International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 5 No. 4 (2016), pp. 23-25.

www.theoryofgroups.ir
www.ui.ac.ir

A REMARK ON GROUP RINGS OF PERIODIC GROUPS

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Communicated by Victor Bovdi

Abstract. A positive solution of the problem of the existence of nontrivial pairs of zero-divisors in group rings of free Burnside groups of sufficiently large odd periods $n > 10^{10}$ obtained previously by S. V. Ivanov and R. Mikhailov extended to all odd periods $n \geq 665$.

1. Introduction

Let G be an arbitrary group, $h \in G$ be an element of finite order $n > 1$ and $X, Y \in \mathbb{Z}[G]$, where $\mathbb{Z}[G]$ is the group ring of G over the integers. Then the following equalities

$$X(1-h) \cdot (1 + h + \cdots + h^{n-1})Y = 0,$$

$$X(1 + h + \cdots + h^{n-1}) \cdot (1-h)Y = 0$$

are hold in $\mathbb{Z}[G]$. Hence, $X(1-h)$ and $(1 + h + \cdots + h^{n-1})Y$, $X(1 + h + \cdots + h^{n-1})$ and $(1-h)Y$ are left and right zero-divisors of $\mathbb{Z}[G]$ (unless one of them is 0 itself) which called in [2] trivial pairs of zero-divisors associated with h. Equivalently, $A, B \in \mathbb{Z}[G]$, with $AB = 0$, $A, B \neq 0$, is a trivial pair of zero-divisors in $\mathbb{Z}[G]$ if there are $X, Y \in \mathbb{Z}[G]$ and $h \in G$ of finite order $n > 1$ such that either $A = X(1-h)$ and $B = (1 + h + \cdots + h^{n-1})Y$ or $A = X(1 + h + \cdots + h^{n-1})$ and $B = (1-h)Y$.

Let $B(m, n)$ be the free Burnside group of rank m and exponent n, that is, $B(m, n)$ is the quotient F_m/F_m^n of a free group F_m of rank m. S.V.Ivanov asked the following question [[1], Problem 11.36d]: Suppose $m \geq 2$ and odd $n \gg 1$. Is it true that every pair of zero-divisors in $\mathbb{Z}[B(m, n)]$ is trivial, i.e., if $AB = 0$ in $\mathbb{Z}[B(m, n)]$, then $A = XC$, $B = DY$, where $X, Y, C, D \in \mathbb{Z}[B(m, n)]$ such that $CD = 0$ and the set $supp(C) \cup supp(D)$ is contained in a cyclic subgroup of $B(m, n)$?

MSC(2010): Primary: 20F50, 20C07; Secondary: 20E06.
Keywords: group ring, free Burnside group, zero divisor.
Received: 15 October 2014, Accepted: 11 May 2015.

In the paper [2] the authors reject this conjecture, proving the existence of nontrivial pairs of zero-divisors in group rings of free Burnside groups of odd exponent $n > 10^{10}$. A slight modification of the proof of Ivanov and Mikhailov with the use the monograph [3] allows to strengthen this result. The following statement is true.

Theorem 1.1. *Let $B(m,n)$ be the free Burnside group of rank $m \geq 2$ and odd exponent $n \geq 665$, and a_1, a_2 be free generators of $B(m,n)$. Denote $c := a_1a_2a_1^{-1}a_2^{-1}$ and let*

$$A := (1 + c + \cdots + c^{n-1})(1 - a_1a_2a_1^{-1}), B := (1 - a_1)(1 + a_2 + \cdots + a_2^{n-1}).$$

Then $AB = 0$ in $\mathbb{Z}[B(m,n)]$ and A, B is a nontrivial pair of zero-divisors in $\mathbb{Z}[B(m,n)]$.

The following Lemma 1.2, proved in [2], play a key role in the prove of Theorem 1.1.

Lemma 1.2. *Suppose that G is a group, $a, b \in G$, $d := aba^{-1}$, the elements $c := aba^{-1}b^{-1}$ and b have an order $n > 1$, the cyclic subgroups $\langle c\rangle, \langle ab^i\rangle$ are nontrivial, antinormal and $d \notin \langle c\rangle, c^jd \notin \langle ab^i\rangle$ for all $i, j \in \{0, 1, \ldots, n-1\}$. Then equalities*

$$(1.1) \qquad (1 + c + \cdots + c^{n-1})(1 - d) = XC, \quad (1 - a)(1 + b + \cdots + b^{n-1}) = DY,$$

where $X, Y \in \mathbb{Z}[G], C, D \in \mathbb{Z}[H]$, H is a cyclic subgroup of G, and $CD = 0$, are impossible.

Recall that a subgroup K of a group G is called antinormal if, for every $g \in G \setminus K$ is true the equality $gKg^{-1} \cap K = 1$.

Proof. Let $F_m = \langle x_1, x_2, \ldots, x_m\rangle$ be a free group of rank m with free generators $x_1, x_2, \ldots, x_m$ and $B(m,n) = F_m/F_m^n$ be the free m-generator Burnside group of exponent n, where F_m^n is the subgroup generated by all nth powers of elements of F_m.

Let $a_1, a_2, dots, a_m$ be free generators of $B(m,n)$, where a_i is the image of x_i, $i = 1, \ldots, m$, under the natural homomorphism $F_m \mapsto B(m,n) = F_m/F_m^n$.

