ON THE COMMUTATIVITY DEGREE IN FINITE MOUFANG LOOPS

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ABSTRACT. The *commutativity degree*, $Pr(G)$, of a finite group G (i.e. the probability that two (randomly chosen) elements of G commute with respect to its operation)) has been studied well by many authors. It is well-known that the best upper bound for $Pr(G)$ is $\frac{5}{8}$ for a finite non–abelian group G.

In this paper, we will define the same concept for a finite non–abelian *Moufang loop* M and try to give a best upper bound for $Pr(M)$. We will prove that for a well-known class of finite Moufang loops, named *Chein loops*, and its modifications, this best upper bound is $\frac{23}{32}$. So, our conjecture is that for any finite Moufang loop M, $Pr(M) \leq \frac{23}{32}$.

Also, we will obtain some results related to the $Pr(M)$ and ask the similar questions raised and answered in group theory about the relations between the structure of a finite group and its commutativity degree in finite Moufang loops.

1. Introduction

A set Q with one binary operation is a quasigroup if the equation $xy = z$ has a unique solution in Q whenever two of the three elements $x, y, z \in Q$ are specified. A loop is a quasigroup with a neutral element 1 satisfying $1x = x1 = x$ for every x. Moufang loops are loops in which any of the (equivalent) Moufang identities $((xy)x)z = x(y(xz))$, $x(y(zy)) = ((xy)z)y$, $(xy)(zx) = x((yz)x)$, $(xy)(zx) = (x(yz))x$ holds.

Moufang loops are certainly the most studied loops. They arise naturally in algebra (as the multiplicative loop of octonions , and in projective geometry (Moufang planes)). Although Moufang loops are generally non–associative, they retain many properties of groups that we know and love. For instance: *(i)* every x is accompanied by its two-sided inverse x^{-1} such that $xx^{-1} = x^{-1}x = 1$, *(ii)*

any two elements generate a subgroup (this property is called diassociativity), *(iii)* in finite Moufang loops, the order of an element divides the order of the loop, and, as has been shown recently in [12], the order of a subloop divides the order of the loop, *(iv)* every finite Moufang loop of odd order is solvable. Also, there are Sylow and Hall theorems for finite Moufang loops. For more details see [2, 20, 8, 12, 13].

On the other hand, many essential tools of group theory are not available for Moufang loops. The lack of associativity makes presentations very awkward and hard to calculate, and permutation representations in the usual sense impossible.

Let A be a finite algebraic structure with at least one binary operation like as "·". Then, one may ask: *What is the probability that two (randomly chosen) elements of A commute (with respect to the operation "·")?* A formal answer is

$$Pr(A) = \frac{\mid \{(x,y) \in A^2 \mid xy = yx\} \mid}{\mid A^2 \mid}.$$

For a finite group A it is proved that $Pr(A) = \frac{k(A)}{|A|}$, where, $k(A)$ is the number of conjugacy classes of A (see [11, 16, 14] for example). The computational results on $Pr(A)$ are mainly due to Gustafson [11] who shows that $Pr(A) \leqslant \frac{5}{8}$ for a finite non–abelian group A, and MacHale [16] who proves this inequality for a finite non–abelian ring. Also, the author of this paper and his colleagues have shown in [1] that the $\frac{5}{8}$ is not an upper bound for $Pr(A)$, where A is a finite non–abelian semigroup and/or monoid.

Now, let M be a finite non–abelian Moufang loop. In this paper, we will ask the same question for M and try to give a best upper bound for $Pr(M)$. Also, we will obtain some results related to the $Pr(M)$ and ask the similar questions raised and answered in group theory about the relations between the structure of a finite group and its commutativity degree in finite Moufang loops.

2. **On the Classification of Moufang loops**

To use in the next section, here we summarize the spectrum of Moufang loops: For which orders n is there a non–associative Moufang loop?

The following Theorem has been proved by Chein and Rajah in [5] and makes an important corollary.

Theorem 2.1. [5, Theorem 2.2]. *Every Moufang loop L of order $2m$, m odd, which contains a normal abelian subgroup M of order m is a group.*   □

We can now settle the question of for which values of $n = 2m$ must every Moufang loop of order n be a group.

