most ω steps.) Also simple groups have no proper non-trivial quasinormal subgroups.

In 1962 Itô and Szép ([4]) proved that *if H is a quasinormal subgroup of a finite group G, then H/H_G is nilpotent*. Here H_G is the largest normal subgroup of G lying in H, i.e. the *core* of H in G.

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Rather curiously, even as recently as 1967, the only examples that had appeared were abelian. Then in that year Thompson found *examples in finite p-groups* (p *odd*) *of class* 2. See [10]. Six years later, Maier and Schmid proved that with the same hypotheses, H/H_G *lies in the hypercentre of* G/H_G, improving Itô and Szép's result considerably (see [6]).

The climax seems to have appeared in 1982 when Berger and Gross proved the following ([1]):- *For each prime* p *and integer* n*, there is a finite* p*-group* $G = HX$ *with* H *qn* G*,* H *is core-free, the exponent of* H *is* p^{n-1} *and* X *is cyclic of order* p^n*. Also these groups are universal in the sense that any finite* p*-group with a similar factorization into subgroups with the same properties, embeds in* G*.* However, in 2011 it was shown in [2] that these groups G have remarkably few quasinormal subgroups lying in H. Indeed when p is odd, *they can have exponent only* p*,* p^{n-2} *and* p^{n-1}*.*

This is rather unfortunate, because quasinormal subgroups of finite p-groups are invariant under projectivities, i.e. subgroup lattice isomorphisms. Normal subgroups are not invariant. So we cannot find much about subgroup lattices here. The reason why quasinormal subgroups are invariant in finite p-groups is because a subgroup is quasinormal if and only if it is modular *and* subnormal. (See [8], 5.1.1 Theorem.) Of course modular subgroups are invariant under projectivities. So we attempt to generalize the concept of quasinormality in a useful way.

We keep the product idea. Also we stay with finite p-groups. For, when H *qn* G, then the complexities of the embedding of H in G reduce to the case when G is a p-group, i.e. when H is a modular subgroup. Thus let H be a subgroup of a finite p-group G such that

$$\langle H, K\rangle = HKH$$

for all subgroups K of G. Then in fact H *qn* G. For, assuming $G \neq 1$, there is a non-trivial central element of G of the form hk with $h \in H$ and $k \in K$. So by induction on order, we have

$$\langle H, K\rangle = HK\langle hk\rangle = H\langle hk\rangle K = HK.$$

Of course the same argument applies if we assume that $\langle H, K\rangle = KHK$. Therefore we go one step further. We say that

$$H \text{ is } 4\text{-quasinormal in } G$$

if $\langle H, K\rangle = HKHK$ for all *cyclic* subgroups K of G, and we write

$$H \; qn_4 \; G.$$

When defining quasinormal subgroups, we only need to consider cyclic subgroups K. If we strengthen our definition here to include *all* subgroups K, then we will say that

$$H \text{ is strongly } 4\text{-quasinormal in } G.$$

There is one very simple observation to make at the start:-

Every subgroup of a nilpotent group of class $\leqslant 2$ *is* 4*-quasinormal.*

For, if H and K are subgroups of a nilpotent group of class $\leqslant 2$ with K cyclic, then

$$\langle H, K\rangle = H[H, K]K = HKHK.$$

With regard to quasinormal subgroups H of any group G, a lot of information has been discovered when H is cyclic. For example, in this case *all subgroups of H are quasinormal in G* (see [8], 5.2.11 Lemma). Also when H is finite of odd order or even infinite, then $[H, G]$ is abelian and H acts on $[H, G]$ as a group of power automorphisms. When H is finite of even order, then the structure of $[H, G]$ and the action of H on it are both very precisely determined (see [9]). So for the remainder of this paper we consider *cyclic* 4-quasinormal subgroups. Also we shall work within finite p-groups and it is convenient to assume that $p \geqslant 5$.