From definition of free Burnside group it follows that if $G = \langle g_1, g_2\rangle$ is generated by elements g_1, g_2 and G has exponent n, i.e. $G^n = \{1\}$, then G is a homomorphic image of $B(m,n)$ for all $m \geq 2$. Since $g_1g_2g_1^{-1}g_2^{-1}$, g_2, $g_1g_2^i$, $i = 0, \ldots, n-1$, are elementary periods of rank 1 (see def. in [3] I, 4.10), therefore elements $a_1a_2a_1^{-1}a_2^{-1}$, a_2, $a_1a_2^i$, $i = 0, \ldots, n-1$, have order n in $B(m,n)$ for all $m \geq 2$.

In addition $a_1a_2a_1^{-1} \notin \langle [a_1, a_2]\rangle$, because in other case after abelanization contradiction would take place. Analogously $[a_1, a_2]^j a_1a_2a_1^{-1} \notin \langle a_1a_2^i\rangle$ in $B(m,n)$ for all $i, j \in \{0, 1, \ldots, n-1\}$.

Lets proof that for all odd $n \geq 665$ every maximal cyclic subgroup of $B(m,n)$ is antinormal in $B(m,n)$. Let intersection $\langle a\rangle \cap \langle xax^{-1}\rangle$ of cyclic subgroups $\langle a\rangle$ and $\langle xax^{-1}\rangle$ is nontrivial subgroup, i.e. $\langle a\rangle \cap \langle xax^{-1}\rangle \neq \{1\}$, where $x \notin \langle a\rangle$. It means, that element x normalize some nontrivial cyclic subgroup generated by some power a^k of element a. Therefore subgroup generated by elements x and a^k is finite subgroup of group $B(m,n)$. By famous theorem of S. I. Adian see [3, Theorem VII.1.8] all finite subgroups of group $B(m,n)$ are cyclic. Hence elements x and a^k are in the same cyclic subgroup of group $B(m,n)$. Since subgroup generated by an element a is maximal subgroup, contradiction with $x \notin \langle a\rangle$ condition follows. Therefore maximal cyclic subgroup of group $B(m,n)$ is antinormal. Cyclic subgroups $\langle [a_1, a_2]\rangle$, $\langle a_1a_2^i\rangle$, $i = 0, \ldots, n-1$, are of order n in $B(m,n)$ as they are generated by images of elementary periods of rank 1. Since $B(m,n)$ has exponent n (i.e. all nth powers of elements of $B(m,n)$ equals to 1), therefore these subgroups $\langle [a_1,a_2]\rangle$, $\langle a_1a_2^i\rangle$, $i=0,\dots,n-1$, are maximal cyclic subgroups.

According to the above approval about antinormal maximal subgroups, these subgroups are antinormal. Now we can see that all conditions of Lemma 1.2 are satisfied for elements $a=a_1$, $b=a_2$, $c=[a_1,a_2]$, $d=a_1a_2a_1^{-1}$ of $B(m,n)$. Hence, Lemma 1.2 applies and yields that equalities (1.1) are impossible. Furthermore, it is easy to see that $(1+c+\cdots+c^{n-1})(1-d)\neq 0$ because $c^id\neq 1$, $i=0,\dots,n-1$, and $(1-a)(1+b+\cdots+b^{n-1})\neq 0$ because $ab^j\neq 1$, $j=0,\dots,n-1$. Finally, repeating the actions of [2], we show that

$$(1+c+\cdots+c^{n-1})(1-d)(1-a)(1+b+\cdots+b^{n-1})=0.$$

Given the brevity of the proof, we present it below. Note $d=cb$ and $da=ab$, hence, assuming that $i_1,j_1,\dots,i_4,j_4$ are arbitrary integers that satisfy $0\leq i_1,j_1,\dots,i_4,j_4\leq n-1$, we have

$$\begin{aligned}
&(1+c+\cdots+c^{n-1})(1-d)(1-a)(1+b+\cdots+b^{n-1})=\\
&=\left(\sum_{i_1}c^{i_1}-\sum_{i_2}c^{i_2}d-\sum_{i_3}c^{i_3}a+\sum_{i_4}c^{i_4}da\right)\left(\sum_{j_1}b^{j_1}\right)=\\
&=\left(\sum_{i_1}c^{i_1}-\sum_{i_2}c^{i_2}cb-\sum_{i_3}c^{i_3}a+\sum_{i_4}c^{i_4}ab\right)\left(\sum_{j_1}b^{j_1}\right)=\\
&=\sum_{i_1,j_1}c^{i_1}b^{j_1}-\sum_{i_2,j_2}c^{i_2+1}b^{j_2+1}-\sum_{i_3,j_3}c^{i_3}ab^{j_3}+\sum_{i_4,j_4}c^{i_4}ab^{j_4+1}=0.
\end{aligned}$$

Thus $(1+c+\cdots+c^{n-1})(1-d)$ and $(1-a)(1+b+\cdots+b^{n-1})$ is a pair of zero-divisors in $\mathbb{Z}[B(m,n)]$ which is nontrivial by Lemma 1.2. Theorem 1.1 is proved. $\square$

Acknowledgments

This work was supported by the State Science Committee of the Ministry of Education and Science of the Republic of Armenia in the framework of scientific grant no. 15T-1A258.

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