Corollary 2.2. [5, Corollary 2.3]. *Every Moufang loop of order $2m$ is associative if and only if every group of order m is abelian.*   □

So, by the above corollary, a non–associative Moufang loop of order $2m$ exists if and only if a non–abelian group of order m exists. Hence, a non–associative Moufang loop of order 2^k exists if and

only if $k > 3$, and for every odd $m > 1$ there is a non–associative Moufang loop of order $4m$. Here is the case $2m$, m odd:

Theorem 2.3. [5, Corollary 2.4]. *Every Moufang loop of order $2m$, $m > 1$ odd, is associative if and only if $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where $p_1 < \cdots < p_k$ are odd primes and where*

   (i) $\alpha_i \leq 2$, *for all* $i = 1, \ldots, k$,
   (ii) $p_j \not\equiv 1 \ (mod \ p_i)$, *for any* i *and* j,
   (iii) $p_j^2 \not\equiv 1 \ (mod \ p_i)$, *for any* i *and* j *with* $\alpha_j = 2$.

$\square$

Concerning odd orders, we have:

Theorem 2.4. [15, Theorems 1 and 2]. *Every Moufang loop of order $p^\alpha q_1^{\alpha_1} \cdots q_k^{\alpha_k}$ is associative if $p < q_1 < \cdots < q_k$ are odd primes, and if one of the following conditions holds:*

   (i) $\alpha \leq 3$ *and* $\alpha_i \leq 2$, *for all* $i = 1, \ldots, k$,
   (ii) $p \geq 5$, $\alpha \leq 4$ *and* $\alpha_i \leq 2$, *for all* $i = 1, \ldots, k$.

$\square$

So, by above theorems, for any prime p, none of the Moufang loops of order p, p^2, p^3 are non–associative; none of the Moufang loops of order p^4 are non–associative unless $p = 2, 3$. By Theorem 2.3, for odd primes $p < q$ a non–associative Moufang loop of order pq^3 exists if and only if $q \equiv 1 \ (mod \ p)$. In [10] and [18], Goodaire, May, Raman, Nagy and Vojtěchovský classified all non–associative Moufang loops of order $3^4 = 81$ and 64. They proved that there are 4262 pairwise nonisomorphic non–associative Moufang loops of order 64, and there are 5 pairwise nonisomorphic non–associative Moufang loops of order 81, 2 of which are commutative. All 5 of these loops are isotopes of the 2 commutative ones. In [17], Nagy and Valsecchi have shown that there are precisely 4 non–associative Moufang loops of order p^5 for every prime $p \geq 5$. Finally, Slattery and Zenisek classified all non–associative Moufang loops of order 243 in [21] So, the classification of non–associative Moufang loops of order p^4 and p^5 is now complete. Much more is known but the problem is open in general.

Table 1 gives the number of pairwise nonisomorphic non–associative Moufang loops of order n for every $1 \leq n \leq 64$ and $n = 81, 243$ for which at least one non–associative Moufang loop exists.

3. Commutativity degree in some classes of Moufang loops

There is a class of non–associative Moufang loops, first defined by Chein [3], that is well understood. Let G be a group of order n, and let u be a new element. Define multiplication $\circ$ on $G \cup Gu$ by $g \circ h = gh$, $g \circ hu = (hg)u$, $gu \circ h = (gh^{-1})u$, $gu \circ hu = h^{-1}g$, where $g, h \in G$. The resulting loop $(G \cup Gu, \circ) = M(G, 2)$ is a Moufang loop. It is non–associative if and only if G is non–abelian.

Let $\pi(m)$ be the number of isomorphism types of non–associative Moufang loops of order at most m, and let $\sigma(m)$ be the number of non–associative loops of the form $M(G, 2)$ of order at most m. Then, according to Chein's classification [4], $\pi(31) = 13$, $\sigma(31) = 8$, $\pi(63) = 158$, $\sigma(63) = 50$. This

TABLE 1. The number $M(n)$ of non–associative Moufang loops of order n.

n	12	16	20	24	28	32	36	40	42
M(n)	1	5	1	5	1	71	4	5	1
n	44	48	52	54	56	60	64	81	243
M(n)	1	51	1	2	4	5	4262	5	72

demonstrates eloquently the abundance of loops of type $M(G, 2)$ among Moufang loops of small order, and chiefly thanks to this fact, they play a prominent role in the classification of Moufang loops. It has been proved that $M(G, 2)$ is isomorphic to $M(H, 2)$ if and only if G is isomorphic to H. Thus, we obtain as many non–associative Moufang loops of order $2n$ as there are non–abelian groups of order n.