We denote terms of the lower central series of G by $\gamma_i(G)$, $i \geqslant 1$, and terms of the upper central series by $\zeta_i(G)$, $i \geqslant 0$.

2. Cyclic 4-Quasinormal Subgroups

Let $H\ qn_4\ G = \langle H, K\rangle$, a finite p-group with H and K both cyclic. When H and K both have order p, then clearly $|G| \leqslant p^3$. So when $p = 2$, then G is either elementary abelian of rank $\leqslant 2$ or dihedral of order 8. When p is odd, then G *is either elementary abelian of rank* $\leqslant 2$ *or extraspecial of order* p^3. As always in initial stages of this type, we assume that p is odd. We shall also assume that

$$p \geqslant 5.$$

The reason for this is easy to explain. For, $|G/G^p| \leqslant p^3$ and since $p \geqslant 5$, it follows from [3], 10.3 Satz, that

$$G \textit{ is regular.}$$

Thus regular groups become relevant and we recall some of their properties.

Let G be a finite p-group. Then we define, for $i \geqslant 1$,

$$\Omega_i(G) = \langle g | g \in G,\ g^{p^i} = 1\rangle$$

and let $|\Omega_i(G)/\Omega_{i-1}(G)| = p^{\omega_i}$. Also we define

$$\mho_i(G) = \langle g^{p^i} | g \in G\rangle.$$

Then if G is regular, $|G/\Omega_i(G)| = |\mho_i(G)|$ for all i and $\omega_1 \geqslant \omega_2 \geqslant \ldots$ (see [3], 10.7 Satz). Also

$$\Omega_i(G) = \{g | g \in G,\ g^{p^i} = 1\},\ \ \mho_i(G) = \{g^{p^i} | g \in G\}$$

for all i. (See [3], 10.5 Hauptsatz.) Not surprisingly, these results have a significant application to the theory of 4-quasinormal subgroups.

Lemma 2.1. *Let G be a finite p-group, $p \geqslant 5$, generated by cyclic subgroups H and K such that $|G/\mho_1(G)| \leqslant p^3$. Let $H = \langle h\rangle$, $K = \langle k\rangle$ and $C = \langle [h, k]\rangle$. Then $G = HCK$.*

Proof. Clearly $G/\mho_1(G)$ is either elementary abelian of rank $\leqslant 2$ or the extraspecial group E_p of order p^3 and, as we saw above, G is regular. Let

$$|H| = p^m, |K| = p^n \textit{ and } |C| = p^\ell.$$

We have $HG' \lhd G$. Therefore if $m > \ell$, the regularity of G implies that $\Omega_1(H) \lhd G$. Then by induction on $|G|$, we may assume that H is core-free in G. Similarly we may assume that K is core-free in G and then $m = n = \ell \geqslant 2$. Thus $G/\mho_1(G)$ is extraspecial of order p^3 and $\mho_1(G) = H^pC^pK^p \lhd G$. Then $H^pCK^p \lhd G$ and so $\mho_{\ell-1}(H^pCK^p) = \Omega_1(C) \lhd G$. Now by induction on $|G|$, we have $G = HCK\Omega_1(C)$ and hence $G = HCK$. □

Our first main result now follows easily.

Theorem 2.2. *Let H be a cyclic 4-quasinormal subgroup of a finite p-group G with $p \geqslant 5$. Then every subgroup of H is 4-quasinormal in G.*

Proof. It suffices to show that if $J = \langle X, K\rangle$ is a finite p-group, $p \geqslant 5$, with X and K cyclic and $|J/\mho_1(J)| \leqslant p^3$, then $J = XKXK$.