Lemma 3.1. *In every Moufang loop $M(G, 2)$, we have $\forall g, h \in G$:*

(i) *$gu \circ h = h \circ gu$ if and only if $h^2 = 1$;*

(ii) *$gu \circ hu = hu \circ gu$ if and only if $(g^{-1}h)^2 = 1$.*

Also, if $gh = hg$ then $gu \circ hu = hu \circ gu$ if and only if $g^2 = h^2$.

Proof. The proof is clear by the definition of $M(G, 2)$. □

By the above lemma, only the trivial element 1 and the elements of order 2 in G are commutative with all of the elements of Gu and so we have the following formula for the commutativity degree of $M(G, 2)$:

Theorem 3.2. *Let G be a finite group, t be the number of involutions of G and $k(G)$ be the number of conjugacy classes of G. Then the commutativity degree of $M(G, 2)$ is:*

$$Pr(M(G, 2)) = \frac{k(G) + 3(t + 1)}{4|G|}.$$

If the order of G is odd, then the formula will be simply:

$$Pr(M(G, 2)) = \frac{k(G) + 3}{4|G|}.$$

Proof. Let $S = \{(x, y) \in M(G, 2) \times M(G, 2) \mid xy = yx\}$. Consider the multiplication table of $M(G, 2)$. It will be like the following:

·	G	Gu
G	$G \circ G$	$G \circ Gu$
Gu	$Gu \circ G$	$Gu \circ Gu$

In $G \circ G$ part of the table, there are precisely $k(G)|G|$ distinct ordered pairs in S and since there are $t+1$ columns in $Gu \circ G$ with $x^2 = 1$ in G and $t+1$ rows in $G \circ Gu$ with $x^2 = 1$ in G (corresponding to identity element and involutions in $M(G, 2)$), so by Lemma 3.1(i), there are $(t+1)|G|$ distinct pairs in S in each case. Also, by Lemma 3.1(ii), there are $(t+1)|G|$ distinct ordered pairs (g, h) in S such that $(g^{-1}h)^2 = 1$. So, we obtain:

$$Pr(M(G, 2)) = \frac{k(G)|G| + 3(t+1)|G|}{4|G|^2} = \frac{k(G) + 3(t+1)}{4|G|}.$$

$\square$

Example 3.1. *The following formula for the commutativity degree of finite dihedral groups is well-known:*

$$Pr(D_{2m}) = \begin{cases} \frac{m+6}{4m}, & \forall m \ \ even \\ \frac{m+3}{4m}, & \forall m \ \ odd. \end{cases}$$

Also, we know that the class number and number of involutions of D_{2m} are as follows:

$$k(D_{2m}) = \begin{cases} \frac{m}{2} + 3, & \forall m \ \ even \\ \frac{m+3}{2}, & \forall m \ \ odd. \end{cases}$$

$$t = \begin{cases} m+1, & \forall m \ \ even \\ m, & \forall m \ \ odd. \end{cases}$$

Hence, by Theorem 3.2 we have:

$$Pr(M(D_{2m}, 2)) = \begin{cases} \frac{\frac{m}{2} + 3 + 3(m+2)}{8m}, & \forall m \ \ even \\ \frac{\frac{m+3}{2} + 3(m+1)}{8m}, & \forall m \ \ odd. \end{cases}$$

From the theorem above we have the following fact:

Corollary 3.3. *If G is a group of even order and both of G and $M(G, 2)$ are abelian then they are both elementary abelian. Also, for every non-trivial group G of odd order, $M(G, 2)$ is non–abelian.*

Proof. If both of G and $M(G, 2)$ are abelian then $M(G, 2)$ is associative and so is a group and we have by the above theorem:

$$\begin{aligned} Pr(M(G, 2)) &= \frac{k(G) + 3(t+1)}{4|G|} = \frac{|G| + 3(t+1)}{4|G|} = 1 \\ \Longleftrightarrow \quad 4|G| &= |G| + 3(t+1) \\ \Longleftrightarrow \quad t &= |G| - 1, \end{aligned}$$

where t is the number of involutions of G. So, all of the non-trivial elements of G are involutions and G is an elementary abelian 2-group. Therefore, $M(G, 2)$ is elementary abelian too.