Let $X = \langle x\rangle$, $K = \langle k\rangle$ and $C = \langle [x, k]\rangle$. By Lemma 2.1, $J = XCK$. We argue by induction on $|J|$. As in the proof of Lemma 2.1, we may assume that X and K are both core-free in J. Then again as in Lemma 2.1, $\Omega_1(C) \lhd J$ and $\Omega_1(C)$ lies in $\zeta_1(J)$. By induction $J = XKXK\Omega_1(C)$. Clearly $XKXK = X\{[x^i, k^j]|\text{ all } i, j\}K$. Assuming $x^i \neq 1$, $\Omega_1(C)$ is generated by an element$[x^i, k^r]$, some r. (See [3], 10.6 Satz.) Then for any s,

$$[x^i, k^j][x^i, k^r]^s = [x^i, k^j][x^i, k^{rs}] = [x^i, k^{j+rs}].$$

Hence $J = XKXK$ as required. □

It is appropriate at this point to give some examples of cyclic 4-quasinormal subgroups that appear in finite quotients of just-infinite pro-p-groups. We refer to [5], Theorem 12.2.3. For $p \geqslant 2$, there are two just-infinite pro-p-groups of rank 3, width 2 and obliquity 0 which arise naturally from the two central simple algebras of dimension 4 over $\mathbb{Q}_p$, the field of fractions of the p-adic completion $\mathbb{Z}_p$ of $\mathbb{Z}$. We call these two groups Γ_1 and Γ_2. Now let P be a finite 2-generator p-group such that $\mho_1(P) = \gamma_3(P)$ with $p \geqslant 5$. Let P have class c. Then $P/\gamma_{c-1}(P)$ is isomorphic to $\Gamma_i/\gamma_{c-1}(\Gamma_i)$ for $i = 1$ or 2. (See [5], Theorem 12.2.14.) In all these quotients it is easy to show that all the cyclic subgroups are 4-quasinormal. The factors of the lower central series are elementary abelian of rank alternating between 2 and 1. Thus we obtain examples of finite p-groups $G = \langle h, k\rangle$ with $\langle h\rangle$ qn_4 G. As c increases, the derived length of G increases and so *there is no bound on this derived length.* However, the condition $\mho_1(G) = \gamma_3(G)$ allows only 2 possibilities for G of each order.

As we recalled in Section 1, quasinormal subgroups of finite p-groups are invariant under projectivities. There is no obvious reason why a 4-quasinormal subgroup should have this property. However, cyclic ones do.

Theorem 2.3. *Cyclic qn_4-subgroups of finite p-groups, $p \geqslant 5$, are invariant under index-preserving projectivities.*

Proof. Let $G = \langle h, k \rangle$ be a finite p-group, $p \geqslant 5$, with $H = \langle h \rangle \; qn_4 \; G$ and $K = \langle k \rangle$. Let $\sigma : G \to \bar{G}$ be an index-preserving projectivity. Then

$$\bar{G} = \langle \bar{h}, \bar{k} \rangle.$$

Let $\bar{H} = \langle \bar{h} \rangle = H^{\sigma}$ and $\bar{K} = \langle \bar{k} \rangle = K^{\sigma}$. We have to show that

$$\bar{G} = \bar{H}\bar{K}\bar{H}\bar{K}. \tag{2.1}$$

We proceed by induction on $|G|$ (=$|\bar{G}|$).

Since $|G/\mho_1(G)| \leqslant p^3$ and $p \geqslant 5$, G is regular. In general regularity is not preserved under index-preserving projectivities. However, in our case $\bar{G}$ is regular since $|\bar{G}/\mho_1(\bar{G})| \leqslant p^3$ and $p \geqslant 5$. Let $|H| = |\bar{H}| = p^m$, $|K| = |\bar{K}| = p^n$, $C = \langle [h, k] \rangle$ and $|C| = p^{\ell}$.