Now, let $M(G, 2)$ be abelian, where G is of odd order. Then we have:

$$Pr(M(G, 2)) = \frac{k(G) + 3(t+1)}{4|G|} = \frac{|G| + 3}{4|G|} = 1 \quad \Leftrightarrow |G| = 1$$

$$\Leftrightarrow |M(G, 2)| = 2,$$

a contradiction. So, $M(G, 2)$ must be non–abelian, as required. $\square$

One of the first exercises in group theory is that a group in which all nonidentity elements have order two (so-called involutions) is abelian. An almost equally easy exercise states that a finite group is abelian if at least $\frac{3}{4}$ of its elements have order two. This cannot be improved, as the dihedral group of order eight, as well as its direct product with any elementary abelian group, provides examples of groups in which the number of involutions is strictly less than $\frac{3}{4}$ of the group order. However, it is known that groups in which many elements are involutions are not far from being abelian. So, by these considerations and Theorem 3.2 one can obtain the next result about the best upper bound for $Pr(M(G, 2))$.

Corollary 3.4. *For every finite non–abelian group G, the commutativity degree of $M(G, 2)$ is less than or equal to $\frac{23}{32} = 0.71875$. Also, this is the best upper bound for $Pr(M(G, 2))$.*

Proof. Let G be a non–abelian group. Since G has at most $\frac{5}{8}|G|$ conjugacy classes and less than $\frac{3}{4}|G|$ involutions then:

$$Pr(M(G, 2)) = \frac{k(G) + 3(t + 1)}{4|G|} \leq \frac{\frac{5}{8}|G| + 3 \cdot \frac{3}{4}|G|}{4|G|} = \frac{23}{32}.$$

Now, we can reach the upper bound $\frac{23}{32}$ by $M(D_8, 2)$ where, D_8 is the dihedral group of order 8. $\square$

Remark 3.5. *If G be an abelian finite group, then by the above results $Pr(M(G, 2)) = 1$ if and only if G is elementary abelian. Also, if G is a finite abelian but non-elementary group then $M(G, 2))$ is a group and so*

$$Pr(M(G, 2)) \leq \frac{5}{8}.$$

On the other hand, the above result shows that the $\frac{5}{8} = 0.625$ law for finite groups and rings does not hold for finite Moufang loops. The next question is: "Does the upper bound $\frac{23}{32}$ hold for every finite Moufang loop?" The calculations with some GAP [9] codes (using LOOPS Package [19]), confirms this as follows in the next section. So we try other classes of finite Moufang loops.

Remark 3.6. *If M_1 and M_2 are two finite Moufang loops, then*

$$Pr(M_1 \times M_2) = Pr(M_1) \times Pr(M_2).$$

Particularly, if G be a finite group, then

$$Pr(M_1 \times G) = Pr(M_1) \times Pr(G).$$

Also, let G be a finite abelian group and M be a finite non–associative Moufang loop. Then $M \times G$ will be a non–associative Moufang loop and we have:

$$Pr(M \times G) = Pr(M) \times Pr(G) = Pr(M).$$

Therefore, for all finite Moufang loops $M = M(H, 2)$, (where H is any finit group); and all finite abelian groups G we obtain:

$$Pr(M \times G) = Pr(M) \leq \frac{23}{32}.$$

Let K, A be loops. Then a loop Q is an extension of K by A if A is a normal subloop of Q such that Q/A is isomorphic to K. An extension Q of K by A is central if A is a subloop of $Z(Q)$. Every Moufang loop M of order p^{k+1}, p a prime, is a central extension of a Moufang loop K of order p^k by the $p-$element field $\mathbb{F}_p$.