Suppose that $m > \ell$. As in Lemma 2.1, we then have $\Omega_1(H) \lhd G$. Also by the same argument (using regularity of $\bar{G}$), $\Omega_1(\bar{H}) \lhd \bar{G}$. Then by induction $\bar{G} \equiv \bar{H}\bar{K}\bar{H}\bar{K} \bmod \Omega_1(\bar{H})$, and (1) follows. Thus we may assume that $m = \ell$; and similarly $n = \ell$. In this case $\Omega_1(C) = \mho_{\ell-1}(G) \lhd (G)$. Similarly with $\bar{c} = [\bar{h}, \bar{k}]$, we have

$$\Omega_1(\langle \bar{c} \rangle) = \mho_{\ell-1}(\bar{G}) \lhd \bar{G}.$$

So $\Omega_1(\langle \bar{c} \rangle) = \Omega_1(C)^{\sigma}$. Therefore by induction $\bar{G} \equiv \bar{H}\bar{K}\bar{H}\bar{K} \bmod \Omega_1(\langle \bar{c} \rangle)$ and the result follows as in the proof of Lemma 2.1. □

3. Strongly 4-Quasinormal Subgroups

For our final result, we show that cyclic 4-quasinormal subgroups of the groups we are considering are strongly 4-quasinormal.

Theorem 3.1. *Cyclic qn_4-subgroups of finite p-groups $(p \geqslant 5)$ are strongly 4-quasinormal.*

We begin by considering small cases.

Lemma 3.2. *Let $G = \langle h, k_1, k_2 \rangle$ be a finite p-group (p odd) with h, k_1 and k_2 of order p, $H = \langle h \rangle \; qn_4 \; G$ and $K = \langle k_1, k_2 \rangle$ abelian. Then $[h, k_i] \in \zeta_1(G)$, $i = 1, 2$ and $G = HKHK$.*

Proof. Let $c_i = [h, k_i]$, $i = 1, 2$. Then $|\langle h, k_i \rangle| \leqslant p^3$ and

$$c_i \textit{ commutes with } h \textit{ and with } k_i$$

for $i = 1, 2$. Thus H commutes with h^{k_i}, $i = 1, 2$. Replacing k_i by $k_1 k_2^{-1}$, similarly we see that h commutes with $h^{k_1 k_2^{-1}}$, i.e. h^{k_1} commutes with h^{k_2}. Thus $[c_1, c_2] = 1$. Also replacing k_i by $k_1 k_2$, we have $[h, k_1 k_2] = c_2 c_1^{k_2}$ commutes with $k_1 k_2$. Conjugating by k_2^{-1}, we obtain $[c_2 c_1, k_1 k_2] = 1$. Moreover $[h, k_1 k_2] = [h, k_2 k_1]$, i.e. $c_2 c_1^{k_2} = c_1 c_2^{k_1}$. Thus $(c_1 c_2)^{k_2} = (c_1 c_2)^{k_1}$. Hence $[c_1 c_2, k_1 k_2^{-1}] = 1$ and we have $[c_1 c_2, k_1] = 1$. Therefore $[c_2, k_1] = 1$.

It follows that c_2 and similarly c_1 belong to $\zeta_1(G)$. Thus G has class at most 2 and (since H is cyclic) $G = HKHK$. □

Now we can deal with the case where H has order p.

Lemma 3.3. *Let G be a finite p-group, $p \geqslant 5$, and let H be a subgroup of order p with $H\ qn_4\ G$. Then H is strongly 4-quasinormal in G.*

Proof. We may assume that $G = \langle H, K\rangle$ with K a subgroup and show that

$$G = HKHK. \tag{3.1}$$

Let $H = \langle h\rangle$. Suppose first that K has exponent p and let $c \in \zeta_1(K)$, $c \neq 1$. By Lemma 3.2, $[h, c] \in \zeta_1(G)$. Proceed by induction on $|G|$. If $c \in \zeta_1(G)$, then induction gives $G = HKHK\langle c\rangle = HKHK$. Otherwise let $N = \langle [h, c]\rangle$. Again by induction

$$G = HKHKN = H\{[h^i, k] |\ \text{all}\ k \in K,\ \text{all}\ i\}\langle [h, c]\rangle K.$$

For $p \nmid i$, we have $ir \equiv 1 \bmod p$, some r. Then

$$[h^i, k][h, c]^j = [h^i, k][h^i, c^{jr}] = [h^i, c^{jr}k] = [h^i, k'],$$

$k' \in K$. So $G = HKHK$.