For a finite group G and its normal subgroup N, it is well-known that $k(G) \leq k(N) \cdot k(G/N)$ and so,

$$Pr(G) \leq Pr(N) \cdot Pr(G/N).$$

Also, for any subgroup H of G, we always have $Pr(G) \leq Pr(H) \leq |G:H|^2 \cdot Pr(G)$. But, we don't have these tools in Moufang loops and can only say that:

Proposition 3.7. *For any finite Moufang loop M and its normal subloop N, $Pr(M) \leq Pr(M/N)$. Also, for any finite group G and its subgroup H, $L = M(H,2)$ is a subloop of $M = M(G,2)$ with $|M|/|L| = |G:H|$, and if G is of odd order then:*

$$Pr(M) \leq Pr(L) \leq \frac{|M|^2}{|L|^2} \cdot Pr(M).$$

Proof. The proof of the first part is straightforward. Now, by Theorem 3.2, we have

$$
\begin{aligned}
Pr(M) \quad &= \frac{k(G) + 3}{4|G|} \\
&\leq \frac{|G:H| \cdot k(H) + 3}{4|G|} \\
&= \frac{|G:H|(k(H) + 3)}{4|G|} + \frac{3(1 - |G:H|)}{4|G|} \\
&= Pr(L) + \frac{3(1 - |G:H|)}{4|G|}.
\end{aligned}
$$

So, $Pr(M) \leq Pr(L)$. Again,

$$
\begin{aligned}
Pr(M) \quad &= \frac{k(G) + 3}{4|G|} \\
&\geq \frac{k(H) \cdot |G:H|^{-1} + 3}{4|G|} \\
&= \frac{k(H)}{4|H||G:H|^2} + \frac{3}{4|G|} \\
&= \frac{k(H) + 3}{4|H||G:H|^2} - \frac{3}{4|H||G:H|^2} + \frac{3}{4|G|} \\
&= \frac{Pr(L)}{|G:H|^2} + \frac{3}{4|G|} \cdot \frac{|G:H| - 1}{|G:H|} \\
&\geq \frac{Pr(L)}{|G:H|^2} \\
&= \frac{Pr(L) \cdot |L|^2}{|M|^2}.
\end{aligned}
$$

$\square$

Example 3.2. *In the above proposition, if G is of even order the result is not true: Let $G = S_3$ and $H = A_3$ (symmetric group of degree 3 and its alternating subgroup). Then if we set $M = M(G,2)$ and*

$L = M(H, 2)$, we obtain $Pr(M) = \frac{3+3(1+3)}{4\cdot 6} = \frac{5}{8}$, but $Pr(L) = \frac{3+3(1+0)}{4\cdot 3} = \frac{1}{2}$, and so, $Pr(M) > Pr(L)$. Also, G is a normal subloop of M and $Pr(M) > Pr(G) \cdot Pr(\frac{M}{G}) = \frac{1}{2} \cdot 1 = \frac{1}{2}$.

By the above facts, we obtain the following corollary:

Corollary 3.8. *Let M be a Moufang loop M of order p^{k+1}, p a prime, which is the central extension of a Moufang loop K of order p^k by the $p-$element field $\mathbb{F}_p$. Then $Pr(M) \leq Pr(K)$.*   □

While working on the problem of Hamming distances of groups [6], Drápal discovered two constructions, named central and dihedral modifications, that modify exactly one quarter of the multiplication table of a group and yield another group, often with a different center and thus not isomorphic to the original group. These constructions work for Moufang loops, too, and have been called *Moufang modifications.* Let us first give a brief description of the constructions (see [7] for more details):

Definition 3.9. *Assume that L is a Moufang loop with normal subloop S such that L/S is a cyclic group of order $2m$. Let $h \in S \cap Z(L)$. Let α be a generator of L/S and write $L = \bigcup_{i\in M} \alpha^i$, where $M = \{-m+1, \ldots, m\}$. Let $\sigma : M \to \mathbb{Z}$; be defined by $\sigma(i) = 0$ if $i \in M$, $\sigma(i) = 1$ if $i > m$, and $\sigma(i) = -1$ if $i < -m+1$. Introduce a new multiplication $*$ on L defined by $x * y = xyh^{\sigma(i+j)}$, where $x \in \alpha^i$, $y \in \alpha^j$, $i, j \in M$. Then $(L, *)$ is a Moufang loop, called a cyclic modification of L.*

Now assume that L is a Moufang loop with normal subloop S such that L/S is a dihedral group of order $4m$, with $m \geq 1$. Let M and σ be defined as in the cyclic case. Let $\beta, \gamma \in L/S$ be two involutions of L/S such that $\alpha = \beta\gamma$ generates a cyclic subgroup of L/S of order $2m$. Let $e \in \beta$ and $f \in \gamma$ be arbitrary. Then L can be written as a disjoint union $L = \bigcup_{i\in M} \alpha^i \cup e\alpha^i$ and also $L = \bigcup_{i\in M} \alpha^i \cup \alpha^i f$. Let $G_0 = \bigcup_{i\in M} \alpha^i$, and $G_1 = L \setminus G_0$. Let $h \in S \cap N(L) \cap Z(G_0)$. Introduce a new multiplication $*$ on L defined by $x * y = xyh^{(-1)^r \sigma(i+j)}$, where $x \in \alpha^i \cup e\alpha^i$, $y \in \alpha^j \cup \alpha^j f$, $i, j \in M$, $y \in G_r$, $r \in \{0, 1\}$. Then $(L, *)$ is a Moufang loop, called a dihedral modification of L.

It turns out that these extensions can be used to obtain all non–associative Moufang loops of order at most 64, [7]. In fact, to apply the cyclic and dihedral modifications, it is beneficial to have access to a class of non–associative Moufang loops. Vojtěchovský uses these modifications on non–associative Moufang loops of the form $M(G, 2)$ to get all non–associative Moufang loops of order 64 and then Nagy and Vojtěchovský in [18] prove by central extention that there are no other non–associative Moufang loops of this order.

The structures of Moufang modifications lead us to the following result about their commutativity degree:

Corollary 3.10. *Let $(L, \cdot)$ be a Moufang loop and let $(L, *)$ be obtained from $(L, \cdot)$ by the cyclic or the dihedral construction. Then:*

$$Pr((L, *)) = Pr((L, \cdot)).$$

Proof. By the definition of Moufang modifications, for every $x, y \in L$, the commutator $[x, y] = 1$ in $(L, *)$ if and only if $[x, y] = 1$ in $(L, \cdot)$.   □

4. **Computations of commutativity degree in finite Moufang loops**

Table 2 shows the commutativity degrees of non–associative Moufang loops of order n up to 64, $n = 81$ and $n = 243$. They have been calculated by GAP codes, [9].

TABLE 2. Commutativity degree of Moufang loops of order n less than or equal to 64 and $n = 81, 243$.