Now let K have exponent $\geqslant p^2$. Again argue by induction on $|G|$. We have $K^p \neq 1$ and K^p commutes with H by regularity arguments. Then $K^p \lhd G$ and $G \equiv HKHK \bmod K^p$, by above. Hence (2) follows. □

Proof of Theorem 3.1. Let H be a cyclic qn_4-subgroup of the finite p-group G ($p \geqslant 5$). We may assume that $G = \langle H, K\rangle$ and show that $G = HKHK$. We proceed by induction on $|G|$. Let $H = \langle h\rangle$. By Lemma 3.3, we may assume that $|H| = p^m$, $m \geqslant 2$. Let K have exponent p^n. If $m > n$, regularity arguments show that $\Omega_1(H) = H_1$ centralizes K. So $H_1 \lhd G$ and induction applies. If $m < n$, then again regularity shows that K^{p^m} is normalized by H and so $K^{p^m} \lhd G$. Then as above induction applies. Therefore we may assume that

$$m = n \geqslant 2.$$

Let $W = \Omega_{m-1}(K)$. We may assume that $W < K$, otherwise K is generated by elements of order $\leqslant p^{m-1}$ and then H_1 centralizes K, as above. Choose $x \in K\backslash W$ such that $[x, K] \in W$. Then with $H_1 = \langle h_1\rangle$, we show that

$$[h_1, x] \in \zeta_1(G). \tag{3.2}$$

For, since

(3.3) every element of W commutes with h_1 and its conjugates,

we may assume that $K = \langle x, y\rangle$. Let $c_1 = [h_1, x]$, $d_1 = [h_1, y]$. Arguing as in the proof of Lemma 2.1 (using [3], 10.6 Satz), we have

$$[c_1, x] = [d_1, y] = 1.$$

Also $[c_1, d_1] = 1$, since H_1^G is elementary abelian. In the same way $[h_1, xy] = d_1 c_1^y$ commutes with xy. Thus conjugating by y^{-1}, we see that $d_1 c_1 = c_1 d_1$ commutes with $xy = [y^{-1}, x^{-1}]xy$. Hence, by (4),

$$c_1 d_1 \textit{ commutes with } xy. \tag{3.4}$$

Also $[h_1, xy] = [h_1, yx]$, i.e. $d_1 c_1^y = c_1 d_1^x$ and therefore $(c_1 d_1)^y = (c_1 d_1)^x$. Thus $c_1 d_1$ commutes with yx^{-1} and hence also with y^2, by (5). Then $[c_1, y] = 1$ and (3) follows.

Now suppose that $c_1 \neq 1$. By induction on $|G|$, we have $G = HKHK\langle c_1 \rangle$. Then it suffices to show that for any $k \in K$ and any i,

$$[h^i, k][h_1, x] = [h^i, k'], \tag{3.5}$$

some $k' \in K$. We may assume that $h_1 = h^{ijp^r}$, some r and j with $p \nmid j$. Then $[h_1, x] = [h^{ijp^r}, x] = [h^{ip^r}, x^j] = [h^i, x^{jp^r s}]$, some s with $p \nmid s$, using the regularity of $\langle h, x \rangle$. (Again see [3], 10.6 Satz.) So

$$[h^i, k][h_1, x] = [h^i, x^{jp^r s} k]$$

and we have (6).

Finally we are left with the situation $[h_1, Z] = 1$, where $Z/W = \zeta_1(K/W)$. Then $[H, \mho_{m-1}(Z)] = 1$, again by regularity. So $\mho_{m-1}(Z)$ is a non-trivial normal subgroup of G lying in K and induction on $|G|$ gives the result. □

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