non–associative Moufang loop $M(m,n)$	$Pr(M(m,n)$
$M(12,1)$	$\frac{5}{8}$
$M(16,n), \ 1 \leq n \leq 5$ (res.)	$\frac{23}{32}, \frac{11}{32}, \frac{11}{32}, \frac{23}{32}, \frac{23}{32}$
$M(20,1)$	$\frac{11}{20}$
$M(24,m), \ 1 \leq m \leq 5$ (res.)	$\frac{5}{8}, \frac{1}{3}, \frac{1}{4}, \frac{5}{8}, \frac{1}{4}$
$M(28,1)$	$\frac{29}{56}$
$M(32,n), \ 1 \leq n \leq 71$	$\frac{13}{64}, \frac{11}{32}, \frac{25}{64}, \frac{7}{16}, \frac{17}{32}, \frac{37}{64}, \frac{5}{8}, \frac{23}{32}$
$M(36,n), \ 1 \leq n \leq 4$ (res.)	$\frac{7}{24}, \frac{1}{2}, \frac{1}{2}, \frac{5}{8}$
$M(40,n), \ 1 \leq n \leq 5$ (res.)	$\frac{11}{20}, \frac{7}{40}, \frac{23}{80}, \frac{11}{20}, \frac{7}{40}$
$M(42,1)$	$\frac{2}{21}$
$M(44,1)$	$\frac{43}{88}$
$M(48,m), \ 1 \leq m \leq 51$	$\frac{13}{96}, \frac{5}{32}, \frac{3}{16}, \frac{7}{32}, \frac{1}{4}, \frac{1}{3}, \frac{11}{32}\frac{35}{96}, \frac{3}{8}, \frac{13}{32}, \frac{17}{32}, \frac{5}{8}, \frac{23}{32}$
$M(52,1)$	$\frac{25}{52}$
$M(54,n), \ 1 \leq n \leq 2$ (res.)	$\frac{7}{54}, \frac{7}{54}$
$M(56,n), \ 1 \leq n \leq 4$ (res.)	$\frac{29}{56}, \frac{1}{7}, \frac{29}{56}, \frac{1}{7}$
$M(60,n), \ 1 \leq n \leq 5$ (res.)	$\frac{1}{4}, \frac{9}{40}, \frac{19}{40}, \frac{11}{20}, \frac{5}{8}$
$M(64,m), \ 1 \leq m \leq 4262$	$\frac{17}{128}, \frac{23}{128}, \frac{13}{64}, \frac{29}{128}, \frac{1}{4}, \frac{35}{128}, \frac{19}{64}, \frac{41}{128}, \frac{11}{32}, \frac{47}{128}, \frac{25}{64}, \frac{53}{128},$ $\frac{7}{16}, \frac{59}{128}, \frac{31}{64}, \frac{65}{128}, \frac{17}{32}, \frac{71}{128}, \frac{37}{64}, \frac{77}{128}, \frac{5}{8}, \frac{83}{128}, \frac{23}{32}$
$M(81,n), \ 1 \leq n \leq 5$ (res.)	$1, 1, \frac{11}{27}, \frac{11}{27}, \frac{11}{27}$
$M(243,m), \ 1 \leq m \leq 72$	$1, \frac{11}{27}, \frac{17}{81}, \frac{35}{243}, \frac{83}{243}$

5. **Conclusion and commutativity degree in other classes of loops**

We conclude that the set of commutativity degrees of all non–associative Moufang loops of order 64 is equal to the set of commutativity degrees of all non–associative Moufang loops of the form $M(G, 2)$, where G is a non–abelian group of order 32. Of course, we can not generalize it to all finite Moufang loops, i.e. for a positive integer n, we may not construct all finite Moufang loops of order n by cyclic or dihedral modifications and these constructions do not apply for all n. But by the above results and considerations, it is more likely that $\frac{23}{32}$ may be the best upper bound for the commutativity degree of all finite Moufang loops. So, we state the following conjecture:

Conjecture. Let M be a finite non–commutative Moufang loop. Then $Pr(M) \leq \frac{23}{32}$.

On the other hand, calculations by GAP codes, [9], show that the upper bound $\frac{23}{32}$ is not valid for some other classes of finite loops, such as $CC-$loops, Bol loops, nilpotent loops (which are not Moufang). For example, there are three $CC-$loops of order 9 with commutativity degree $\frac{7}{9}$, there are some left Bol loops of order 16 with commutativity degree $\frac{7}{8}$, there are some nilpotent loops of order 8 with commutativity degree $\frac{7}{8}$ and nilpotent loops of order 10 with commutativity degree $\frac{23}{25}$.

Finally, some problems like those that have been answered about commutativity degree in finite groups arise for Moufang loops, like as:

The structure of finite groups with commutativity degree $\geq \frac{1}{2}$ has been characterized by Lescot in 1995 [14]. Also, we know that if for a finite group G, $Pr(G) = 5/8$, then G is nilpotent.

Question 1: *Can we determine the structure of a given finite Moufang loop by its commutativity degree (such as nilpotency, solvability, simplicity and so on)?*

If G is a non–abelian finite simple group, then $Pr(G) \leq 1/12$, with equality for the alternating group of degree 5, A_5.

Question 2: *Is there a similar upper bound for a non–abelian finite simple Moufang loop?* (For example, the commutativity degree of the Paige loop of order 120, which is simple, is 4/25.)

It is well-known that there are no finite groups G such that $\frac{7}{16} < Pr(G) < \frac{1}{2}$ (see [11] or [16]). So, one may ask:

Question 3: *Can we derive a similar result in finite Moufang loops?*

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